e-ISSN: 2149-4584

Erzincan University Journal of Science and Technology

EJSAT **2025,** 18 (3) 696-702

Research Article

On the Number of Nonempty Subsets in a Given Set

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Received: 27/05/2024**, Revised:** 30/12/2024**, Accepted:** 13/01/2025**, Published:** 31/12/2025

Abstract

Sets contain some numbers according to their properties and can help us to write some series. At this point, thanks to the equations arising from the mathematical series, the existence of certain proofs can be proved in which generalizations can be expressed. In addition, the proofs can be tools or results in reaching new generalizations. In this direction, the aim of this study is to prove a sum sequence for the calculation of all subsets of a set other than emptiest set in which there are ordered sums of its elements except the empty set. In this direction, a different format of the summation series formed by subtracting the empty set from the number of all subsets that form the ordered sums of a set with n elements different from the empty set is obtained and the equality is proved by inductive proof method. Because of, the obtained general equality is presented as a new sum series. It is predicted that new generalizations can be reached thanks to this equality.

Keywords: Set, sum series, ordered sums, subset.

Bir Kümede Sıralı Toplamların Bulunduğu Tüm Alt Kümelerin Toplam Dizisi Üzerine

Öz

Kümeler özelliklerine göre birtakım sayıları ihtiva etmekte ve bazı serileri yazabilmemize yardımcı olabilmektedir. Bu noktada oluşturulan matematiksel serilerden doğan eşitlikler sayesinde genellemelerin ifade edilebileceği birtakım ispatların varlığı kanıtlanabilmektedir. Ayrıca bu ispatlar yeni genellemelere ulaşmada birer araç veya sonuç olabilmektedir. Bu doğrultuda çalışmanın amacı boş kümeden farklı bir kümede boş küme hariç elemanlarının sıralı toplamlarının bulunduğu tüm alt kümelerinin hesabına yönelik bir toplam dizisinin ispatını yapabilmektir. Bu doğrultuda boş kümeden farklı n elemanlı bir kümenin sıralı toplamlarını oluşturan bütün alt kümelerinin sayısından boş küme çıkartılarak oluşturulan toplam serisinin bir farklı formatı elde edilmiş ve tümevarım ispat yöntemiyle eşitlik ispatlanmıştır. Sonuç olarak elde edilen genel eşitlik yeni bir toplam serisi olarak sunulmuştur. Bu eşitlik sayesinde yeni genellemelere ulaşılabileceği öngörülmektedir.

Anahtar Kelimeler: Küme, toplam serisi, sıralı toplamlar, alt küme.

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Cite this Article: Erzincan University Journal of Science and Technology 2025, 18(3)696-702. https://doi.org/10.18185/erzifbed.1490639

1. Introduction and Preliminaries

The concept of the proof in mathematics has been considered extremely important. It has appeared in front of human beings while discussing many mathematical concepts. With the existence of mathematics, the abstract dimension of this science has become more understandable thanks to propositions, theorems and proofs related to these theorems. The first proof recorded in history belongs to the Babylonians. Thales applied the first formal mathematical proof in geometry [1]. This proof is extremely important in revealing the relationship between abstract concepts and concrete concepts. One of the main purposes of different types of the proofs in mathematics is to express the relationships between abstract concepts, to interpret these relationships correctly and to reach new knowledge. At this point, it is possible to obtain new knowledge and advance mathematical knowledge through proofs.

Although the proof condition can be realized in different types, in mathematical theorems involving algebraic expressions and equations, inductive proof is a frequently used type of the proof. As a matter of fact, according to Yıldırım (2011), Euclid's contribution to geometry lies not in presenting original knowledge, but in presenting previously known proofs in a deductive system [2]. In this presentation, from a small number of axioms, postulates and definitions selected as premises, the proofs of all the remaining propositions are given by deductive inference. The propositions proved in this axiomatic system constitute the theorems of the system. It was observed that Euclid, while making proofs, adhered to the requirement of Aristotle, who lived a period before him, that proofs should be made by applying the least number of assumptions [3].

While performing operations on sets, series are used from time to time, and when calculating the sum of subset series, operations can be performed quickly. At this point, number theory comes into play and contributes to mathematics in analyzing certain relationships. One of the important research topics for number theory is to characterize numbers and to evaluate the relations or correlations of numbers with each other. Many problems in number theory can essentially be transformed into the number of subsets of a set with a finite number of elements under certain conditions. Therefore, the result of the problem can be evaluated with different approaches such as reduction, coding, and generating function [4]. In addition, in the proof of the theorem to be constructed in this study, the concept of combination appears as an important element for the solution. Combination: Given $r, n \in N$ and $r \leq n$, each of the subsets of a set A with n elements and r elements is called an r′ combination of a set A [5].

One of the important issues in number theory is to characterize numbers and to examine their relationships and relations with each other. In number theory, many problem situations can actually be transformed into the number of subsets of a finite number of elements with certain conditions [6]. As a matter of fact, one of the main objectives of mathematical science is to extend the mathematical structures studied to larger, inclusive mathematical structures and to obtain general mathematical results by working on these extended structures [7]. Although syntactic or sequential proof is based on the formal inference process, it is said that semantic proof production can be fed by intuitive inferences [8]. Therefore, since mathematics is a cumulative science and there is a relational structure between mathematical concepts, it can be said that it is a necessity to build new knowledge on the previous ones [9]. In this direction, in this study, sum series were handled together with the concept of combination, and a generalization was reached for an equality related to sets by inductive proof.

2. Main Theorems and Proofs

In this section, a different format of the sum series formed by subtracting the empty set from the number of all subsets forming the ordered sums of an n-element set is obtained. In the study, a sum sequence for the calculation of all subsets of a set with ordered sums of its elements except the empty set was proved by inductive proof. In mathematics, induction is a powerful proof technique and has a systematic structure as in other proof techniques [10], [11].

Now we have the following:

Theorem 1: For a series of sums generated by subtracting the empty set from the number of all subsets of a set of n elements that form ordered sums. Then we have

$$\sum_{k=1}^{n} 2^{k-1}(\mathrm{n}-\mathrm{k}) = \sum_{k=2}^{n} \binom{n}{k}. \tag{1.1}$$

Proof: Let $A=\{a_1, a_2, \dots a_n\}$ be a set of n elements. Then for the $n-1$ elements, each of the ordered binary sums selected from this set, we can write

$a_1+a_2, a_1+a_3, a_1+a_4, \dots a_1+a_n$,

and for the $n-2$ elements, we have

$a_2+a_3, a_2+a_4, a_2+a_5, \dots a_2+a_n$.

At last, for one element, we get

$a_{n-1}+a_n$.

Thus, we obtain

$$C(n,2) = \frac{n(n-1)}{2}. \tag{1.2}$$

Now, if we write each of the ordered triple sums selected from this set, then for the $n-2$ elements, we have

$a_1+a_2+a_3, a_1+a_2+a_4, \dots a_1+a_2+a_n,$

and for the $n-3$ elements, we have

$a_1+a_3+a_4, a_1+a_3+a_5, \dots a_1+a_3+a_n,$

At last, for one element, we get

$a_1+a_{n-1}+a_n$,

Also with similar way, for the $n-3$ elements, we have

$a_2+a_3+a_4,\ a_2+a_3+a_5,\ \ldots\ a_2+a_3+a_n,$

and for the $n-4$ elements, we get

$a_2+a_4+a_5,\ a_2+a_4+a_6,\ \ldots\ a_2+a_4+a_n,$

At last, for one element, we get

$a_2+a_{n-1}+a_n.$

Indeed, ın this way the process is executed for the sum of all ternary elements in the last step, for one element, we get

$a_{n-2}+a_{n-1}+a_n.$

Thus, the triple sums are given by

$$C(n,\ 3)=\frac{n(n-1)(n-2)}{3!}\ . \tag{1.3}$$

When the process is executed for the sum of all elements, at each step the sum results in a combination expression and at step n, for one element, we have

$a_1+a_2+\ldots+a_n.$

Thus, with the n elements sum, we get

$$C(n,\ n)=1. \tag{1.4}$$

Now, for given all these sums, the number of terms can be expressed by

$$\sum_{k=2}^{n}\binom{n}{k}. \tag{1.5}$$

But if this sum is considered in terms of each term, then for the number of times we can write

$n-1,\ n-2,\ n-3,\ \ldots\ ,1.$

Thus, we obtain

$1.(n-1)+2.(n-2)+4.(n-3)\ \ldots\ 2^{n-1}[n-(n-1)].$

If we generalize this sum for the grand total, we find

$$\sum_{k=1}^{n} 2^{k-1}(\mathrm{n}-\mathrm{k}). \tag{1.6}$$

Hence we get the following:

$$\sum_{k=1}^{n} 2^{k-1}(\mathrm{n}-\mathrm{k}) = \sum_{k=2}^{n} \binom{n}{k}. \qquad (1.7)$$

Now let prove equation (1.7) by induction method. As is well known that, the number of all subsets of a set of n elements is 2^n. Therefore, we find

$$\sum_{k=1}^{n} \binom{n}{k}=2^{\mathrm{n}}.$$

Also, one can write

$$\sum_{k=1}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

If the derivative of both sides of this equation is taken with respect to k, then we have

$$\sum_{k=1}^{n} k.x^{k-1} = \frac{-(n+1).x^n(1-x)+(1-x^{n+1})}{(1-x)^2}$$

For the $x=2$, we obtain

$$\sum_{k=1}^{n} k.2^{k-1} = \frac{(n+1).2^n+(1-2^{n+1})}{(1-2)^2} = n.2^n+2^n+1-2.2^n=(n-1).2^n+1= \frac{1-2^n}{1-2}=2^n-1=\sum_{k=1}^{n} 2^{k-1}$$

Therefore, we have

$$\sum_{k=1}^{n} 2^{k-1}(n-k) = \sum_{k=2}^{n} \binom{n}{k}. \qquad (1.8)$$

Thus from (1.8) we can write

$$\sum_{k=1}^{n} 2^{k-1}(n) - \sum_{k=1}^{n} 2^{k-1}(k) = 2^n-C(n,0)-C(n,1)$$

and

$$n.\left(\frac{1-2^n}{1-2}\right) - \left[\frac{(n+1).2^n+(1-2^{n+1})}{1}\right] = 2^n-1-n.$$

Using -1 is in the last equation, we get

$$-n+2^n.n-(n+1).2^n-1+2^{n+1}=2^n-1-n.$$

If the expressions -1 and $-n$ on both sides of the equation are simplified. We get

$$2^n.n-(n+1).2^n+2^{n+1}=2^n.$$

Hence we obtain

$$(2^n.n-2^n.n)-2^n+2.2^n=2^n$$

Hence we get

$2^n = 2^n$

Thus, the proof is completed.

3. Conclusion

With the help of numbers and series, certain relationships and generalizations can be reached through consecutive sums performed on a set. When evaluating the relationship between clusters and their subsets, sum series can be utilized from time to time and some inferences can be reached at this point. Therefore, in this study, a different format of the summation series formed by subtracting the empty set from the number of all subsets forming the ordered sums of an *n* element cluster was tried to be determined.

Starting from this point: For the sum series generated by subtracting the empty set from the number of all subsets of set of the *n* elements that form the ordered sums of set of the *n* elements, we get the equation (1.1). Thus if we collect the result, we can give the following:

$$\sum_{k=1}^{n} 2^{k-1}(n) - \sum_{k=1}^{n} 2^{k-1}(k) = \sum_{k=1}^{n} 2^{k-1}(\mathrm{n}-\mathrm{k}) = \sum_{k=2}^{n} \binom{n}{k}$$

Because of the above equation, it is proved by induction that the combinatorial formula obtained for the sum series formed by subtracting the empty set from the number of all subsets forming the ordered sums of a set of *n* elements can actually be expressed by a power series. Thanks to this equality, it is predicted that different proofs in the literature on ordered sums can be carried out and new generalizations can be reached.

Ethics in Publishing

There are no ethical issues regarding the publication of this study

Author Contributions

All authors contributed equally to the study.

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