Turk. J. Math. Comput. Sci.
16(2)(2024) 450–462
© MatDer
DOI : 10.47000/tjmcs.1491105

# Existence Results for a Class of $\psi$-Hilfer Fractional Hybrid Differential Equations

Ayoub Louakar[1,*], Hamid Lmou[1], Ahmed Kajouni[1], Khalid Hilal[1]

[1] *Laboratory of Applied Mathematics and Scientific Competing, Sultan Moulay Slimane University, Beni Mellal, Morocco.*

Received: 28-05-2024 • Accepted: 17-12-2024

Abstract. This study investigates the existence and uniform local attractiveness of solutions for a class of fractional $\psi$-Hilfer hybrid differential equations within Banach algebras. Utilizing advanced hybrid fixed-point theory, we derive results that not only establish conditions for the existence of solutions but also demonstrate their uniform local attractiveness. Our findings offer valuable insights into the behavior of these fractional differential equations and provide a solid theoretical foundation for future research and applications in this field.

*2020 AMS Classification:* 26A33, 34A08, 47H10

**Keywords:** $\psi$-Hilfer fractional derivative, uniformly locally attractive, hybrid fixed-point theory, fractional differential equation.

## 1. Introduction

The idea of derivatives of arbitrary order, which is essential to fractional calculus and provides a great tool for characterizing the inherent properties of many materials and processes, has maintained its appeal to a large number of scientists in recent years, one can see [11, 15]. There are several representations and definitions for the derivative of fractional order; the most widely used ones are Riemann-Liouville and Caputo. In [13], Hilfer discussed fractional time development in physical processes and presented an extension of the Riemann-Liouville fractional derivative and Caputo fractional derivative. It was called a generalized derivative by the author, but the Hilfer fractional derivative was given to it later recently. This operator has two parameters, $\alpha$ and $\beta$, which can be reduced to the definitions of the Riemann-Liouville fractional derivative and Caputo fractional derivative, respectively, if $\beta = 0$ and $\beta = 1$, respectively. Sousa and de Oliveira [26] introduced the new version of the Hilfer fractional derivitive with respect to another function $\psi$. They presented a generalization concerning these derivatives in which they combined several formulations, including the traditional Caputo and Riemann-Liouville operators, and proposed a new fractional differential operator, known as the fractional $\psi$-Hilfer operator.

Fractional differential equations involving the $\psi$-Hilfer fractional derivative have received a lot of attention recently. Various properties of these equations, such as the attractivity of solutions, Ulam-Hyers stability, and the existence and uniqueness of solutions, have been thoroughly investigated by researchers. For instance, Ahmad et al. [3] studied the existence, uniqueness, and stability of implicit switched coupled $\psi$-Hilfer fractional differential equations, while Abdo et al. [1] investigated Ulam-Hyers-Mittag-Leffler stability for $\psi$-Hilfer problems with fractional order and infinite delay.

*Corresponding Author
Email addresses: ayoublouakar007@gmail.com (A. Louakar), hamid.lmou@usms.ma (H. Lmou), Ahmed.kajouni@usms.ma (A. Kajouni), hilalkhalid2005@yahoo.fr (K. Hilal)

Moreover, for coupled systems including the Hilfer fractional Langevin equation with non-local boundary conditions, Hilal et al. [12] examined existence and stability results. This study was extended to the Hilfer Langevin fractional pantograph differential equations, addressing existence and uniqueness, by Lmou et al. [16]. A study of the stability of fractional differential equations with impulsive conditions was presented by Shah et al. [23] in a related work. Si Bachir et al. [24] investigated $\psi$-Hilfer hybrid fractional differential equations in more detail, as well as the existence and attractivity of solutions; Sousa and Capelas Oliveira [25] constructed a Gronwall inequality and used the Hilfer operator to study the Cauchy-type problem.

Researchers have also looked on hybrid fractional differential equations. The fractional derivative of an unknown function hybrid with nonlinearity dependent on it is included in this class of problems. A brief history of fractional differential equations and hybrid differential equations will be given, (see [2, 4, 5, 8–10, 14, 17–22]).

In [6], Dhage and Lakshmikantham initiated the study of the first-order hybrid differential equation

$$\begin{cases} \frac{d}{d\iota}\left(\frac{w(\iota)}{f(\iota,w(\iota))}\right) = g(\iota, w(\iota)), \quad \iota \in [0,T], \\ w(0) = w_0 \in \mathbb{R}, \end{cases}$$

where $f \in C([0,T]\times\mathbb{R},\mathbb{R}^*)$ and $g \in C([0,T]\times\mathbb{R},\mathbb{R})$.

Zhao et al. [27] discussed the following hybrid fractional initial value problem

$$\begin{cases} \mathfrak{D}^p\left(\frac{w(\iota)}{f(\iota,w(\iota))}\right) = g(\iota, w(\iota)), \quad \iota \in [0,T], \\ w(0) = w_0 \in \mathbb{R}, \end{cases}$$

where $\mathfrak{D}^p$ is the Riemann-Liouville fractional derivative of order $0 < p < 1$, $f \in C([0,T]\times\mathbb{R},\mathbb{R}^*)$, and $g \in C([0,T]\times\mathbb{R},\mathbb{R})$.

In [5], an initial value problem was discussed for hybrid fractional differential equations involving $\psi$-Hilfer fractional derivative of the form

$$\begin{cases} {}^H\mathfrak{D}^{p,q;\psi}\left(\frac{w(\iota)}{f(\iota,w(\iota))}\right) = g(\iota, w(\iota)), \quad \iota \in [0,T], \\ \mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}\left(\frac{w(0)}{f(0,w(0))}\right) = w_0 \in \mathbb{R}, \end{cases}$$

where ${}^H\mathfrak{D}^{p,q;\psi}$ is the $\psi$-Hilfer fractional derivative with $0 < p < 1, 0 \le q \le 1, p \le \gamma = p + q - pq < 1$, $f \in C([0,T]\times\mathbb{R},\mathbb{R}^*)$, and $g \in C([0,T]\times\mathbb{R},\mathbb{R})$.

In this work, we study the existence and attractivity of solutions to the following problem

$$\begin{cases} {}^H\mathfrak{D}^{p_1,q_1;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q_2;\psi}_{0^+,\iota}\dfrac{w(\iota)}{\mathcal{G}(\iota,w(\iota))} + f_1(\iota,w(\iota))\right) = f_2(\iota,w(\iota)), \quad \iota \in \mathbb{R}_+ := [0,+\infty), \\ \dfrac{w(0)}{\mathcal{G}(0,w(0))} = 0, \\ \mathfrak{I}^{1-\gamma_1-p_2;\psi}_{0^+,\iota}\left(\dfrac{w(0)}{\mathcal{G}(0,w(0))}\right) = w_0, \quad w_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathfrak{I}^{1-\gamma_1-p_2;\psi}_{0^+,\iota}$ and $\mathfrak{D}^{p_i,q_i;\psi}_{0^+,\iota}$, $i = 1, 2$ represent the Riemann–Liouville fractional integral and the $\psi$-Hilfer fractional derivative in order $p_i$ and type $q_i$ respectively. $0 < p_i < 1, 0 \le q_i \le 1$; $1 < p_1 + p_2 \le 2$. $f_1, f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^*$ are given functions.

This work is divided into five sections. Basic definitions of $\psi$-Hilfer fractional calculus, essential lemmas, and certain fixed-point theorems are given in Section 2. In Section 3, we derive the formula of solution for problem (1.1). During Section 4, we discuss whether or not problem (1.1) has an existence of solutions. Section 5 discusses how attractivity of solutions can be achieved in the above problem.

**Special Cases:**

- For $q_1, q_2 = 1$ and $\psi(\iota) = \iota$, we get hybrid Caputo fractional differential equation of the form

$$ {}^C\mathfrak{D}^{p_1}\left({}^C\mathfrak{D}^{p_2}\frac{w(\iota)}{\mathcal{G}(w,w(\iota))} + f_1(\iota,w(\iota))\right) = f_2(\iota,w(\iota)),$$
$$w(\iota)|_{\iota=0} = 0.$$

- For $q_1, q_2 = 0$ and $\psi(\iota) = \iota$, we get the hybrid differential equation involving Riemann-Liouville fractional derivative of the form

$$^{RL}\mathfrak{D}^{p_1}\left(^{RL}\mathfrak{D}^{p_2}\frac{w(\iota)}{\mathcal{G}(w, w(\iota))} + f_1(\iota, w(\iota))\right) = f_2(\iota, w(\iota)),$$
$$w(\iota)|_{\iota=0} = 0.$$

- For $p_1 = 0, q_2 = 0, f_1 = 0$, and $\psi(\iota) = \iota$, we obtain the following problem

$$^{RL}\mathfrak{D}^{p_2}\left(\frac{w(\iota)}{\mathcal{G}(w, w(\iota))}\right) = f_2(\iota, w(\iota)),$$
$$w(\iota)|_{\iota=0} = 0,$$

  which is studied in [27].
  For $p_2 = 1$, we obtain

$$\frac{d}{d\iota}\left(\frac{w(\iota)}{\mathcal{G}(w, w(\iota))}\right) = f_2(\iota, w(\iota)),$$
$$w(\iota)|_{\iota=0} = 0,$$

  which is investigated in [6] by Dhage and Lakshmikantham.

- Also, a natural consequence of problem (1.1), if we take $f_1(\iota, w(\iota)) = \lambda w(\iota)$, we obtain the famous fractional Langvin equation

$$\begin{cases} ^{H}\mathfrak{D}^{p_1,q_1;\psi}_{0^+,\iota}\left(^{H}\mathfrak{D}^{p_2,q_2;\psi}_{0^+,\iota}\dfrac{w(\iota)}{\mathcal{G}(\iota, w(\iota))} + \lambda\right)w(\iota) = f_2(\iota, w(\iota)), \\ \dfrac{w(0)}{\mathcal{G}(0, w(0))} = 0, \\ \mathfrak{I}^{1-\gamma_1-p_2;\psi}_{0^+,\iota}\left(\dfrac{w(0)}{\mathcal{G}(0, w(0))}\right) = w_0, \quad w_0 \in \mathbb{R}. \end{cases}$$

## 2. Preliminaries

Let $\gamma = \gamma_1 + p_2$. Define on $[a, b], (0 < a < b < \infty)$ the weighted space

$$C_{1-\gamma;\psi}([a, b]) = \left\{w : [a, b] \to \mathbb{R} : (\psi(\iota) - \psi(0))^{1-\gamma} w(\iota) \in C([a, b])\right\},$$

with the norm

$$\|w\|_{C_{1-\gamma;\psi}} = \sup_{\iota\in[a,b]} \left|(\psi(\iota) - \psi(0))^{1-\gamma} w(\iota)\right|.$$

Let $\mathcal{BC} := \mathcal{BC}(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. By $\mathcal{BC}_{1-\gamma} = \mathcal{BC}_{1-\gamma}(\mathbb{R}_+)$, we denote the weighted space of all bounded and continuous functions defined by

$$\mathcal{BC}_{1-\gamma} = \left\{\phi : \mathbb{R}_+ \to \mathbb{R} : (\psi(\iota) - \psi(0))^{1-\gamma}\phi(\iota) \in \mathcal{BC}\right\},$$

with the norm

$$\|\phi\|_{\mathcal{BC}_{1-\gamma}} = \sup_{\iota\in\mathbb{R}_+} \left|(\psi(\iota) - \psi(0))^{1-\gamma} \phi(\iota)\right|.$$

Let $\mathcal{X} = \left(\mathcal{BC}_{1-\gamma}(\mathbb{R}_+), \|\cdot\|_{\mathcal{BC}_{1-\gamma}}\right)$ be a Banach algebra where $(wz)(\iota) = w(\iota)z(\iota), \iota \in \mathbb{R}_+$ is the definition of the product of vectors.
Additionally, we recall some results and properties from the $\psi$-fractional calculus.

**Definition 2.1** ( [26])**.** For $p > 0, w \in \mathbb{L}^1([a, b], \mathbb{R})$, additionally $\psi \in C^n([a, b], \mathbb{R})$, the fractional $\psi$-Riemann-Liouville operator with order $p$ of $w$ can be written as

$$\mathfrak{I}^{p;\psi}_{a^+,\iota} w(\iota) = \frac{1}{\Gamma(p)} \int_a^\iota \psi'(s)(\psi(\iota) - \psi(s))^{p-1} w(s)ds,$$

in which $\psi'(\iota) > 0$, $\forall\iota \in [a, b]$.

**Definition 2.2** ( [26]). For $0 < p < 1$ and $w, \psi \in C^{n-1}([a,b],\mathbb{R})$ with $\psi'(\iota) > 0$, $\forall \iota \in [a,b]$, the fractional $\psi$-Hilfer derivative operator with order $p$ and type $0 \leq q \leq 1$ of $w$ is represented as

$$^{H}\mathfrak{D}^{p,q;\psi}_{a^+,\iota} w(\iota) = \mathfrak{I}^{q(1-p);\psi}_{a^+,\iota}\left(\frac{1}{\psi'(\iota)}\frac{d}{d\iota}\right)\mathfrak{I}^{(1-q)(1-p);\psi}_{a^+,\iota} w(\iota).$$

**Lemma 2.3** ( [26]). *Let $0 < p < 1$, $0 \leq q \leq 1$, $w \in C^1([a,b],\mathbb{R})$, then*

$$\mathfrak{I}^{p;\psi}_{a^+,\iota}\, {}^{H}\mathfrak{D}^{p,q;\psi}_{a^+,\iota} w(\iota) = w(\iota) - \frac{(\psi(\iota)-\psi(a))^{\varrho-1}}{\Gamma(\varrho)}\mathfrak{I}^{1-\varrho;\psi}_{a^+,\iota} w(a),$$

*where $\varrho = p + q(1-p)$.*

**Lemma 2.4** ( [25, 26]). *Let $p, q > 0, \delta > p$ and $w \in C([a,b],\mathbb{R})$. Following that $\forall \iota \in [a,b]$ there is*

*(i)* $\mathfrak{I}^{p;\psi}_{a^+,\iota}\mathfrak{I}^{q;\psi}_{a^+,\iota} w(\iota) = \mathfrak{I}^{p+q;\psi}_{a^+,\iota} w(\iota)$,

*(ii)* ${}^{H}\mathfrak{D}^{p,q;\psi}_{a^+,\iota}\mathfrak{I}^{p;\psi}_{a^+,\iota} w(\iota) = w(\iota)$,

*(iii)* $\mathfrak{I}^{p;\psi}_{a^+,\iota}(\psi(\iota)-\psi(a))^{q-1} = \frac{\Gamma(p)}{\Gamma(p+q)}(\psi(\iota)-\psi(a))^{p+q-1}$,

*(iv)* $\mathfrak{I}^{q;\psi}_{a^+,\iota}(1) = \frac{(\psi(\iota)-\psi(a))^{q}}{\Gamma(q+1)}$,

*(v)* ${}^{H}\mathfrak{D}^{p,q;\psi}_{a^+,\iota}(\psi(\iota)-\psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta-p)}(\psi(\iota)-\psi(a))^{\delta-p-1}$,

*(vi)* ${}^{H}\mathfrak{D}^{p,q;\psi}_{a^+,\iota}(\psi(\iota)-\psi(a))^{\delta-1} = 0, \quad 0 < \delta < 1.$

**Lemma 2.5** ( [24]). *Let $0 \leq \gamma < 1$ and $f \in C_{1-\gamma;\psi}[a,b]$. Then,*

$$\mathfrak{I}^{p;\psi}_{a^+,\iota} f(a) = \lim_{\iota \to a+}\mathfrak{I}^{p;\psi}_{a^+,\iota} f(\iota) = 0, \quad 0 \leq 1-\gamma < p.$$

**Theorem 2.6** ( [7]). *Assume that the Banach algebra $\mathcal{X}$ has a non-empty closed, convex, and bounded subset $\mathcal{S}$. Take $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{B} : \mathcal{S} \to \mathcal{X}$ be two operators in which*
*(i) $\mathcal{A}$ is Lipschitzian with a Lipschitz constant $\alpha$;*
*(ii) $\mathcal{B}$ is completely continuous;*
*(iii) $w = \mathcal{A}w\mathcal{B}z \Longrightarrow w \in \mathcal{S}$ for all $z \in \mathcal{S}$;*
*(iv) $\alpha\mathcal{M} < 1$ where $\mathcal{M} = \sup\{\|Bw\| : w \in \mathcal{S}\}$.*
*Hence, one exists $w \in \mathcal{S}$, in which $w = \mathcal{A}w\mathcal{B}w$.*

## 3. Formula of Solutions

**Lemma 3.1.** *Take $h_1, h_2 \in C(J,\mathbb{R})$; $J := [0,d]$, $d > 0$. Then, it follows that the problem :*

$$\begin{cases} {}^{H}\mathfrak{D}^{p_1,q_1;\psi}_{0^+,\iota}\left({}^{H}\mathfrak{D}^{p_2,q_2;\psi}_{0^+,\iota}\dfrac{w(\iota)}{\mathcal{G}(\iota,w(\iota))} + h_1(\iota)\right) = h_2(\iota), a.e. \quad \iota \in J, \\ \dfrac{w(0)}{\mathcal{G}(0,w(0))} = 0, \\ \mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}\left(\dfrac{w(0)}{\mathcal{G}(0,w(0))}\right) = w_0, \quad \gamma = \gamma_1 + p_2, \end{cases} \tag{3.1}$$

*is equivalent to the equation*

$$w(\iota) = \mathcal{G}(\iota,w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)}(\psi(\iota)-\psi(0))^{\gamma-1} + \mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota} h_2(\iota) - \mathfrak{I}^{p_2;\psi}_{0^+,\iota} h_1(\iota)\right\}. \tag{3.2}$$

*Proof.* Taking the fractional integral operator of order $p_1$ on each side of (3.1). Then, utilizing Lemma 2.3, we arrive at

$$^{H}\mathfrak{D}^{p_2,q_2;\psi}_{0^+,\iota}\frac{w(\iota)}{\mathcal{G}(\iota,w(\iota))} + h_1(\iota) = \mathfrak{I}^{p_1;\psi}_{0^+,\iota} h_2(\iota) + e_1\frac{(\psi(\iota)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}. \tag{3.3}$$

Utilizing again Lemma 2.3, we can get by taking fractional integral operator of order $p_2$ on each side of (3.3)

$$\frac{w(\iota)}{\mathcal{G}(\iota,w(\iota))} = \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} h_2(\iota) - \mathfrak{I}^{p_2;\psi}_{0^+,\iota} h_1(\iota) + e_1\frac{(\psi(\iota)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} + e_2\frac{(\psi(\iota)-\psi(0))^{\gamma_2-1}}{\Gamma(\gamma_2)}, \tag{3.4}$$

where $e_1$ and $e_2$ are arbitrary constants.
In (3.4), the boundary condition $w(0) = 0$ leads to $e_2 = 0$, and therefore we get

$$\frac{w(\iota)}{\mathcal{G}(\iota, w(\iota))} = \Im_{0^+,\iota}^{p_1+p_2;\psi} h_2(\iota) - \Im_{0^+,\iota}^{p_2;\psi} h_1(\iota) + e_1 \frac{(\psi(\iota) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}. \tag{3.5}$$

In addition, if we combine the condition $\Im_{0^+,\iota}^{1-\gamma;\psi}\left(\frac{w(0)}{\mathcal{G}(0,w(0))}\right) = w_0$ with the value of (3.5), we obtain $e_1 = w_0$.
We substitute $e_1$ in (3.5), we obtain

$$w(\iota) = \mathcal{G}(\iota, w(\iota)) \left\{ \frac{w_0}{\Gamma(\gamma)} (\psi(\iota) - \psi(0))^{\gamma-1} + \Im_{0^+,\iota}^{p_1+p_2;\psi} h_2(\iota) - \Im_{0^+,\iota}^{p_2;\psi} h_1(\iota) \right\}.$$

On the other hand, suppose $w$ can be the unique solution satisfying (3.2). Then, (3.2) may expressed as

$$\frac{w(\iota)}{\mathcal{G}(\iota, w(\iota))} = \frac{w_0}{\Gamma(\gamma)} (\psi(\iota) - \psi(0))^{\gamma-1} + \Im_{0^+,\iota}^{p_1+p_2;\psi} h_2(\iota) - \Im_{0^+,\iota}^{p_2;\psi} h_1(\iota). \tag{3.6}$$

Taking the fractional $\psi$-Hilfer derivative operator ${}^{H}\mathfrak{D}_{a^+,\iota}^{p_2,q_2;\psi}$ on both sides of (3.6), and using Lemma 2.4. Then, taking fractional $\psi$-Hilfer derivative operator ${}^{H}\mathfrak{D}_{a^+,\iota}^{p_1,q_1;\psi}$ again, we otain

$${}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q_1;\psi} \left( {}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q_2;\psi} \frac{w(\iota)}{\mathcal{G}(\iota, w(\iota))} \right) = h_2(\iota) - {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q_1;\psi} h_1(\iota).$$

So, it follows

$${}^{H}\mathfrak{D}_{a^+,\iota}^{p_1,q_1;\psi} \left( {}^{H}\mathfrak{D}_{a^+,\iota}^{p_2,q_2;\psi} \frac{w(\iota)}{\mathcal{G}(\iota, w(\iota))} + h_1(\iota) \right) = h_2(\iota), \quad \iota \in \mathbb{R}_+.$$

Now, we will show that $w$ satisfies the boundary conditions. To do this, we have $\frac{w(0)}{\mathcal{G}(0,w(0))} = 0$, and from (3.6) and Lemma 2.4, we get $\Im_{0^+,\iota}^{1-\gamma;\psi}\left(\frac{w(0)}{\mathcal{G}(0,w(0))}\right) = w_0$.
Hence, the proof is complete. □

## 4. Existence Results

We present the following hypothesis in order to show the existence, and attractivity of solution.
$(H_1)$: $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^*$ is continuous and bounded with bound $\|\mathcal{G}\| = \sup_{(\iota,w)\in\mathbb{R}_+\times\mathbb{R}} |\mathcal{G}(\iota, w)|$. Additionally, there exists a function $\omega \in C(\mathbb{R}_+, \mathbb{R})$ with

$$|\mathcal{G}(\iota, w) - \mathcal{G}(\iota, z)| \leq \omega(\iota)|w - z|,$$

$\iota \in \mathbb{R}_+$ and $w, z \in \mathbb{R}$.
$(H_2)$: $f_1, f_2 : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ fulfill the condition of caratheodory ( i.e. continuous in $w$ for all $\iota \in \mathbb{R}$ and mesurable in $\iota$ for all $w \in \mathbb{R}$) and $\omega_1, \omega_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are continuous functions achieving the following requirements

$$|f_1(\iota, w)| \leq \omega_1(\iota) \text{ and } |f_2(\iota, w)| \leq \omega_2(\iota), \forall(\iota, w) \in \mathbb{R}_+ \times \mathbb{R}.$$

Further, assume that

$$\lim_{\iota\to\infty} (\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right)(\iota) = 0,$$

$${\omega_2}^* = \sup_{\iota\to\infty} (\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right)(\iota) < \infty,$$

and

$$\lim_{\iota\to\infty} (\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_2;\psi} \omega_1\right) \iota) = 0,$$

$${\omega_1}^* = \sup_{\iota\to\infty} (\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_2;\psi} \omega_1\right) \iota) < \infty.$$

Using Theorem 2.6, we will now discuss the existence result.

**Theorem 4.1.** *Suppose that* $(H_1)$ *and* $(H_2)$ *are valid. Further if* $\|\omega\| \left\{ |\frac{w_0}{\Gamma(\gamma)}| + {\omega_1}^* + {\omega_2}^* \right\} < 1$. *Then, there is at least one solution to problem (1.1) on* $\mathcal{X}$.

*Proof.* Define $\mathcal{S} = \{w \in \mathcal{X} : \|w\|_{\mathcal{B}C_{1-\gamma}} \leqslant \eta\}$, where $\|\mathcal{G}\| \left\{|\frac{w_0}{\Gamma(\gamma)}| + {\omega_1}^* + {\omega_2}^*\right\} < \eta$. We have that $\mathcal{S}$ is a bounded subset of $\mathcal{X}$ which is closed and convex.
Define $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{X}$ and $\mathcal{B} : \mathcal{S} \longrightarrow \mathcal{X}$ by

$$(\mathcal{A}w)(\iota) = \mathcal{G}(\iota, w(\iota)), \tag{4.1}$$

and

$$(\mathcal{B}w)(\iota) = \frac{w_0}{\Gamma(\gamma)} (\psi(\iota) - \psi(0))^{\gamma-1} + \Im_{0^+,\iota}^{p_1+p_2;\psi} f_2(\iota, w(\iota)) - \Im_{0^+,\iota}^{p_2;\psi} f_1(\iota, w(\iota)). \tag{4.2}$$

The problem (1.1) is thus equivalent to the operator equation.

$$w(\iota) = \mathcal{A}w(\iota)\mathcal{B}w(\iota), \forall \iota \in \mathbb{R}_+.$$

Now, we demonstrate that $\mathcal{A}$ and $\mathcal{B}$ fulfill all requirements of Theorem 2.6 on $\mathcal{X}$.
**Step 1:** $\mathcal{A}$ is Lipschitz condition.
Take $w, z \in \mathcal{X}$ and $\iota \in \mathbb{R}_+$. By (H1), we get

$$\begin{aligned}\left|(\psi(\iota) - \psi(0))^{1-\gamma} (\mathcal{A}w(\iota) - \mathcal{A}z(\iota))\right| &= \left|(\psi(\iota) - \psi(0))^{1-\gamma} (\mathcal{G}(\iota, w(\iota)) - \mathcal{G}(\iota, z(\iota)))\right| \\ &\leq \omega(\iota) \left|(\psi(\iota) - \psi(0))^{1-\gamma} (w(\iota) - z(\iota))\right| \\ &\leq \|\omega\| \|w - z\|_{\mathcal{B}C_{1-\gamma}}.\end{aligned}$$

Therefore,

$$\|\mathcal{A}w - \mathcal{A}z\|_{\mathcal{B}C_{1-\gamma,\psi}} \leq \|\omega\| \|w - z\|_{\mathcal{B}C_{1-\gamma}}. \tag{4.3}$$

The next step is to demonstrate the completely continuous of $\mathcal{B} : \mathcal{S} \to \mathcal{X}$.
To do this, we will demonstrate the continuity, uniform boundedness, and equicontinuousness of $\mathcal{B}$.
**Step 2:** $\mathcal{B}$ is continuous.
Consider a sequence $w_n \to w$ in $\mathcal{S}$. Then, for each $\iota \in \mathbb{R}_+$, we have

$$\begin{aligned}|(\mathcal{B}w_n)(\iota) - (\mathcal{B}w)(\iota)| \leq & \frac{1}{\Gamma(p_1 + p_2)} \int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) \left|f_2(s, w_n(s)) - f_2(s, w(s))\right| ds \\ &+ \frac{1}{\Gamma(p_2)} \int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) \left|f_1(s, w_n(s)) - f_1(s, w(s))\right| ds,\end{aligned}$$

where $\mathcal{N}_\psi^p(\iota, s) = \psi'(s)(\psi(\iota) - \psi(s))^p, \quad p = p_2, p_1 + p_2$.
Then,

$$\begin{aligned}|(\psi(\iota) - \psi(0))^{1-\gamma} (\mathcal{B}w_n)(\iota) - (\mathcal{B}w)(\iota)| \leq & \frac{(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) \left|f_2(s, w_n(s)) - f_2(s, w(s))\right| ds \\ &+ \frac{(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) \left|f_1(s, w_n(s)) - f_1(s, w(s))\right| ds.\end{aligned} \tag{4.4}$$

**Case 1:** $\iota \in [0, \epsilon]$, $\epsilon > 0$. $f_1$ and $f_2$ are continuous by using the Lebesgue dominated convergence theorem and $w_n \to w$ as $n \to \infty$. Eq. (4.4) gives us

$$\|\mathcal{B}w_n - \mathcal{B}w\|_{\mathcal{B}C_{1-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.$$

**Case 2:** $\iota \in [\epsilon, \infty)$. Then, based on the hypotheses and (4.4), we obtain

$$\begin{aligned}\|\mathcal{B}w_n - \mathcal{B}w\|_{\mathcal{B}C_{1-\gamma}} \leq & \frac{2(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s)\omega_2(s) ds \\ &+ \frac{2(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s)\omega_1(s) ds.\end{aligned}$$

It follows that

$$\begin{aligned}\|\mathcal{B}w_n - \mathcal{B}w\|_{\mathcal{B}C_{1-\gamma}} \leq & 2(\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right) \iota) \\ &+ 2(\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_2;\psi} \omega_1\right) \iota).\end{aligned} \tag{4.5}$$

Since $w_n \to w$ as $n \to \infty$, $(\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right) \iota) \to 0$ as $\iota \to \infty$, and $(\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right) \iota) \to 0$ as $\iota \to \infty$ , it follows from (4.5) that

$$\|\mathcal{B}w_n - \mathcal{B}w\|_{\mathcal{B}C_{1-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.$$

**Step 3:** $\mathcal{B}(\mathcal{S} = \{\mathcal{B}w : w \in \mathcal{S}\}$ is equicontinuous and uniformly bounded on every compact subsets $[0, \epsilon]$ of $\mathbb{R}_+$, $\epsilon > 0$. Firstly, we show the uniform boundedness of $\mathcal{B}$.
Take $w \in \mathcal{S}$, we have

$$|(\mathcal{B}w)(\iota)| \leq \left|\frac{w_0}{\Gamma(\gamma)} (\psi(\iota) - \psi(0))^{\gamma-1}\right| + \frac{1}{\Gamma(p_1 + p_2)} \int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) |f_2(s, w(s))| ds + \frac{1}{\Gamma(p_2)} \int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) |f_1(s, w(s))| ds.$$

Then,

$$| (\psi(\iota) - \psi(0))^{1-\gamma} \mathcal{B}w)(\iota)| \leq \left|\frac{w_0}{\Gamma(\gamma)}\right| + \frac{(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s)\omega_2(s) ds + \frac{(\psi(\iota) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s)\omega_1(s) ds.$$

By using $(H_2)$ and taking supremum over $\iota$, we arrive at

$$\|\mathcal{B}w\|_{\mathcal{BC}_{1-\gamma}} \leq \left|\frac{w_0}{\Gamma(\gamma)}\right| + \omega_1{}^* + \omega_2{}^*. \tag{4.6}$$

Hence, $\mathcal{B}(\mathcal{S})$ is uniformly bounded.
It can now be shown that $\mathcal{B}(\mathcal{S})$ is equicontinuous. Take, $\iota_1, \iota_2 \in [0, \epsilon]$, $\iota_1 < \iota_2$ and $w \in \mathcal{S}$

$$\begin{aligned}
&\left|(\psi(\iota_2) - \psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_2) - (\psi(\iota_1) - \psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_1)\right| \\
&\leq \left|\frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_0^{\iota_2} \mathcal{N}_\psi^{p_1+p_2}(\iota_2, s) f_2(s, w(s)) ds - \frac{(\psi(\iota_1) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_0^{\iota_1} \mathcal{N}_\psi^{p_1+p_2}(\iota_1, s) f_2(s, w(s)) ds \right.\\
&\quad \left. + \frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_0^{\iota_2} \mathcal{N}_\psi^{p_2}(\iota_2, s) f_1(s, w(s)) ds - \frac{(\psi(\iota_1) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_0^{\iota_1} \mathcal{N}_\psi^{p_2}(\iota_1, s) f_1(s, w(s)) ds\right| \\
&\leq \frac{1}{\Gamma(p_1 + p_2)} \int_0^{\iota_1} \left|(\psi(\iota_2) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_1+p_2}(\iota_2, s) - (\psi(\iota_1) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_1+p_2}(\iota_1, s)\right| |f_2(s, w(s))| ds \\
&+ \frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_{\iota_1}^{\iota_2} \mathcal{N}_\psi^{p_1+p_2}(\iota_2, s) |f_2(s, w(s))| ds \\
&+ \frac{1}{\Gamma(p_2)} \int_0^{\iota_1} \left|(\psi(\iota_2) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_2}(\iota_2, s) - (\psi(\iota_1) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_2}(\iota_1, s)\right| |f_1(s, w(s))| ds \\
&+ \frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_{\iota_1}^{\iota_2} \mathcal{N}_\psi^{p_2}(\iota_2, s) |f_1(s, w(s))| ds.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|(\psi(\iota_2) - \psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_2) - (\psi(\iota_1) - \psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_1)\right| \\
&\leq \frac{1}{\Gamma(p_1 + p_2)} \int_0^{\iota_1} \left|(\psi(\iota_2) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_1+p_2}(\iota_2, s) - (\psi(\iota_1) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_1+p_2}(\iota_1, s)\right| \omega_2(s) ds \\
&+ \frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_1 + p_2)} \int_{\iota_1}^{\iota_2} \mathcal{N}_\psi^{p_1+p_2}(\iota_2, s)\omega_2(s) ds \\
&+ \frac{1}{\Gamma(p_2)} \int_0^{\iota_1} \left|(\psi(\iota_2) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_2}(\iota_2, s) - (\psi(\iota_1) - \psi(0))^{1-\gamma} \mathcal{N}_\psi^{p_2}(\iota_1, s)\right| \omega_1(s) ds \\
&+ \frac{(\psi(\iota_2) - \psi(0))^{1-\gamma}}{\Gamma(p_2)} \int_{\iota_1}^{\iota_2} \mathcal{N}_\psi^{p_2}(\iota_2, s)\omega_1(s) ds.
\end{aligned}$$

Then, based on the continuity of the functions $\omega_1$ and $\omega_2$, by defining $\omega_{1*} = \sup_{\iota\in[0,\epsilon]} \omega_1(\iota)$ and $\omega_{2*} = \sup_{\iota\in[0,\epsilon]} \omega_2(\iota)$, we obtain

$$\begin{aligned}
&\left|(\psi(\iota_2)-\psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_2)-(\psi(\iota_1)-\psi(0))^{1-\gamma}(\mathcal{B}w)(\iota_1)\right|\\
&\leq \frac{\omega_{2*}}{\Gamma(p_1+p_2)}\int_0^{\iota_1}\left|(\psi(\iota_2)-\psi(0))^{1-\gamma}\,\mathcal{N}_{\psi}^{p_1+p_2}(\iota_2,s)-(\psi(\iota_1)-\psi(0))^{1-\gamma}\,\mathcal{N}_{\psi}^{p_1+p_2}(\iota_1,s)\right|ds\\
&+\frac{\omega_{2*}\,(\psi(\iota_2)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2+1)}\,(\psi(\iota_2)-\psi(\iota_1))^{p_1+p_2}\\
&+\frac{\omega_{1*}}{\Gamma(p_2)}\int_0^{\iota_1}\left|(\psi(\iota_2)-\psi(0))^{1-\gamma}\,\mathcal{N}_{\psi}^{p_2}(\iota_2,s)-(\psi(\iota_1)-\psi(0))^{1-\gamma}\,\mathcal{N}_{\psi}^{p_2}(\iota_1,s)\right|ds\\
&+\frac{\omega_{1*}\,(\psi(\iota_2)-\psi(0))^{1-\gamma}}{\Gamma(p_2+1)}\,(\psi(\iota_2)-\psi(\iota_1))^{p_2}\longrightarrow 0,\quad \text{as}\quad \iota_1\longrightarrow\iota_2,
\end{aligned}$$

the Arzela-Ascoli theorem thus shows that $\mathcal{B}(\mathcal{S})$ is relatively compact, and hence a compact operator. From the continuity and compactness of $\mathcal{B}:\mathcal{S}\longrightarrow\mathcal{X}$, it is completely continuous.

**Step 4:** To prove $w\in\mathcal{X}, w=\mathcal{A}w\mathcal{B}z\Longrightarrow w\in\mathcal{S}$ for all $z\in\mathcal{S}$.

Take $w\in\mathcal{X}$ and $z\in\mathcal{S}$ such that $w=\mathcal{A}w\mathcal{B}z$.

$$\begin{aligned}
|\,(\psi(\iota)-\psi(0))^{1-\gamma}\,(\mathcal{B}w)(\iota)| &=|\,(\psi(\iota)-\psi(0))^{1-\gamma}\,(\mathcal{A}w\mathcal{B}z)(\iota)|\\
&=\Bigg|(\psi(\iota)-\psi(0))^{1-\gamma}\,\mathcal{G}(\iota,w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)}\,(\psi(\iota)-\psi(0))^{\gamma-1}\right.\\
&\quad\left.+\frac{1}{\Gamma(p_1+p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_1+p_2}(\iota,s)f_2\,(s,z(s))\,ds+\frac{1}{\Gamma(p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_2}(\iota,s)f_1\,(s,z(s))\,ds\right\}\Bigg|.\\
&=\Bigg|\mathcal{G}(\iota,w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)}+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_1+p_2}(\iota,s)f_2\,(s,z(s))\,ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_2}(\iota,s)f_1\,(s,z(s))\,ds\right\}\Bigg|.
\end{aligned}$$

Then,

$$\begin{aligned}
|\,(\psi(\iota)-\psi(0))^{1-\gamma}\,(\mathcal{B}w)(\iota)| &\leq|\mathcal{G}(\iota,w(\iota))|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_1+p_2}(\iota,s)\omega_2(s)ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathcal{N}_{\psi}^{p_2}(\iota,s)\omega_1(s)ds\right\}\\
&\leq\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+(\psi(\iota)-\psi(0))^{1-\gamma}\left(\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}\omega_2\right)\iota)+(\psi(\iota)-\psi(0))^{1-\gamma}\left(\mathfrak{I}_{0^+,\iota}^{p_2;\psi}\omega_1\right)\iota)\right\}.
\end{aligned}$$

Taking supremum over $\iota$, we arrive at

$$\begin{aligned}
\|\mathcal{B}w\|_{\mathcal{BC}_{1-\gamma}}&\leq\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+{\omega_1}^*+{\omega_2}^*\right\}\leq\eta\\
&\Longrightarrow w\in\mathcal{S}.
\end{aligned}$$

**Step 5:** In conclusion, we demonstrate that for $\mathcal{M}=\sup\{\|\mathcal{B}w\|_{\mathcal{BC}_{1-\gamma}}:w\in\mathcal{S}\}$, we have $\alpha\mathcal{M}<1$.

Using (4.6), we get

$$\begin{aligned}
\mathcal{M}&=\sup\{\|\mathcal{B}w\|_{\mathcal{BC}_{1-\gamma}}:w\in\mathcal{S}\}\\
&\leq\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+{\omega_1}^*+{\omega_2}^*\right\},
\end{aligned}$$

with the aid of (4.3), we arrive at $\alpha=\|\omega\|$. Thus, as a result of $\|\omega\|\left\{|\frac{w_0}{\Gamma(\gamma)}|+{\omega_1}^*+{\omega_2}^*\right\}<1$, we get

$$\alpha\mathcal{M}\leq\|\omega\|\mathcal{M}\leq\|\omega\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+{\omega_1}^*+{\omega_2}^*\right\}<1.$$

Since all of the requirements of Theorem 2.6 are satisfied, the operator equation $w = \mathcal{A}w\mathcal{B}w$ hence has a solution in $\mathcal{S}$. ◻

## 5. Attractivity of Solutions

In this part, the Attractivity of solutions to problem (1.1) will be discussed. To show that, we need the following definition

**Lemma 5.1** ( [24]). *Solutions of equation $(Kw)(\iota) = w(\iota)$ are locally attractive if a ball $B(w_0, \eta)$ exists in the space $\mathcal{BC}$ in which, any solutions $z = z(\iota)$ and $\sigma = \sigma(\iota)$ to the above equations belonging to $B(w_0, \eta) \cap \Lambda$, can be written as follows*

$$\lim_{\iota \to \infty}(w(\iota) - \sigma(\iota)) = 0. \tag{5.1}$$

*The solutions are considered uniformly locally attractive if the limit (5.1) is uniform with respect to $B(w_0, \eta) \cap \wedge$, where $\phi \neq \wedge \subset \mathcal{BC}$. (or that the solutions are locally asymptotically stable; equivalent).*

**Theorem 5.2.** *Under the hypotheses of Theorem 4.1. Then, the solutions of the problem (1.1) are uniformly locally attractive on $\mathbb{R}_+$.*

Let us now show that the solution is uniformly locally attractive on $\mathbb{R}_+$.

*Proof.* Define the operator $\mathcal{K} : \mathcal{S} \to \mathcal{X}$ as

$$\begin{aligned}(\mathcal{K}w)(\iota) =& \mathcal{G}(\iota, w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)}(\psi(\iota)-\psi(0))^{1-\gamma} + \frac{1}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w(s))\, ds \right.\\ &\left. + \frac{1}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w(s))\, ds\right\}.\end{aligned}$$

Keep in mind that $\mathcal{K}w(\iota) = \mathcal{A}w(\iota)\mathcal{B}w(\iota), \forall \iota \in \mathbb{R}_+$, where $\mathcal{A}$ and $\mathcal{B}$ are given in (4.1) and (4.2), respctively.
Given the conditions of this theorem, we suppose that $w_*$ is a solution to problem (1.1).
Take $w \in \mathcal{B}\left(w_*, 2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right| + 2\omega_1{}^* + 2\omega_2{}^*\right\}\right)$, we get

$$\begin{aligned}&|(\psi(\iota)-\psi(0))^{1-\gamma}(\mathcal{K}w)(\iota) - (\psi(\iota)-\psi(0))^{1-\gamma} w_*(\iota)|\\ =& \left|\mathcal{G}(\iota, w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w(s))\, ds + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w(s))\, ds\right\}\right.\\ &-\mathcal{G}(\iota, w_*(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w_*(s))\, ds\right.\\ &\left.\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w_*(s))\, ds\right\}\right|.\end{aligned}$$

Then,

$$\begin{aligned}&|(\psi(\iota)-\psi(0))^{1-\gamma}(\mathcal{K}w)(\iota) - (\psi(\iota)-\psi(0))^{1-\gamma} w_*(\iota)|\\ \leq& \left|\mathcal{G}(\iota, w(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w(s))\, ds + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w(s))\, ds\right\}\right.\\ &-\mathcal{G}(\iota, w_*(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w(s))\, ds + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w(s))\, ds\right\}\\ &+\mathcal{G}(\iota, w_*(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w(s))\, ds + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w(s))\, ds\right\}\\ &-\mathcal{G}(\iota, w_*(\iota))\left\{\frac{w_0}{\Gamma(\gamma)} + \frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota, s) f_2(s, w_*(s))\, ds\right.\\ &\left.\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota, s) f_1(s, w_*(s))\, ds\right\}\right|.\end{aligned}$$

Then,

$$
\begin{aligned}
&|(\psi(\iota)-\psi(0))^{1-\gamma}(\mathcal{K}w)(\iota)-(\psi(\iota)-\psi(0))^{1-\gamma}w_*(\iota)| \\
&\leq |\mathcal{G}(\iota,w(\iota))-\mathcal{G}(\iota,w_*(\iota))|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\,|f_2(s,w(s))|\,ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\,|f_1(s,w(s))|\,ds\right\}\\
&\quad+|\mathcal{G}(\iota,w_*(\iota))|\left\{\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\,|f_2(s,w(s))-f_2(s,w_*(s))|\,ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\,|f_1(s,w(s))-f_1(s,w_*(s))|\,ds\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|(\psi(\iota)-\psi(0))^{1-\gamma}(\mathcal{K}w)(\iota)-(\psi(\iota)-\psi(0))^{1-\gamma}w_*(\iota)|\\
&\leq 2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\omega_2(s)ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\omega_1(s)ds\right\}+2\|\mathcal{G}\|\left\{\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\omega_2(s)ds\right.\\
&\quad\left.+\frac{(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\omega_1(s)ds\right\}\\
&\leq 2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+2(\psi(\iota)-\psi(0))^{1-\gamma}\left(\Im_{0^+,\iota}^{p_1+p_2;\psi}\omega_2\right)(\iota)+2(\psi(\iota)-\psi(0))^{1-\gamma}\left(\Im_{0^+,\iota}^{p_2;\psi}\omega_1\right)(\iota)\right\}.
\end{aligned}
$$

Taking supremum over $\iota$, we arrive at

$$
\|\mathcal{K}w-w_*\|_{\mathcal{BC}_{1-\gamma}}\leq 2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+2\omega_1{}^*+2\omega_2{}^*\right\}.
$$

Consequently, we deduce that $\mathcal{K}$ is a continuous function such that

$$
\mathcal{K}\left(\mathcal{B}\left(w_*,2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+2\omega_1{}^*+2\omega_2{}^*\right\}\right)\right)\subset\mathcal{B}\left(w_*,2\|\mathcal{G}\|\left\{\left|\frac{w_0}{\Gamma(\gamma)}\right|+2\omega_1{}^*+2\omega_2{}^*\right\}\right).
$$

Moreover, if $w$ is a solution of problem (1.1), then

$$
\begin{aligned}
&|w(\iota)-w_*(\iota)|=|(\mathcal{K}w)(\iota)-(\mathcal{K}w_*)(\iota)|\\
&\leq|\mathcal{G}(\iota,w(\iota))-\mathcal{G}(\iota,w_*(\iota))|\left\{(\psi(\iota)-\psi(0))^{\gamma-1}\left|\frac{w_0}{\Gamma(\gamma)}\right|+\frac{1}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\,|f_2(s,w(s))|\,ds\right.\\
&\quad\left.+\frac{1}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\,|f_1(s,w(s))|\,ds\right\}\\
&\quad+|\mathcal{G}(\iota,w_*(\iota))|\left\{\frac{1}{\Gamma(p_1+p_2)}\int_0^\iota \mathcal{N}_\psi^{p_1+p_2}(\iota,s)\,|f_2(s,w(s))-f_2(s,w_*(s))|\,ds\right.\\
&\quad\left.+\frac{1}{\Gamma(p_2)}\int_0^\iota \mathcal{N}_\psi^{p_2}(\iota,s)\,|f_1(s,w(s))-f_1(s,w_*(s))|\,ds\right\}\\
&\leq 2\|\mathcal{G}\|\left\{(\psi(\iota)-\psi(0))^{\gamma-1}\left|\frac{w_0}{\Gamma(\gamma)}\right|+2\left(\Im_{0^+,\iota}^{p_1+p_2;\psi}\omega_2\right)(\iota)+2\left(\Im_{0^+,\iota}^{p_2;\psi}\omega_1\right)(\iota)\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|w(\iota)-w_*(\iota)|\\
&\leq 2\|\mathcal{G}\|\left\{(\psi(\iota)-\psi(0))^{\gamma-1}\left|\frac{w_0}{\Gamma(\gamma)}\right|+2\frac{(\psi(\iota)-\psi(0))^{1-\gamma}\left(\Im_{0^+,\iota}^{p_1+p_2;\psi}\omega_2\right)(\iota)}{(\psi(\iota)-\psi(0))^{1-\gamma}}+2\frac{(\psi(\iota)-\psi(0))^{1-\gamma}\left(\Im_{0^+,\iota}^{p_2;\psi}\omega_1\right)(\iota)}{(\psi(\iota)-\psi(0))^{1-\gamma}}\right\}.
\end{aligned}
\tag{5.2}
$$

Using (5.2) and the fact that

$$\lim_{\iota\to\infty}(\psi(\iota)-\psi(0))^{1-\gamma}\left(\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}\omega_2\right)\iota)=0,$$

and

$$\lim_{\iota\to\infty}(\psi(\iota)-\psi(0))^{1-\gamma}\left(\mathfrak{I}_{0^+,\iota}^{p_2;\psi}\omega_1\right)\iota)=0.$$

It follows that

$$\lim_{\iota\to\infty}|w(\iota)-w_*(\iota)|=0.$$

Lemma 5.1 implies that solutions of problem (1.1) are uniformly locally attractive. □

## 6. Examples

**Example 6.1.** Taking the following problem

$$\begin{cases} {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q_1;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q_2;\psi}\dfrac{w(\iota)}{\mathcal{G}(\iota,w(\iota))}+f_1(\iota,w(\iota))\right)=f_2(\iota,w(\iota)), \quad \iota\in\mathbb{R}_+,\\ \dfrac{w(0)}{\mathcal{G}(0,w(0))}=0,\\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}\left(\dfrac{w(0)}{\mathcal{G}(0,w(0))}\right)=1, \end{cases} \tag{6.1}$$

where $p_1=\frac{1}{4}$,$p_2=\frac{2}{5}$, $q_1=\frac{5}{6}$, $q_2=1$, and $\psi(\iota)=\iota^2$.
We take

$$f_1(\iota,w)=\frac{e^{-\iota^2}|w|}{13(1+\iota^2)^3(1+|w|)},$$

$$f_2(\iota,w)=\frac{\iota^3\sin(\iota)}{(1+\iota^2)^2(1+|w|)}e^{-3\iota^2},$$

$$\mathcal{G}(\iota,w(\iota))=\frac{\iota^2|w|}{8(1+|w|)}+5\iota+1,$$

where $f_1$, $f_2$ and $\mathcal{G}$ satisfy $(H_1)$ and $(H_2)$ of Theorem 4.1 with

$$\omega_1(\iota)=\frac{e^{-\iota^2}}{13(1+\iota^2)^3},$$

$$\omega_2(\iota)=\frac{\iota^3\sin(\iota)}{(1+\iota^2)^2}e^{-3\iota^2}.$$

Moreover, we have

$$\lim_{\iota\to\infty}\frac{\iota^{\frac{-11}{20}}}{\Gamma(\frac{13}{20})}\int_0^\iota 2s(\iota^2-s^2)^{\frac{-7}{20}}\omega_2(s)ds=0,$$

and

$$\lim_{\iota\to\infty}\frac{\iota^{\frac{-11}{20}}}{\Gamma(\frac{2}{5})}\int_0^\iota 2s(\iota^2-s^2)^{\frac{-3}{5}}\omega_1(s)ds=0.$$

So, by Theorem 4.1 and 5.2 there is at least one solution to problem (6.1). Further, the solutions are uniformly locally attractive on $\mathbb{R}_+$.

**Example 6.2.** Taking the following problem

$$\begin{cases} {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q_1;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q_2;\psi}\dfrac{w(\iota)}{\mathcal{G}(\iota,w(\iota))}+f_1(\iota,w(\iota))\right)=f_2(\iota,w(\iota)), \quad \iota\in\mathbb{R}_+,\\ \dfrac{w(0)}{\mathcal{G}(0,w(0))}=0,\\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}\left(\dfrac{w(0)}{\mathcal{G}(0,w(0))}\right)=1, \end{cases} \tag{6.2}$$

where $p_1 = \frac{1}{6}, p_2 = \frac{1}{4}, q_1 = \frac{5}{8}, q_2 = \frac{7}{10}$, and $\psi(\iota) = \iota^4 + 1$. Let

$$f_1(\iota, w) = \frac{e^{-\iota}|w|}{9(1+\iota^2)^2(1+|w|)},$$
$$f_2(\iota, w) = \frac{\iota^3 \cos(\iota)}{(1+\iota^2)^3(1+|w|)} e^{-3\iota},$$
$$\mathcal{G}(\iota, w(\iota)) = \frac{\iota^2|w|}{10(1+|w|)} + 3\iota + 1,$$

where $f_1$, $f_2$, and $\mathcal{G}$ satisfy $(H_1)$ and $(H_2)$ of Theorem 4.1 with

$$\omega_1(\iota) = \frac{e^{-\iota}}{9(1+\iota^2)^2},$$
$$\omega_2(\iota) = \frac{\iota^3 \cos(\iota)}{(1+\iota^2)^3} e^{-3\iota}.$$

Moreover, we verify that

$$\lim_{\iota\to\infty} (\psi(\iota) - \psi(0))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_1+p_2;\psi} \omega_2\right)(\iota) = \lim_{\iota\to\infty} \frac{\iota^{-0.05}}{\Gamma\left(\frac{5}{12}\right)} \int_0^{\iota} 4s^3 \left(\iota^4 - s^4\right)^{-\frac{7}{12}} \omega_2(s) ds = 0,$$

and

$$\lim_{\iota\to\infty} (\psi(\iota) - \psi(s))^{1-\gamma} \left(\Im_{0^+,\iota}^{p_2;\psi} \omega_1\right)(\iota) = \lim_{\iota\to\infty} \frac{\iota^{-0.05}}{\Gamma\left(\frac{1}{4}\right)} \int_0^{\iota} 4s^3 \left(\iota^4 - s^4\right)^{-\frac{3}{4}} \omega_1(s) ds = 0.$$

Thus, by Theorem 5.2, the solutions of the problem (6.2) are uniformly locally attractive on $\mathbb{R}_+$.

## Conclusion

In this work, we developed and validated certain conditions guaranteeing the existence and uniform local attractiveness of solutions for a class of $\psi$-Hilfer fractional differential equations. Our findings provide an important theoretical advance by utilizing a sophisticated hybrid fixed point theory framework within Banach algebras. This method not only offers a strong basis for the study of fractional differential equations, but it additionally generates novel possibilities for further study. Our results shed light on the possibility of using similar methods in various fields of practical and mathematical sciences and advance our understanding of the dynamics of these complex problems.

## Acknowledgements

The authors are thankful to the referee's thoughtful comments on the manuscript, which helped to improve it.

## Funding

Not applicable.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

All authors contributed equally to this work.

## References


[1] Abdo, M.S., Panchal, S.K., Wahash, H.A., *Ulam-Hyers-Mittag-Leffler stability for a $\psi$-Hilfer problem with fractional order and infinite delay*, Results in Applied Mathematics, **7**(2020), 100–115.

[2] Ahmad, B., Ntouyas, S.K., *An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions*, Abstract and Applied Analysis, **2014**(2014).

[3] Ahmad, M., Zada, A., Wang, X., *Existence, uniqueness and stability of implicit switched coupled fractional differential equations of $\psi$-Hilfer type*, International Journal of Nonlinear Sciences and Numerical Simulation, **21**(3-4)(2020), 327–337.

[4] Atshan, S.M., Hamoud, A.A., *Qualitative analysis of ABR-fractional Volterra-Fredholm system*, Nonlinear Functional Analysis and Applications, (2024), 113–130.

[5] Dhage, B.C., Ntouyas, S.K., *Existence results for boundary value problems for fractional hybrid differential inclusions*, Topological Methods in Nonlinear Analysis, **44**(2014), 229–238.

[6] Dhage, B.C., Lakshmikantham, V., *Basic results on hybrid differential equations*, Nonlinear Analysis: Hybrid Systems, **4**(2010), 414–424.

[7] Dhage, B.C., *On $\alpha$-condensing mappings in Banach algebras*, Math. Stud., **63**(1994), 146–152.

[8] Dhage, B.C., *Existence and attractivity theorems for nonlinear hybrid fractional integrodifferential equations with anticipation and retardation*, Cubo (Temuco), **22**(3)(2020), 325–350.

[9] Hamoud, A.A., Mohammed, N.M., *Existence and uniqueness of solutions for the neutral fractional integro differential equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms, **29**(2022), 49–61.

[10] Hamoud, A.A., Khandagale, A.D., Shah, R., Ghadle, K.P., *Some new results on Hadamard neutral fractional nonlinear Volterra-Fredholm integro-differential equations*, Discontinuity, Nonlinearity, and Complexity, **12**(04)(2023), 893–903.

[11] Herrmann, R., Fractional Calculus for Physicist, World Scientific, Singapore, 2014.

[12] Hilal, K., Kajouni, A., Lmou, H.,*Existence and stability results for a coupled system of Hilfer fractional Langevin equation with non local integral boundary value conditions*, Filomat, **37**(2023), 1241–1259.

[13] Hilfer, R., Applications of Fractional Calculus in Physics, World Scientific Publishing Co. Inc, River Edge, NJ, Singapore, 2000.

[14] Jameel, S.A.M., Rahman, S.A., Hamoud, A.A., *Analysis of Hilfer fractional Volterra-Fredholm system*, Nonlinear Functional Analysis and Applications, (2024), 259–273.

[15] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and aplications of fractional differential equations*, in NorthHoland Mathematics Studies, vol. 24, Elsevier, Amsterdam, Netherlands, 2006.

[16] Lmou, H., Hilal, K., Kajouni, A., *Existence and uniqueness results for Hilfer Langevin fractional pantograph differential equations and inclusions*, International Journal of Difference Equations, **18**(1)(2023), 145–162.

[17] Lmou, H., Hilal, K., Kajouni, A., *A new result for $\psi$-Hilfer fractional pantograph-type Langevin equation and inclusions*, Journal of Mathematics, (2022).

[18] Louakar, A., Kajouni, A., Hilal, K., Lmou, H., *A class of implicit fractional $\psi$-Hilfer Langevin equation with time delay and impulse in the weighted space*, Communications in Advanced Mathematical Sciences, **7**(2)(2024), 88–103.

[19] Mahmudov, N., Matar, M.M., *Existence of mild solution for hybrid differential equations with arbitrary fractional order*, TWMS J. Pure Appl. Math., **8**(2)(2017), 160–169.

[20] Mali, A.D., Kucche, K.D., Vanterler da Costa Sousa, J., *On coupled system of nonlinear $\Psi$-Hilfer hybrid fractional differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, **24**(4)(2023), 1425–1445.

[21] More, P.M., Karande, B.D., Amrutrao, P.S., *Locally attractivity solution of coupled fractional integral equation in Banach algebras*, Asian Research Journal of Mathematics, **19**(1)(2023), 45–55.

[22] Shabna, M.S., Ranjin, M.C., *On existence of $\psi$-Hilfer hybrid fractional differential equations*, South East Asian International Journal of Mathematics and Mathematical Sciences, **16**(2020).

[23] Shah, K., Ali, A., Bushnaq, S., *HyersUlam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions*, Mathematical Methods in the Applied Sciences, **41**(17)(2018), 8329–8343.

[24] Si Bachir, F., Abbas, S., Benbachir, M., Benchohra, M., N'Guérékata, G.M., *Existence and attractivity results for $\psi$-Hilfer hybrid fractional differential equations*, Cubo., **23**(2021), 145–159.

[25] Sousa, J.V.C., Capelas Oliveira, E., *A Gronwall inequality and the Cauchy type problem by means of Hilfer operator*, Diff. Equ. and Appl., **11**(1)(2019), 87–106.

[26] Sousa, J.V.D.C., Capelas de Oliveira, E.C., *On the $\psi$-Hilfer fractional derivative*, Commun Nonlinear Sci Numer Simulat, **60**(2018), 72–91.

[27] Zhao, Y., Sun, S., Han, Z., Li, Q., *Theory of fractional hybrid differential equations*, Computers and Mathematcs with Applications, **62**(3)(2011), 1312–1324.