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A Nonstandard Finite Difference Scheme for a Mathematical Model Presenting the Climate Change on the Oxygen-plankton System

Zahraa Makki Farhan ALJAMMALI [1], İlkem TURHAN ÇETİNKAYA [2*]

[1]*Kütahya Dumlupınar University, Graduate Education Institute, Department of Mathematics, Kutahya, Turkey.*

[2]*Kütahya Dumlupınar University, Science and Art Faculty, Department of Mathematics, Kutahya, Turkey.*

(ORCID: 0009-0000-8077-4118) (ORCID: 0000-0002-5520-310X)

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Abstract

This paper presents a mathematical model describing climate change in the oxygen-plankton system. The model consists of a system of non-linear ordinary differential equations. The Nonstandard Finite Difference (NSFD) method is applied to discretize the non-linear system. The stability of the continuous and discrete model is presented for the given parameters in the literature. Similar results for stability are obtained in both continuous and discrete models. The model is solved by the Runge–Kutta–Fehlberg (RKF45) method, and the numerical results are compared in graphical forms. Moreover, the comparison of numerical results obtained by the NSFD method, the Euler method and the fourth order Runge-Kutta (RK4) method is presented in tabular form. Furthermore, the efficiency of the NSFD method compared to classical methods such as the Euler method and the RK4 method for the bigger step sizes is shown in tabular forms.

1. Introduction

Climate change and global warming are serious threats to ecological life. One of the results of climate change and global warming is the increasing of the sea surface temperature in the oceans. Therefore, the photosynthetic production rate of phytoplankton changes. Mathematical models describing climate change are developed. Some of them can be summarized as follows. Sekerci and Petrovskii [1] consider a model of coupled plankton-oxygen dynamics under climate change. The stability of the steady states is analyzed and detailed numerical simulations are presented in [1]. Moreover, they analyze both analytically and numerically a model of the oxygen-phytoplankton-zooplankton dynamics in [2]. The climate models are presented by Priyadarshini and Veeresha [3]. The stability analysis is presented, and the Adams Predictor-Corrector method is applied to obtain the numerical results in [3]. Mondal et al. [4] give a detailed analysis of the dynamics of the oxygen-plankton model with a modified Holling type II functional response. Furthermore, the continuous model given in [1] is reconsidered in the fractional cases in [5-7]. A new delayed plankton–oxygen dynamical model is presented by Xu et al. [8]. They explore bifurcation and stability. Gökçe [9] analyzes the stability of a mathematical model of oxygen–phytoplankton interactions. Chowdhury et al. [10] consider a coupled model presenting plankton-oxygen dynamics and investigate the model using analytical techniques and numerical simulations.

Remarkable mathematical methods can be encountered in solving mathematical models. In this study, we prefer the Nonstandard Finite Difference (NSFD) method developed by Mickens [11-15] considering the advantages over the classical and standard finite difference methods. The NSFD method can be considered as a generalization of the usual discrete models of differential equations. In many cases, it removes the numerical instabilities

*Corresponding author: *ilkem.turhan@dpu.edu.tr*
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encountered in usual finite difference schemes. The method can be applied to both ordinary and partial differential equations. Moreover, the NSFD method preserves dynamic consistency. By virtue of the advantages of NSFD schemes, many researchers have been studying NSFD schemes for solving mathematical models. The detailed studies presented about the NSFD schemes can be found in [16, 17]. Moreover, there have been numerous studies of the NSFD schemes in the literature, recently. For example, Khan et al. [18] consider a nonlinear mathematical model of COVID-19. The NSFD schemes are constructed, local and global stability are studied in [18]. Zhang et al. [19] examine an epidemic model for waterborne disease, where the NSFD method is applied, and stability analysis is presented. Yang et al. [20] construct an NSFD scheme for a diffusive within-host virus dynamics model. Globally asymptotically stability of the model under the effect of virus-to-cell and cell-to-cell transmissions is studied in [20]. Kocabıyık et al. [21] and Dang and Hoang [22] consider computer virus models in the view of NSFD schemes. Moreover, Hoang et al. [23] construct the NSFD schemes for a modified epidemiological model of computer viruses. More studies about NSFD schemes can be found in [24-37]. This study constructing an NSFD scheme for an ecological model consists of five sections. The study begins with a brief introduction section. The second section presents a mathematical model of the oxygen-plankton system. The third section gives the method used in the study. The fourth section gives the stability of the model and numerical simulation. Lastly, the final section belongs to the conclusion and suggestions section.

2. Definition of the Model

In this paper, a simplified mathematical model of the climate change on the oxygen-plankton system given in [1,2,5] is considered. The continuous model is defined as

$$\frac{df}{dt}=T\left(1-\frac{f(t)}{f(t)+1}\right)g(t)-\frac{\delta g(t)f(t)}{f(t)+f_2} -\frac{\nu f(t)s(t)}{f(t)+f_3}-f(t), \tag{1a}$$

$$\frac{dg}{dt}=\left(\frac{Gf(t)}{f(t)+f_1}-g(t)\right)g(t)-\frac{g(t)s(t)}{g(t)+\tilde{h}} -\sigma g(t), \tag{1b}$$

$$\frac{ds}{dt}=\left(\frac{\eta f^2(t)}{f^2(t)+f_4^2}\right)\left(\frac{g(t)s(t)}{g(t)+\tilde{h}}\right)-\xi s(t) \tag{1c}$$

subject to initial conditions

$$f(0)=f_0,\ g(0)=g_0,\ s(0)=s_0. \tag{2}$$

The unknowns of the system of non-linear ordinary differential equations (1a)-(1c) presented by $f(t)$, $g(t)$, and $s(t)$ denote the oxygen concentration, phytoplankton density, and zooplankton density, respectively. The parameters T, δ, ν, G, σ, η, and ξ denote the rate of oxygen production by phytoplankton, phytoplankton respiration coefficient, zooplankton respiration coefficient, rate of phytoplankton maximum growth, phytoplankton natural mortality rate, zooplankton feeding efficiency, and zooplankton natural mortality, respectively. Moreover, the parameters $\tilde{h}$, f_1, f_2, f_3, and f_4 denote the half saturation values of the phytoplankton predation, phytoplankton growth, respiration by phytoplankton, respiration by zooplankton, and zooplankton feeding efficiency, respectively.

3. Material and Method

This section presents the discrete model of Eqs. (1a)-(1c) with the help of the NSFD method which has many advantages compared to classical methods. The NSFD method leads to determining a convenient denominator function that can be chosen instead of step size. Therefore, unlike classical methods, the convergent schemes that satisfy positivity conditions can be constructed by the NSFD method for bigger step sizes. Moreover, while numerical instabilities may be encountered in standard finite difference methods, the NSFD method removes the numerical instabilities. Detailed explanations about the rules of the NSFD method, determining denominator functions, and the benefits of the methods take place in [11-15].

The continuous model (1a)-(1c) can be discretized using the discretizing procedure of the NSFD method given in [11-15]. Thus, the following substitutions i)-iii) are employed to discrete the continuous model (1a)-(1c), respectively:

i) $g(t)\to g(n)$, $\dfrac{f(t)g(t)}{f(t)+1}\to\dfrac{f(n+1)g(n)}{f(n)+1}$, $\dfrac{g(t)f(t)}{f(t)+f_2}\to\dfrac{g(n)f(n+1)}{f(n)+f_2}$,

$\frac{f(t)s(t)}{f(t)+f_3} \to \frac{f(n+1)s(n)}{f(n)+f_3}$ and $f(t) \to f(n+1)$.

ii) $\frac{f(t)g(t)}{f(t)+f_1} \to \frac{f(n)g(n)}{f(n)+f_1}$, $g^2(t) \to g(n+1)g(n)$,

$\frac{g(t)s(t)}{g(t)+\tilde{h}} \to \frac{g(n+1)s(n)}{g(n)+\tilde{h}}$ and $g(t) \to g(n+1)$.

iii)

$$\frac{f^2(t)g(t)s(t)}{\left(f^2(t)+f_4^2\right)\left(g(t)+\tilde{h}\right)} \to \frac{f^2(n)g(n)s(n)}{\left(f^2(n)+f_4^2\right)\left(g(n)+\tilde{h}\right)}$$

and $s(t) \to s(n+1)$.

Therefore, the following discrete model preserving positivity condition is obtained for the mathematical model describing climate change in the oxygen-plankton system:

$$f(n+1) = \frac{f(n)+\varphi_1 Tg(n)}{1+\varphi_1\left(1+\frac{Tg(n)}{f(n)+1}+\frac{\delta g(n)}{f(n)+f_2}+\frac{\nu s(n)}{f(n)+f_3}\right)}, \quad (3a)$$

$$g(n+1) = \frac{g(n)\left(1+\frac{\varphi_2 Gf(n)}{f(n)+f_1}\right)}{1+\varphi_2\left(g(n)+\frac{s(n)}{g(n)+\tilde{h}}+\sigma\right)}, \quad (3b)$$

$$s(n+1) = \frac{s(n)\left(1+\frac{\varphi_3 \eta f^2(n)g(n)}{\left(f^2(n)+f_4^2\right)\left(g(n)+\tilde{h}\right)}\right)}{1+\xi\varphi_3}, \quad (3c)$$

where φ_1, φ_2 and φ_3 denote the denominator functions constructed as

$\varphi_1 = e^h - 1$,

$\varphi_2 = \frac{e^{\sigma h}-1}{\sigma}$,

and

$\varphi_3 = \frac{e^{\xi h}-1}{\xi}$.

The following lemma can be given for the positivity condition of the discrete system (3a)-(3c):

Lemma 3.1. All solutions of discrete system (3a)-(3c) satisfy the positivity condition under the assumption of positive initial conditions f_0, g_0, s_0, and positive parameters T, δ, ν, G, σ, η, ξ, $\tilde{h}$, f_1, f_2, f_3, f_4 and h.

Proof. Assume that the initial conditions f_0, g_0, s_0 and the parameters T, δ, ν, G, σ, η, ξ, $\tilde{h}$, f_1, f_2, f_3, f_4, and h are positive. Then, it is obvious from the discrete system (3a)-(3c) and their corresponding denominator functions that, all solutions of discrete system (3a)-(3c) are positive.

3.1. Stability Analysis

In this section, some theorems used during the stability analysis of the continuous model (1a)-(1c) and the discrete model (3a)-(3c) are given.

For an autonomous differential equation defined as

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \quad (4)$$

the linearized system can be determined for an equilibrium point E as

$$\frac{dy}{dt} = J(E)y,$$

where $J(E)$ is the Jacobian matrix of the system (4) at the equilibrium point E.

Theorem 3.1 gives a condition for the stability of a continuous system.

Theorem 3.1. [38] Assume that all eigenvalues of the Jacobian matrix of the system (4) have negative reel parts. Then the equilibrium point E is asymptotically stable.

Theorem 3.2 (The Schur-Cohn Criterion, n=3). Let us consider the discrete system defined by

$x(k+1) = Ax(k)$.

Assume that the characteristic polynomial of the matrix A is

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \quad (5)$$

where a_1, a_2 and a_3 are constants. The zeros of the characteristic polynomial (5) lie inside the unit disk if and only if the following conditions hold [39]:

i) $1+a_1+a_2+a_3>0$.

ii) $1-a_1+a_2-a_3>0$.

iii) $|a_1+a_3|<1+a_2$ and $|a_2-a_1a_3|<1-a_3^2$.

Thus, it can be deduced that if the conditions i)-iii) of Schur-Cohn criterion are satisfied then the discrete system (3a)-(3c) is locally asymptotically stable at the equilibrium point.

4. Results and Discussion

This section is devoted to showing the stability analysis of the model and numerical results for the following parameters [5]:

$$\sigma = 0.1,\ f_1 = 0.7,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ G = 1.8,\ \delta = 1,\ \nu = 0.01,\ \eta = 0.7,\ \tilde{h} = 0.1,\ \xi = 0.1. \quad (6)$$

Firstly, the stability of the continuous model (1a)-(1c) will be examined for the parameter $T = 1.8$ in addition to the parameters (6). Eleven equilibrium points are obtained for the continuous model (1a)-(1c). Considering Theorem 3.1, it can be concluded that only the trivial equilibrium and the equilibrium point obtained as

$$E = E(f, g, h) = \begin{pmatrix} 0.455479762381921, 0.493302878184821 \\ , 0.0689661098622050 \end{pmatrix} \quad (7)$$

are asymptotically stable.

Here, only the stability of the non-trivial equilibrium point E defined by Eq. (7) will be shown.
The Jacobian matrix of the continuous system (1a)-(1c) at the equilibrium point (7) is obtained as

$$J(E) = \begin{bmatrix} -1.65234318346793 & 0.923764295710813 & -0.00312941322960424 \\ 0.465542685604508 & -0.396654077620652 & -0.831452022774700 \\ 0.0250797518364091 & 0.0023563816122754 & 0 \end{bmatrix}.$$

Corresponding eigenvalues of the Jacobian matrix $J(E)$ determined as

$\lambda_1 = -1.93764307516937,$

$\lambda_{2,3} = -0.0556770929596066 \mp 0.0923582563050207i$

have negative real parts. Thus, it can be deduced from Theorem 3.1 that the equilibrium point (7) of the continuous model (1a)-(1c) is asymptotically stable.

From now on, the stability of the discrete model (3a)-(3c) will be examined. In addition to parameters (6), the step size is considered as $h = 0.001$. The continuous model (1a)-(1c) and the discrete model (3a)-(3c) possess the same equilibrium points. Thus, considering Theorem 3.2, the equilibrium point (7) and the trivial equilibrium point of the discrete model are asymptotically stable.
Now, we will show the stability of the model at the equilibrium point (7):
The Jacobian matrix of the discrete system (3a)-(3c) is determined as

$$J(E) = \begin{bmatrix} 0.998350048513008 & 0.000922427186186707 & -0.00000312488353706965 \\ 0.000465235841788792 & 0.999603607361028 & -0.000830904004474441 \\ 0.0000250784978914492 & 0.0000023562637972002 & 1 \end{bmatrix}. \quad (8)$$

The characteristic polynomial of the Jacobian matrix (8) is as

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \quad (9)$$

where the constants of characteristic polynomial (9) are as

$a_1 = -2.99795365587404,$
$a_2 = 2.99590753866671,$
$a_3 = -0.997953882770182.$

Next, we check the conditions of Theorem 3.2.

i. $1 + a_1 + a_2 + a_3 = 2.2488 \times 10^{-11} > 0$.

ii. $1 - a_1 + a_2 - a_3 = 7.99181507731093 > 0$.

iii. Since $|a_1 + a_3| = 3.99590753864422$ and $1 + a_2 = 3.99590753866671$, the condition $|a_1 + a_3| < 1 + a_2$ is satisfied. Similarly, since $1 - a_3^2 = 0.004088047863918$ and $|a_2 - a_3 a_1| = 0.00408804742215$, the condition $1 - a_3^2 > |a_2 - a_3 a_1|$ is satisfied, too.

Thus, it can be deduced from the Schur-Cohn criterion (Theorem 3.2), the discrete system (3a)-(3c) is locally asymptotically stable.

Hereinafter, the numerical results obtained by the NSFD and the RKF45 methods are presented in graphical forms. In addition to the parameters (6), the step size is chosen as $h = 0.01$, and the positive initial conditions are considered as $f_0 = 0.4$, $g_0 = 0.36$, $s_0 = 0.12$. The parameter defining the rate of oxygen production by phytoplankton T is chosen as $T = 1.8$, $T = 2$, and $T = 2.2$ to be able to compare the obtained numerical results with the studies in the literature. Moreover, the numerical results obtained by the Euler, RK4, and NSFD methods are given in tabular forms and the effectiveness of the NSFD method is presented.

Figure 1 presents the variation in oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 1.8$. It can be concluded from Figure 1 that the system approaches the extinction state rapidly. Thereby, the plankton-oxygen system is not sustainable for $T = 1.8$.

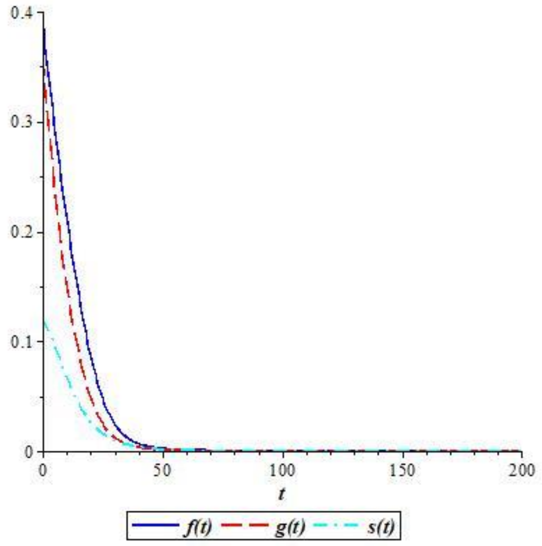

Figure 1. NSFD solution for oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 1.8$.

Figure 2 presents the variation in oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 2$. It can be concluded from Figure 2 that the system approaches the coexistence state for $T = 2$.

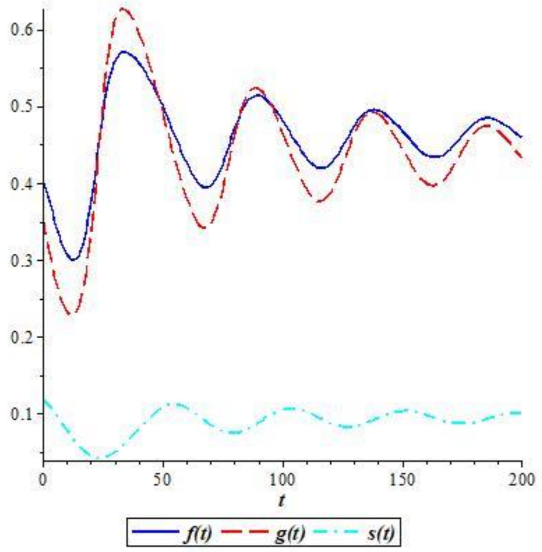

Figure 2. NSFD solution of oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 2$.

Figure 3 presents the variation in oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 2.2$. It can be concluded from Figure 3 that the system shows a few oscillations before approaching the extinction state for $T = 2.2$.

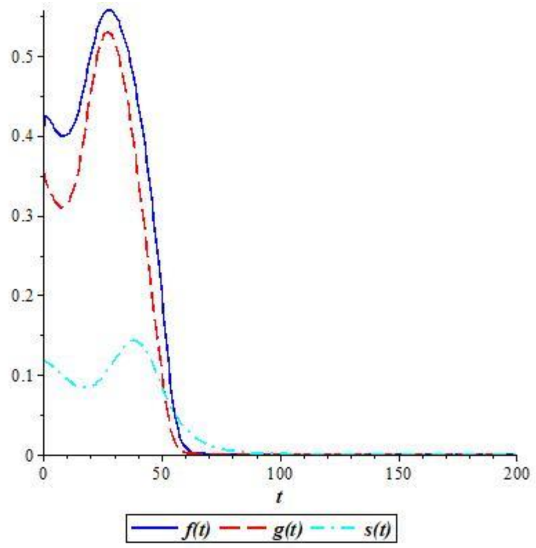

Figure 3. NSFD solution of oxygen concentration $f(t)$, phytoplankton density $g(t)$, and zooplankton density $s(t)$ for $T = 2.2$.

The model (1a)-(1c) has been solved numerically by using the Adams predictor-collector method in [3]. One can see that Figures 1-3 are the same as Figure 8, Figure 9, and Figure 11 provided in [3], respectively. Moreover, the model (1a)-(1c) takes into account the effect of global warming through the parameter T [3]. Thus, detailed interpretations of the effects of global warming can be found in [1, 3, 5, 40].

Figure 4 presents the numerical comparison of the NSFD method with the RKF45 method for the parameters (6) and $T = 1.8$. It can be concluded from Figure 4 that the numerical results for the NSFD method are in good agreement with the RKF45 method.

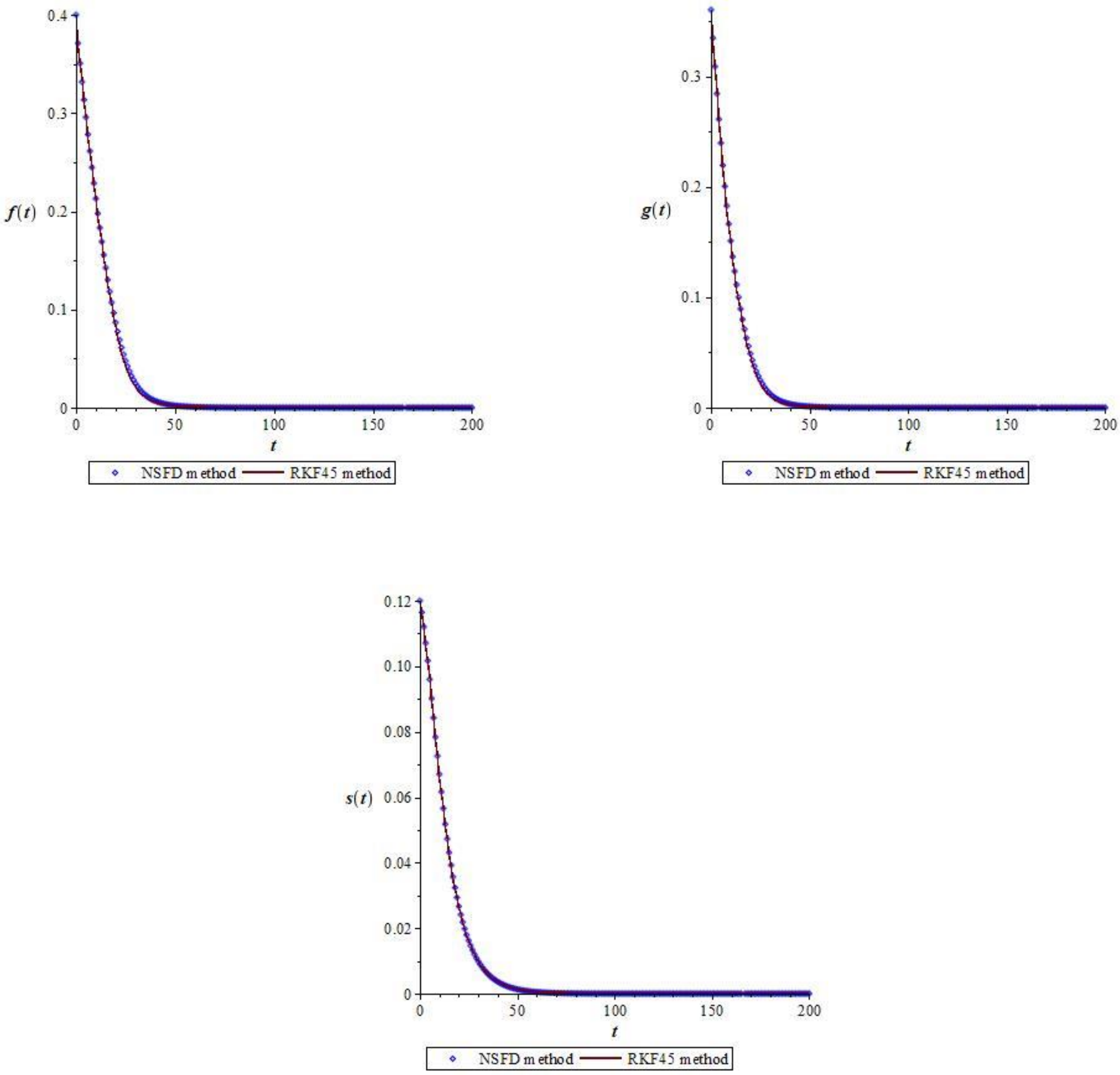

Figure 4. Numerical comparison of NSFD method with RKF45 method. ($T = 1.8$)

Moreover, the numerical comparison of the NSFD method with the Euler and RK4 methods is given in Table 1 for the step size $h = 0.001$ and the parameter $T = 2.2$. Thus, the accuracy of the numerical results obtained by the NSFD method is verified.

Table 1. Numerical comparison of the NSFD, Euler, and RK4 methods for the step size $h = 0.001$ ($T = 2.2$).

	Euler			RK4			NSFD		
t	$f(t)$	$g(t)$	$s(t)$	$f(t)$	$g(t)$	$s(t)$	$f(t)$	$g(t)$	$s(t)$
0	0.4	0.36	0.12	0.4	0.36	0.12	0.4	0.36	0.12
1	0.42490	0.34497	0.1177	0.42489	0.34497	0.1177	0.42489	0.34498	0.1177
5	0.40838	0.31779	0.10864	0.40838	0.31779	0.10864	0.40840	0.31781	0.10864
10	0.40011	0.31291	0.09521	0.40012	0.31292	0.09522	0.40015	0.31294	0.09522
50	0.20067	0.10209	0.08609	0.20077	0.10218	0.08608	0.20132	0.10259	0.08621
100	0.000004	0.00001	0.00059	0.000004	0.000001	0.00059	0.000004	0.000001	0.00059

Table 2 presents the numerical results at $t = 700$ obtained by the Euler, RK4, and NSFD methods for different step sizes ($T = 2.2$). Thus, the system approaches the extinction state for the bigger step sizes by the NSFD method. Table 2 confirms the advantage of the NSFD method over the Euler and RK4 methods.

Table 2. Numerical results at $t = 700$ obtained by the Euler, RK4, and NSFD methods for different step sizes ($T = 2.2$).

	Euler			RK4			NSFD		
h	$f(t)$	$g(t)$	$s(t)$	$f(t)$	$g(t)$	$s(t)$	$f(t)$	$g(t)$	$s(t)$
0.01	3.4×10^{-32}	1.3×10^{-32}	5×10^{-30}	3.7×10^{-32}	1.5×10^{-32}	5.2×10^{-30}	3.9×10^{-32}	1.5×10^{-32}	5.2×10^{-30}
0.1	1.5×10^{-32}	6.3×10^{-33}	3.7×10^{-30}	3.7×10^{-32}	1.5×10^{-32}	5.2×10^{-30}	6.1×10^{-32}	2.5×10^{-32}	6.1×10^{-30}
1	3.6×10^{-37}	1.4×10^{-37}	1.7×10^{-31}	3.6×10^{-32}	1.4×10^{-32}	5.2×10^{-30}	10^{-30}	7.5×10^{-31}	2.6×10^{-29}
5	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	4.2×10^{-27}	10^{-27}	6.3×10^{-27}
10	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	Float (undefined)	10^{-23}	2.2×10^{-24}	4×10^{-24}

Similarly, Table 3 presents the comparison of three methods for the stability of the equilibrium point E for different step sizes. The NSFD method is the most appropriate for the bigger step sizes compared with the Euler and RK4 methods.

Table 3. Comparison of stability of the equilibrium point E for different step sizes. ($T = 2$)

h	Euler	RK4	NSFD
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
1	Convergence	Convergence	Convergence
5	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence
100	Divergence	Divergence	Convergence

5. Conclusion and Suggestions

The study aims to present an NSFD scheme for the model describing climate change in the oxygen-plankton system. The continuous model is discretized through the NSFD schemes and solved numerically. The numerical results are compared by the RKF45 method to present the accuracy of the NSFD method. Moreover, the numerical result obtained by the Euler, and the RK4 methods are presented to compare with the NSFD method. The effectiveness of the considered method for the bigger step sizes is presented in Table 2 and Table 3.

It can be concluded from Figures 1-3 that when the parameter defining the rate of oxygen production by phytoplankton is $T=1.8$, $T=2$ and $T=2.2$, the plankton-oxygen system is not sustainable, the system approaches the coexistence state, and the system shows a few oscillations before approaching the extinction state, respectively.

This study implies that the NSFD method is easy to apply to the system of nonlinear ordinary differential equations. It preserves positivity conditions and is efficient even for bigger step sizes. In future studies, the NSFD schemes can be applied to the other versions of oxygen-plankton models.

Contributions of the authors

The authors' contributions to the paper are equal.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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