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Congruence of Curves in Weyl-Otsuki Spaces

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Abstract

In this paper, we study the congruence of curves in Weyl-Otsuki spaces using Ricci's coefficients of that congruence in the orthogonal case. We first prove that Ricci's coefficients λ_{abc} determine the regular general connection of an Otsuki space. Then, we give the condition for these coefficients in Weyl-Otsuki spaces to be skew-symmetric in the first two indices as in Riemannian spaces. We obtain the necessary and sufficient conditions for the curves of congruence to be geodesic, normal, and irrotational. Finally, we prove that if a congruence satisfies the equation, $''T^{h}_{kj} - 2\,'T^{h}_{i[j}U^{i}_{n|}U_{n|k]} + \varphi_{[k}\delta^{h}_{j]} = 0,$ and any two of the conditions to be geodesic, normal, and irrotational, then it also satisfies the other third one.

Keywords: Weyl-Otsuki spaces; General connections; Ricci's coefficients; Congruence of curves; Geodesic curves.

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Weyl-Otsuki Uzaylarında Kongrüans Eğrileri

Öz

Bu makalede Weyl-Otsuki uzaylarında kongrüans eğrilerini bu eğrilerin ortogonal olması durumda Ricci katsayılarını kullanarak inceledik. İlk olarak, λ_{abc} Ricci katsayılarının bir Otsuki uzayının regüler genel koneksiyonunu belirlediğini gösterdik. Ardından Riemann uzaylarda olduğu gibi Weyl-Otsuki uzaylarında bu katsayıların ilk iki indisine göre ters-simetrik olma koşulunu verdik. Kongrüans eğrilerinin, sırasıyla, jeodezik, normal ve irrotasyonel olması için gerek ve yeter koşulları elde ettik. Son olarak bir kongrüans eğrisinin $"T^l_{ji} - 2'T^l_{k[i}V^k_{n|}V_{n|j]} + \gamma_{[j}\delta^l_{i]} = 0$ denklemi ile birlikte jeodezik, normal ve irrotasyonel olma koşullarından herhangi ikisini sağlaması durumunda diğer üçüncü koşulu da sağladığını kanıtladık.

Anahtar Kelimeler: Weyl-Otsuki uzayları; Genel koneksiyonlar; Ricci katsayıları; Kongrüans eğrileri; Jeodezik eğriler.

1. Introduction

The theory of Otsuki spaces is based on the notion of regular general connection, introduced by T. Otsuki [1]. He gave the theoretical foundation for general connections and showed that they are the generalizations of the classical connections, for instance, the affine, projective, and conformal connections [2, 3]. The general connections were first noticed by A. Moor and were linked with Weyl spaces [4]. These spaces are then called Weyl-Otsuki spaces. Then D.F. Nadj obtained curvatures [5] and the Frenet formulas [6] of the Weyl-Otsuki spaces and also studied Riemann-Otsuki spaces, which are the special cases of the Weyl-Otsuki spaces [7-9]. The general connections were also introduced into vector bundles by N. Abe [10, 11], into general relativity by H. Nagayama [12, 13], and the theory of black holes by T. Otsuki [14-16].

The coefficients of an affine connection on an orthonormal basis are called Ricci's coefficients. Since these coefficients can determine an affine connection, they have great importance in studying some geometric properties of a Riemannian space, such as the parallelism of the unit tangent vector field of an orthogonal ennuple and the conditions for the curves of an orthogonal ennuple to be normal or to be irrotational. Moreover, since Ricci's coefficients with respect to an affine connection are skew-symmetric with its first two indices, they provide an easier investigation of the above geometric properties for Riemannian spaces. But this is not the case for Weyl-Otsuki spaces. The purpose of the paper is to get some conditions on Weyl-Otsuki spaces, which will be called co-recurrence conditions, for a proper investigation of the above geometric properties of these spaces.

2. Preliminaries

In this section, we will introduce the notion of regular general connection and its properties. Then we will define the co-recurrence condition on Weyl-Otsuki spaces to obtain the geometric properties of these spaces.

Definition 1. A ***regular general connection***[1] of an n-dimensional space M_n is defined as any cross-section Γ of the vector bundle $T(M_n) \otimes \mathcal{D}^2(M_n)$ over M_n, where $T(M_n)$ and $\mathcal{D}^2(M_n)$ are tangent bundle of order 1 and cotangent bundle of order 2 of M_n respectively, [1]. In a coordinate neighborhood, Γ is written as

$$\Gamma = \partial u_k \otimes \left(P_i^k d^2 u^i + \Gamma_{ij}^k du^i \otimes du^j \right),$$

where $P = \left(P_i^k \right)$ is an isomorphism of $T(M_n)$.

If P is the identity transformation, then Γ becomes a classical affine connection. So, general connections[2] are the generalizations of the classical connections, for instance, the affine, projective, and conformal connections.

It follows from Definition 1 that there exist a $(1,1)$-tensor $Q = \left(Q_i^k \right)$ such that $P^{-1} = Q,$ since $\det\left(P_i^k \right) \neq 0$. Therefore Γ and P determine two affine connections $'\Gamma$ and $''\Gamma$ which are called ***contravariant and covariant part of*** Γ**,** respectively, in the following way:

$$'\Gamma_{ij}^k = Q_l^k \Gamma_{ij}^l \quad \text{and} \quad ''\Gamma_{ij}^k = \left(\Gamma_{lj}^k - \partial_j P_l^k \right) Q_i^l .$$

Using the above equations, we can define the ***basic covariant differential*** of a (p,q)-tensor $U = \left(U_{i_1 \ldots i_r}^{j_1 \cdots j_s} \right)$ with respect to a regular general connection Γ by

[1] It was called Otsuki connection by Nadj [7].

[2] If P is only a homomorphism, then Γ is called a general connection.

$$\nabla U^{j_1\cdots j_s}_{i_1\ldots i_r}=\nabla_k U^{j_1\cdots j_s}_{i_1\ldots i_r}du^k,$$
$$\nabla_k U^{j_1\cdots j_s}_{i_1\ldots i_r}=\partial_k U^{j_1\cdots j_s}_{i_1\ldots i_r}+\sum_{p=1}^{s}{}'\Gamma^{j_p}_{hk}U^{j_1\cdots j_{p-1}hj_{p+1}\cdots j_s}_{i_1\ldots i_r}-\sum_{q=1}^{r}{}''\Gamma^{h}_{i_q k}U^{j_1\cdots j_s}_{i_1\ldots i_{q-1}hi_{q+1}\ldots i_r}. \tag{1}$$

Using (1), one can easily see that the basic covariant differentiation of the tensor product of any two tensors obeys the classical rule;

$$\nabla_k\left(U^{j_1\cdots j_s}_{i_1\ldots i_r}V^{j_{s+1}\cdots j_{s+p}}_{i_{r+1}\ldots i_{r+q}}\right)=\left(\nabla_k U^{j_1\cdots j_s}_{i_1\ldots i_r}\right)V^{j_{s+1}\cdots j_{s+p}}_{i_{r+1}\ldots i_{r+q}}+U^{j_1\cdots j_s}_{i_1\ldots i_r}\left(\nabla_k V^{j_{s+1}\cdots j_{s+p}}_{i_{r+1}\ldots i_{r+q}}\right).. \tag{2}$$

It is well-known that the covariant differentiations and the contractions are commutative operators in classical differential geometry. This is due to the fact that the covariant derivative of the identity transformation I is constant. Hence, if we use (1) for the identity transformation

$$\nabla_j\delta^k_i={}'\Gamma^k_{ij}-{}''\Gamma^k_{ij}, \tag{3}$$

Then, we have the relations between basic covariant differentiation and contractions as follows:

$$\delta^i_j\left(\nabla_k U^{j_1\cdots j_s j}_{i_1\ldots i_r i}\right)=\nabla_k\left(U^{j_1\cdots j_s j}_{i_1\ldots i_r i}\delta^i_j\right)+U^{j_1\cdots j_s j}_{i_1\ldots i_r i}\nabla_k\delta^i_j. \tag{4}$$

Now, we will give the curvature and the torsion tensors for the affine connections ${}'\Gamma$ and ${}''\Gamma$. Let ${}'R^k_{ilj}$, ${}''R^k_{ilj}$ and ${}'T^k_{ij}$, ${}''T^k_{ij}$ be the components of the curvature and the torsion tensors of ${}'\Gamma$ and ${}''\Gamma$, respectively. Then, we have

$${}'R^k_{ilj}=\partial_l\,{}'\Gamma^k_{ij}-\partial_j\,{}'\Gamma^k_{il}+{}'\Gamma^k_{hl}\,{}'\Gamma^h_{ij}-{}'\Gamma^k_{hj}\,{}'\Gamma^h_{il}$$
$${}''R^k_{ilk}=\partial_l\,{}''\Gamma^k_{ij}-\partial_j\,{}''\Gamma^k_{il}+{}''\Gamma^k_{hl}\,{}''\Gamma^h_{ij}-{}''\Gamma^k_{hj}\,{}''\Gamma^h_{il} \tag{5}$$

and

$${}'T^k_{ij}={}'\Gamma^k_{ij}-{}'\Gamma^k_{ji},\qquad {}''T^k_{ij}={}''\Gamma^k_{ij}-{}''\Gamma^k_{ji}. \tag{6}$$

Using the equations (1)-(6), we obtain the following ***Ricci formulas:***

$$2\nabla_{[k}\nabla_{l]}U^{j_1\cdots j_s}_{i_1\ldots i_r}=\nabla_k\nabla_l U^{j_1\cdots j_s}_{i_1\ldots i_r}-\nabla_l\nabla_k U^{j_1\cdots j_s}_{i_1\ldots i_r}$$
$$=\sum_{p=1}^{s}{}'R^{j_p}_{hkl}U^{j_1\ldots h\ldots j_s}_{i_1\ldots i_r}-\sum_{q=1}^{r}{}''R^{h}_{i_q kl}U^{j_1\cdots j_s}_{i_1\ldots h\ldots i_r}-{}''T^h_{lk}\nabla_h U^{j_1\cdots j_s}_{i_1\ldots i_r} \tag{7}$$

A. Moor [4] introduced the concept of Weyl-Otsuki space by associating the theory of Otsuki space with the Weyl metric as follows:

Definition 2. Let M_n be an n-dimensional space with a regular general connection Γ and a Riemannian metric $g=\left(g_{ij}\right)$. Then M_n is called a ***Weyl-Otsuki space*** if the covariant differential of the metric tensor with respect to Γ holds the ***recurrence equation***,

$$\nabla_k g_{ij} = \varphi_k g_{ij}, \tag{8}$$

for a covariant vector field $\varphi=\left(\varphi_k\right)$. In this case, g is also called ***recurrent metric tensor***.

Remark 1. We will denote the Weyl-Otsuki space with regular general connection Γ, Riemannian metric g and covariant vector field $\varphi=\left(\varphi_k\right)$ by a quadruple $\left(M_n,\Gamma,g,\varphi\right)$.

Unlike an affine connection, the ***co-recurrence equation*** $\nabla_k g^{ij} = -\varphi_k g^{ij}$ does not hold for a regular general connection. Now, we will give the condition that the metric tensor satisfies the co-recurrence equation for regular general connections. In virtue of (2), (4) and (8), we obtain

$$\nabla_j \delta_i^k = \nabla_j\left(g^{kl}g_{li}\right) = g_{li}\nabla_j g^{kl} + \varphi_j\delta_i^k - g^{kh}g_{li}\nabla_j\delta_h^l$$

or equivalently

$$\nabla_j g^{kl} = -\varphi_j g^{kl} + g^{ki}\nabla_j\delta_i^l + g^{li}\nabla_j\delta_i^k.$$

Hence, we have the following lemma:

Proposition 1. Let $\left(M_n,\Gamma,g,\varphi\right)$ be a Weyl-Otsuki space. Then the co-recurrence equation

$$\nabla_k g^{ij} = -\varphi_k g^{ij}, \tag{9}$$

holds in a Weyl-Otsuki space if and only if

$$g^{ih}\nabla_k\delta_h^j + g^{jh}\nabla_k\delta_h^i = 0.$$

3. Congruence of Curves in Weyl-Otsuki Spaces

In this section, we will show that Ricci's coefficients determine the connection in Weyl-Otsuki spaces as well as in Riemann spaces. We will use Ricci's coefficients to examine the condition for congruence of curves to be geodesic. Moreover, if the co-recurrence equation is satisfied, we will provide the properties of these coefficients in Weyl-Otsuki spaces.

Definition 3. Let $U=\left(U^1,\ldots,U^n\right)$ be a vector field on a Weyl-Otsuki space $\left(M_n,\Gamma,g,\varphi\right)$. The system of differential equations $\frac{dx^1}{U^1}=\ldots=\frac{dx^n}{U^n}$ admits $n-1$ independent solutions $\psi^i\left(x^1,\ldots,x^n\right)=c^i$, $\left(i=1,\ldots,n-1\right)$, where c's are constants [17]. If we substitute any point $p\in M_n$ in the last equations, the constants c are determined so that these $n-1$ equations define a curve through p. Since one can define such a curve through each point, U determines a family of curves, one of which passes through each point of that space. This family of curves is called a ***congruence*** of curves in a Weyl-Otsuki space[3]. An ***orthogonal ennuple*** in a Weyl-Otsuki space consists of n mutually orthogonal congruences of curves[4].

Let $U_{a|}$ [5], $\left(a=1,...,n\right)$, be the unit tangents to the n congruences of an orthogonal ennuple. The contravariant and covariant components of $U_{a|}$ will be denoted by $U^{i}_{a|}$ and $U_{a|i}$ respectively. Since the n congruences are mutually orthogonal, we have the relations

$$g_{ij}U^{i}_{a|}U^{j}_{b|}=\delta_{ab} \qquad \text{or} \qquad U^{i}_{a|}U_{b|i}=\delta_{ab}. \tag{10}$$

Since $U^{i}_{a|}$ is the cofactor of $U_{a|i}$ in the determinant $\left|U_{a|i}\right|$ divided by the value of that determinant, we have

$$\sum_a U^{i}_{a|}U_{a|j}=\delta^i_j. \tag{11}$$

Definition 4. Let $U_{a|}$, $\left(a=1,...,n\right)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space $\left(M_n,\Gamma,g,\varphi\right)$ [17]. The derived vector of $U_{a|}$ in the direction of $U_{c|}$ has components $\left(\nabla_k U^{j}_{a|}\right)U^{k}_{c|}$; and the projection of this vector on $U_{b|}$ is a scalar invariant, denoted by η_{abc}, so that

$$\eta_{abc}=\left(\nabla_k U^{j}_{a|}\right)U_{b|j}U^{k}_{c|}. \tag{12}$$

[3] In the other words a congruence is the set of integral curves determined by a vector field.

[4] These definitions are the generalizations of the definitions in Riemannian spaces, [17].

[5] The subscript a, followed by a bar distinguishing one congruence from another, and having no significance of covariance.

The invariants η_{abc} are called ***Ricci's coefficients.***

Lemma 1. Let $U_{a|}$, $(a=1,...,n)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space (M_n,Γ,g,φ) Then the Ricci's coefficients η_{abc} determine a classical affine connection $'\Gamma$.

Proof. In virtue of (11) and (12) we have

$$\nabla_k U^j_{a|} = \sum_{b,c} \eta_{abc} U^j_{b|} U_{c|k} \ . \tag{13}$$

Multiplying this equation by $U^k_{c|}$, and using (10), we get

$$U^k_{c|} \nabla_k U^j_{a|} = \sum_b \eta_{abc} U^j_{b|} \ . \tag{14}$$

Now, if we multiply the equation (13) by $U_{a|i}$, and use (1), then we obtain

$$\sum_a \left(\nabla_k U^j_{a|}\right) U_{a|i} = \sum_a \left(\partial_k U^j_{a|} + {}'\Gamma^j_{lk} U^l_{a|}\right) U_{a|i} = \sum_{a,b,c} \eta_{abc} U_{a|i} U^j_{b|} U_{c|k} \ ,$$

by summing with respect to a. From the last equation we have

$${}'\Gamma^j_{ik} = -\sum_a U_{a|i} \partial_k U^j_{a|} + \sum_{a,b,c} \eta_{abc} U_{a|i} U^j_{b|} U_{c|k} \ , \tag{15}$$

which yields us the result. ■

Moreover, for a given isomorphism P of $T(M_n)$, a regular general connection Γ can be determined by $\Gamma^j_{ki} = P^j_l \, {}'\Gamma^l_{ki}$. Hence, using (15) we have the following theorem:

Theorem 1. Let $U_{a|}$, $(a=1,...,n)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space (M_n,Γ,g,φ). Then, for any isomorphism P of $T_p(M_n)$, the Ricci's coefficients η_{abc} determine a regular general connection Γ.

We note that in Riemannian spaces, Ricci's coefficients η_{abc} are skew-symmetric in the indices a and b but this is not the case in Weyl-Otsuki spaces unless the co-recurrence condition holds. In fact, using (2), (4), and (9), we obtain

$$\begin{aligned}\nabla_k U^j_{a|} &= \nabla_k \left(U_{a|l} g^{jl} \right) = g^{jl} \nabla_k U_{a|l} + U_{a|l} \nabla_k g^{jl} - U_{a|l} g^{ji} \nabla_k \delta^l_i \\ &= g^{jl} \nabla_k U_{a|l} - \varphi_k g^{jl} U_{a|l} - U_{a|l} g^{ji} \nabla_k \delta^l_i .\end{aligned}$$

Substituting the last equation in (12), we have

$$\begin{aligned}\eta_{abc} &= \left(g^{jl} \nabla_k U_{a|l} - \varphi_k g^{jl} U_{a|l} - U_{a|l} g^{ji} \nabla_k \delta^l_i \right) U_{b|j} U^k_{c|} \\ &= \left(U^l_{b|} \nabla_k U_{a|l} - \varphi_k \delta_{ab} - U_{a|l} U^i_{b|} \nabla_k \delta^l_i \right) U^k_{c|},\end{aligned}$$

or equivalently

$$U^l_{b|} \nabla_k U_{a|l} = \sum_c \eta_{abc} U_{c|k} + \varphi_k \delta_{ab} + U_{a|l} U^i_{b|} \nabla_k \delta^l_i . \tag{16}$$

On the other hand, using (2) and (4), we have

$$\nabla_k \delta_{ab} = \nabla_k \left(U^j_{a|} U_{b|j} \right) = U_{b|j} \nabla_k U^j_{a|} + U^j_{a|} \nabla_k U_{b|j} - U^i_{a|} U_{b|j} \nabla_k \delta^j_i = 0.$$

Multiplying the last equation by $U^k_{c|}$, we get

$$\begin{aligned}0 &= U^k_{c|} U_{b|j} \nabla_k U^j_{a|} + U^k_{c|} U^j_{a|} \nabla_k U_{b|j} - U^k_{c|} U^i_{a|} U_{b|j} \nabla_k \delta^j_i \\ &= \eta_{abc} + U^k_{c|} \left(\sum_d \eta_{bad} U_{d|k} + \varphi_k \delta_{ab} + U_{b|l} U^i_{a|} \nabla_k \delta^l_i \right) - U^k_{c|} U^i_{a|} U_{b|j} \nabla_k \delta^j_i \\ &= \eta_{abc} + \eta_{bac} + \varphi_k \delta_{ab} U^k_{c|}\end{aligned}$$

from (12) and (16). Hence, we obtain

$$\eta_{abc} + \eta_{bac} = -\varphi_k \delta_{ab} U^k_{c|}, \tag{17}$$

and the following result:

Proposition 2. Let $U_{a|}$, $\left(a = 1, \dots, n\right)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space $\left(M_n, \Gamma, g, \varphi\right)$. If the co-recurrence equation is satisfied, then Ricci's coefficients η_{abc} are skew-symmetric in the indices a and b with $a \neq b$, i.e.,

$$\eta_{abc} = -\eta_{bac}, \ (a \neq b). \tag{18}$$

Now, we can give another version of Ricci formulas using Ricci's coefficients for Weyl-Otsuki spaces. Let $U_{a|}$, $(a = 1,...,n)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space $(M_n, \Gamma, g, \varphi)$. Taking the basic covariant differentiation of (13) with respect to u^k, and using (12), (2), (4), (8), we have

$$\begin{aligned}
\nabla_i \nabla_k U^j_{a|} &= \nabla_i \left(\sum_{e,f} \eta_{aef} U^j_{e|} U_{f|k} \right) = \sum_{e,f} \left(U^j_{e|} U_{f|k} \partial_i \eta_{aef} + \eta_{aef} U_{f|k} \nabla_i U^j_{e|} + \eta_{aef} U^j_{e|} \nabla_i U_{f|k} \right) \\
&= \sum_{e,f} U^j_{e|} U_{f|k} \partial_i \eta_{aef} + \sum_{e,f} \eta_{aef} U_{f|k} \left(\sum_{r,s} \eta_{ers} U^j_{r|} U_{s|i} \right) + \sum_{e,f} \eta_{aef} U^j_{e|} \nabla_i \left(U^l_{f|} g_{lk} \right) \\
&= \sum_{e,f} U^j_{e|} U_{f|k} \partial_i \eta_{aef} + \sum_{e,f} \eta_{aef} U_{f|k} \left(\sum_{r,s} \eta_{ers} U^j_{r|} U_{s|i} \right) \\
&\quad + \sum_{e,f} \eta_{aef} U^j_{e|} \left(g_{lk} \nabla_i U^l_{f|} + U^l_{f|} \nabla_i g_{lk} - U^h_{f|} g_{lk} \nabla_i \delta^l_h \right) \\
&= \sum_{e,f} U^j_{e|} U_{f|k} \partial_i \eta_{aef} + \sum_{e,f} \eta_{aef} U_{f|k} \left(\sum_{r,s} \eta_{ers} U^j_{r|} U_{s|i} \right) \\
&\quad + \sum_{e,f} \eta_{aef} U^j_{e|} \left(g_{lk} \sum_{r,s} \eta_{frs} U^l_{r|} U_{s|i} + U^l_{f|} \varphi_i g_{lk} - U^h_{f|} g_{lk} \nabla_i \delta^l_h \right) \\
&= \sum_{e,f} U^j_{e|} U_{f|k} \partial_i \eta_{aef} + \sum_{e,f} \eta_{aef} U_{f|k} \left(\sum_{r,s} \eta_{ers} U^j_{r|} U_{s|i} \right) \\
&\quad + \sum_{e,f} \eta_{aef} U^j_{e|} \left(\sum_{r,s} \eta_{frs} U_{r|k} U_{s|i} + \varphi_i U_{f|k} - U^h_{f|} g_{lk} \nabla_i \delta^l_h \right).
\end{aligned}$$

If we compute $\nabla_k \nabla_i U^j_{a|}$ and subtract it from the above equation, we obtain

$$\begin{aligned}\nabla_i\nabla_k U_{a|}^j-\nabla_k\nabla_i U_{a|}^j &= \sum_{e,f} U_{e|}^j U_{f|k}\partial_i\eta_{aef}+\sum_{e,f}\eta_{aef}U_{f|k}\left(\sum_{r,s}\eta_{ers}U_{r|}^j U_{s|i}\right)\\ &+\sum_{e,f}\eta_{aef}U_{e|}^j\left(\sum_{r,s}\eta_{frs}U_{r|k}U_{s|i}+\varphi_i U_{f|k}-U_{f|}^h g_{lk}\nabla_i\delta_h^l\right)\\ &-\sum_{e,f}U_{e|}^j U_{f|i}\partial_k\eta_{aef}-\sum_{e,f}\eta_{aef}U_{f|i}\left(\sum_{r,s}\eta_{ers}U_{r|}^j U_{s|k}\right)\\ &-\sum_{e,f}\eta_{aef}U_{e|}^j\left(\sum_{r,s}\eta_{frs}U_{r|i}U_{s|k}+\varphi_k U_{f|i}-U_{f|}^h g_{li}\nabla_k\delta_h^l\right)\\ &=2\sum_{e,f}U_{e|}^j U_{f|\,[k}\partial_{i]}\eta_{aef}+2\sum_{e,f,r,s}\eta_{aef}\eta_{ers}U_{r|}^j U_{f|\,[k}U_{s|i]}\\ &+2\sum_{e,f}\eta_{aef}U_{e|}^j\left(\sum_{r,s}\eta_{frs}U_{r|\,[k}U_{s|i]}+U_{f|\,[k}\varphi_{i]}-U_{f|}^h g_{l[k}\nabla_{i]}\delta_h^l\right).\end{aligned}$$

Multiplying the last equation by $U_{b|j}U_{c|}^k U_{d|}^i$ and summing with respect to i,j,k we get

$$\begin{aligned}\left(\nabla_i\nabla_k U_{a|}^j-\nabla_k\nabla_i U_{a|}^j\right)U_{b|j}U_{c|}^k U_{d|}^i &= 2U_{b|j}U_{c|}^k U_{d|}^i\sum_{e,f}U_{e|}^j U_{f|\,[k}\partial_{i]}\eta_{aef}\\ &+2U_{b|j}U_{c|}^k U_{d|}^i\sum_{e,f,r,s}\eta_{aef}\eta_{ers}U_{r|}^j U_{f|\,[k}U_{s|i]}\\ &+2U_{b|j}U_{c|}^k U_{d|}^i\sum_{e,f}\eta_{aef}U_{e|}^j\left(\sum_{r,s}\eta_{frs}U_{r|\,[k}U_{s|i]}\right.\\ &\left.+U_{f|\,[k}\varphi_{i]}-U_{f|}^h g_{l[k}\nabla_{i]}\delta_h^l\right)\\ &=2U_{[d\;|}^i\partial_i\eta_{abc]}+2\sum_e\eta_{ae[c}\eta_{ebd]}+2\sum_f\eta_{abf}\eta_{f[cd]}\\ &+2\eta_{ab[c}\varphi_i U_{d]\,|}^i-2\sum_f\eta_{abf}U_{f|}^h U_{[c\;|l}U_{d]\,|}^i\nabla_{i]}\delta_h^l.\end{aligned}$$

Consequently, if we write the Ricci formulas (7) in this equation, then we have

$$\begin{aligned}\frac{1}{2}\left({}'R_{hik}^j U_{a|}^h-{}''T_{ki}^h\nabla_h U_{a|}^j\right)U_{b|j}U_{c|}^k U_{d|}^i &= U_{[d\;|}^i\partial_i\eta_{abc]}+\sum_e\eta_{ae[c}\eta_{ebd]}+\sum_f\eta_{abf}\eta_{f[cd]}\\ &+2\eta_{ab[c}\varphi_i U_{d]\,|}^i-\sum_f\eta_{abf}U_{f|}^h U_{[c\;|l}U_{d]\,|}^i\nabla_{i]}\delta_h^l.\end{aligned}$$

Now, we will give the condition for congruence of curves to be geodesic in Weyl-Otsuki spaces. Let s_a be the arc-length parameter of one of the curves $u=u(s_a)$ of the ennuple whose unit tangent is $U_{a|}=du/ds_a$. Then, using (14), we have

$$\frac{\nabla U_{a|}^{j}}{ds_a}=\left(\nabla_k U_{a|}^{j}\right)\frac{du^k}{ds_a}=U_{a|}^{k}\nabla_k U_{a|}^{j}=\sum_b \eta_{aba}U_{b|}^{j}.$$

Hence we obtain the following result:

Theorem 2. Let $U_{a|}$, $(a=1,...,n)$, be the unit tangents to the congruences of an orthogonal ennuple in a Weyl-Otsuki space (M_n,Γ,g,φ). Then, the curves of the congruence are geodesics if and only if $\eta_{aba}=0$, $(b=1,...,n)$.

4. Normal Orthogonal Ennuples

In this section, we will express the condition of an orthogonal ennuple in a Weyl-Otsuki space in which the co-recurrence equation is satisfied to be normal regarding Ricci's coefficients.

Definition 5. [17] An orthogonal ennuple that intersects orthogonally with a family of hypersurfaces, $\phi=const.$ is called a ***normal orthogonal ennuple***.

Let $U_{a|}$ be the unit tangent to the congruences of an orthogonal ennuple in a Weyl-Otsuki space in which the co-recurrence equation is satisfied, and $\phi=const.$ be a family of hypersurfaces. Then this ennuple is normal when the gradient of ϕ at each point has the direction of the vector $U_{a|}$. This condition is expressed as

$$\frac{\partial_1\phi}{U_{a|1}}=\frac{\partial_2\phi}{U_{a|2}}=...=\frac{\partial_n\phi}{U_{a|n}}.$$

But the necessary and sufficient condition for the existence of such a function ϕ is

$$U_{a|j}\left(\partial_i U_{a|k}-\partial_k U_{a|i}\right)+U_{a|k}\left(\partial_j U_{a|i}-\partial_i U_{a|j}\right)+U_{a|i}\left(\partial_k U_{a|j}-\partial_j U_{a|k}\right)=0.$$

Now, let a congruence of an orthogonal ennuple with unit tangent $U_{n|}$ be normal. Using (1) in the last equation, we have

$$\begin{aligned}&U_{n|j}\left(\nabla_i U_{n|k}+{}''\Gamma_{ki}^{h}U_{n|h}-\nabla_k U_{n|i}-{}''\Gamma_{ik}^{h}U_{n|h}\right)\\&+U_{n|k}\left(\nabla_j U_{n|i}+{}''\Gamma_{ij}^{h}U_{n|h}-\nabla_i U_{n|j}-{}''\Gamma_{ji}^{h}U_{n|h}\right)\\&+U_{n|i}\left(\nabla_k U_{n|j}+{}''\Gamma_{jk}^{h}U_{n|h}-\nabla_j U_{n|k}-{}''\Gamma_{kj}^{h}U_{n|h}\right)=0.\end{aligned}$$

If we multiply this equation by $U_{a|}^{j}U_{b|}^{i}$, $(a,b=1,2,...,n-1)$, in virtue of (16), (3), and (6), we get

$$\begin{aligned}
0 &= U_{a|}^{j}U_{b|}^{i}U_{n|k}\left(\nabla_j U_{n|i} + {}''\Gamma_{ij}^{h}U_{n|h} - \nabla_i U_{n|j} - {}''\Gamma_{ji}^{h}U_{n|h}\right) \\
&= U_{n|k}\Big(\eta_{nba} + \varphi_j \delta_{nb} U_{a|}^{j} + U_{n|h}U_{b|}^{i}U_{a|}^{j}\nabla_j \delta_i^h + {}''\Gamma_{ij}^{h}U_{n|h}U_{a|}^{j}U_{b|}^{i} \\
&\qquad - \eta_{nab} - \varphi_i \delta_{na} U_{b|}^{i} - U_{n|h}U_{a|}^{j}U_{b|}^{i}\nabla_i \delta_j^h - U_{a|}^{j}U_{b|}^{i}\,{}''\Gamma_{ji}^{h}U_{n|h}\Big) \\
&= U_{n|k}\Big(\eta_{nba} + U_{n|h}U_{b|}^{i}U_{a|}^{j}\nabla_j \delta_i^h + {}''\Gamma_{ij}^{h}U_{n|h}U_{a|}^{j}U_{b|}^{i} \\
&\qquad - \eta_{nab} - U_{n|h}U_{a|}^{j}U_{b|}^{i}\nabla_i \delta_j^h - U_{a|}^{j}U_{b|}^{i}\,{}''\Gamma_{ji}^{h}U_{n|h}\Big) \\
&= U_{n|k}\left(\eta_{nba} - \eta_{nab} + {}'T_{ij}^{h}U_{a|}^{j}U_{b|}^{i}U_{n|h}\right).
\end{aligned}$$

Hence, we have the following result:

Theorem 4. In a Weyl-Otsuki space in which the condition (9) holds, the necessary and sufficient condition that the congruences $U_{n|}$ of an orthogonal ennuple be normal is that

$$\eta_{nab} - \eta_{nba} = {}'T_{ij}^{h}U_{a|}^{j}U_{b|}^{i}U_{n|h}, \qquad (a,b=1,2,...,n-1). \tag{19}$$

If all the congruences of an orthogonal ennuple are normal, all the invariants η_{abc} with three distinct indices must be zero. Using the equation (18) in (19), we have

$$\begin{aligned}
\eta_{abc} &= \eta_{acb} + {}'T_{ij}^{h}U_{b|}^{j}U_{c|}^{i}U_{a|h} = -\eta_{cab} + {}'T_{ij}^{h}U_{b|}^{j}U_{c|}^{i}U_{a|h} \\
&= -\eta_{cba} - {}'T_{ij}^{h}U_{a|}^{j}U_{b|}^{i}U_{c|h} + {}'T_{ij}^{h}U_{b|}^{j}U_{c|}^{i}U_{a|h} \\
&= \eta_{bca} + {}'T_{ij}^{h}\left(U_{b|}^{j}U_{c|}^{i}U_{a|h} - U_{a|}^{j}U_{b|}^{i}U_{c|h}\right) \\
&= \eta_{bac} + {}'T_{ij}^{h}U_{c|}^{j}U_{a|}^{i}U_{b|h} + {}'T_{ij}^{h}\left(U_{b|}^{j}U_{c|}^{i}U_{a|h} - U_{a|}^{j}U_{b|}^{i}U_{c|h}\right) \\
&= -\eta_{abc} + {}'T_{ij}^{h}\left(U_{b|}^{j}U_{c|}^{i}U_{a|h} + U_{c|}^{j}U_{a|}^{i}U_{b|h} - U_{a|}^{j}U_{b|}^{i}U_{c|h}\right).
\end{aligned}$$

Consequently, we obtain the following result:

Theorem 5. In a Weyl-Otsuki space in which the condition (9) holds, the necessary and sufficient condition that all the congruences of an orthogonal ennuple be normal is that

$$\eta_{abc} = \frac{1}{2}\,{}'T_{ij}^{h}\left(U_{b|}^{j}U_{c|}^{i}U_{a|h} + U_{c|}^{j}U_{a|}^{i}U_{b|h} - U_{a|}^{j}U_{b|}^{i}U_{c|h}\right),$$
$$(a,b,c=1,2,...,n;\ a,b,c \text{ unequal}).$$

5. Curl of an Orthogonal Ennuple

In this section, we will give the condition for an orthogonal ennuple to be irrotational in Weyl-Otsuki spaces in terms of Ricci's coefficients. We will also express the relationship between the geodesic, normal, and irrotational conditions of an orthogonal ennuple in Weyl-Otsuki spaces in which the co-recurrence equation is satisfied.

Definition 6. [17] The curl of the unit tangent to an orthogonal ennuple is briefly called ***the curl of the orthogonal ennuple***. If the curl of an orthogonal ennuple vanishes identically, the ennuple will be described as ***irrotational***.

Consider the nth congruence of an orthogonal ennuple whose unit tangent vector is $U_{n|}$ in a Weyl-Otsuki space in which the condition (9) holds. Putting $a=n$ in (16), multiplying it by $U_{b|i}$ and summing with respect to b, we obtain

$$\nabla_k U_{n|i} = \sum_{b,c} \eta_{nbc} U_{b|i} U_{c|k} + \varphi_k U_{n|i} + U_{n|l} \nabla_k \delta_i^l. \tag{20}$$

Since the curl of the unit tangent to the nth congruence of an orthogonal ennuple is the tensor, whose components are $\nabla_k U_{n|j} - \nabla_j U_{n|k}$, then from (20), these components have the values

$$\nabla_k U_{n|j} - \nabla_j U_{n|k} = \sum_{b,c} \left(\eta_{nbc} - \eta_{ncb}\right) U_{b|j} U_{c|k} + 2\left(\nabla_{[k} \delta_{j]}^l + \varphi_{[k} \delta_{j]}^l\right) U_{n|l}.$$

This double sum may be separated into two sums as follows. In the first, let b and c take the values $1, 2, ..., n-1$; and, in the second, let either or both take the value n. Then we have

$$\begin{aligned} \nabla_k U_{n|j} - \nabla_j U_{n|k} = & \sum_{b,c=1}^{n-1} \left(\eta_{nbc} - \eta_{ncb}\right) U_{b|j} U_{c|k} + \sum_{b=1}^{n-1} \left(\eta_{nbn} - \eta_{nnb}\right) U_{b|j} U_{n|k} \\ & + \sum_{c=1}^{n-1} \left(\eta_{nnc} - \eta_{ncn}\right) U_{n|j} U_{c|k} + 2\left(\nabla_{[k} \delta_{j]}^l + \varphi_{[k} \delta_{j]}^l\right) U_{n|l}. \end{aligned} \tag{21}$$

Now, we will give the relationship between the geodesic, normal, and irrotational conditions of an orthogonal ennuple in Weyl-Otsuki spaces where the co-recurrence equation is satisfied.

Theorem 6. Let $U_{n|}$ be the unit tangent to the n th congruence of an orthogonal ennuple in a Weyl-Otsuki space $(M_n, \Gamma, g, \varphi)$ in which the condition (9) holds. If this congruence satisfies

$$"T^h_{kj} - 2\,'T^h_{i[j}U^i_{n|}U_{n|k]} + \varphi_{[k}\delta^h_{j]} = 0, \tag{22}$$

and any two of the following conditions, then it will also satisfy the other third one;

i) that it be a normal congruence, **ii)** that it be a geodesic congruence, **iii)** that it be irrotational.

Proof. Case 1. Suppose that the n th congruence is both normal and geodesic. Using (19), we can write the first term on the right-hand side of (21) as

$$\begin{aligned}
\sum_{b,c=1}^{n-1}(\eta_{nbc} - \eta_{ncb})U_{b|j}U_{c|k} &= \sum_{b,c=1}^{n-1}{'T^h_{il}}U^l_{b|}U^i_{c|}U_{n|h}U_{b|j}U_{c|k} \\
&= {'T^h_{il}}\left(\delta^l_j - U^l_{n|}U_{n|j}\right)\left(\delta^i_k - U^i_{n|}U_{n|k}\right)U_{n|h} \\
&= {'T^h_{il}}\left(\delta^l_j\delta^i_k - \delta^l_jU^i_{n|}U_{n|k} - \delta^i_kU^l_{n|}U_{n|j} + U^l_{n|}U_{n|j}U^i_{n|}U_{n|k}\right)U_{n|h} \\
&= \left({'T^h_{kj}} - {'T^h_{ij}}U^i_{n|}U_{n|k} - {'T^h_{kl}}U^l_{n|}U_{n|j} + {'T^h_{il}}U^l_{n|}U_{n|j}U^i_{n|}U_{n|k}\right)U_{n|h} \\
&= \left({'T^h_{kj}} - {'T^h_{ij}}U^i_{n|}U_{n|k} + {'T^h_{ik}}U^i_{n|}U_{n|j}\right)U_{n|h} \\
&= \left({'T^h_{kj}} - 2\,{'T^h_{i[j}}U^i_{n|}U_{n|k]}\right)U_{n|h}.
\end{aligned} \tag{23}$$

Since the congruence is also a geodesic curve, then we have

$$\eta_{nbn} = \eta_{ncn} = 0,$$

from Theorem 2. So (21) can be written as

$$\begin{aligned}
\nabla_kU_{n|j} - \nabla_jU_{n|k} &= \left({'T^h_{kj}} - 2\,{'T^h_{i[j}}U^i_{n|}U_{n|k]}\right)U_{n|h} - \sum_{b=1}^{n-1}\eta_{nnb}U_{b|j}U_{n|k} \\
&\quad + \sum_{c=1}^{n-1}\eta_{nnc}U_{n|j}U_{c|k} + 2\left(\nabla_{[k}\delta^l_{j]} + \varphi_{[k}\delta^l_{j]}\right)U_{n|l} \\
&= \left("T^h_{kj} - 2\,{'T^h_{i[j}}U^i_{n|}U_{n|k]} + 2\varphi_{[k}\delta^l_{j]}\right)U_{n|h} \\
&\quad + \sum_{b=1}^{n-1}\eta_{nnb}\left(U_{n|j}U_{b|k} - U_{b|j}U_{n|k}\right).
\end{aligned}$$

If we use (17) for the last term on the right-hand side of the above equation, then we have

$$\begin{aligned}\sum_{b=1}^{n-1}\eta_{nnb}\left(U_{n|j}U_{b|k}-U_{b|j}U_{n|k}\right) &= -\frac{1}{2}\varphi_i\sum_{b=1}^{n-1}U^i_{b|}\left(U_{n|j}U_{b|k}-U_{b|j}U_{n|k}\right)\\ &= -\frac{1}{2}\varphi_i U_{n|j}\left(\delta^i_k - U^i_{n|}U_{n|k}\right)+\frac{1}{2}\varphi_i U_{n|k}\left(\delta^i_j - U^i_{n|}U_{n|j}\right) \\ &= \varphi_{[j}U_{n|k]}.\end{aligned} \tag{24}$$

So we obtain

$$\nabla_k U_{n|j} - \nabla_j U_{n|k} = \left({}''T^h_{kj} - 2\,{}'T^h_{i[j}U^i_{n|}U_{n|k]} + \varphi_{[k}\delta^l_{j]}\right)U_{n|h}.$$

Hence, a normal and geodesic congruence that satisfies equation (22) is irrotational.

Case 2. Let the nth congruence be both normal and irrotational. At first, if we substitute (23) in (21) and use (17), (24) as in case 1, then we get

$$\nabla_k U_{n|j} - \nabla_j U_{n|k} = \left({}''T^h_{kj} - 2\,{}'T^h_{i[j}U^i_{n|}U_{n|k]} + \varphi_{[k}\delta^l_{j]}\right)U_{n|h} + \sum_{b=1}^{n-1}\eta_{nbn}\left(U_{b|j}U_{n|k} - U_{n|j}U_{b|k}\right).$$

Now, since the congruence is irrotational and satisfies (22), then the last equation yields

$$\sum_{b=1}^{n-1}\eta_{nbn}\left(U_{b|j}U_{n|k} - U_{n|j}U_{b|k}\right) = 0.$$

Then, this congruence is geodesic by Theorem 2.

Case 3. Let the nth congruence be both geodesic and irrotational. By the definition of irrotational curve and Theorem 2, (21) takes the form

$$\sum_{b,c=1}^{n-1}\left(\eta_{nbc}-\eta_{ncb}\right)U_{b|j}U_{c|k} + \sum_{b=1}^{n-1}\eta_{nnb}\left(U_{n|j}U_{b|k} - U_{b|j}U_{n|k}\right) + 2\left(\nabla_{[k}\delta^l_{j]} + \varphi_{[k}\delta^l_{j]}\right)U_{n|l} = 0.$$

If we use (3), (6) and (24) in the last equation, then we have

$$\sum_{b,c=1}^{n-1}\left(\eta_{nbc}-\eta_{ncb}\right)U_{b|j}U_{c|k} + \left({}'T^l_{jk} - {}''T^l_{jk} + \varphi_{[k}\delta^l_{j]}\right)U_{n|l} = 0.$$

Since the congruence satisfies (22), we obtain

$$\sum_{b,c=1}^{n-1}\left(\eta_{nbc}-\eta_{ncb}\right)U_{b|j}U_{c|k}+\left({}'T_{jk}^{l}+2\,{}'T_{i[j}^{l}U_{n|}^{i}U_{n|k]}\right)U_{n|l}=0.$$

Finally, by multiplying the last equation by $V_{b|}^{i}V_{c|}^{j}$ we get

$$\begin{aligned}\eta_{nbc}-\eta_{ncb}&=-\left({}'T_{jk}^{l}+2\,{}'T_{i[j}^{l}U_{n|}^{i}U_{n|k]}\right)U_{n|l}U_{b|}^{j}U_{c|}^{k}\\&=-\left({}'T_{jk}^{l}U_{b|}^{j}U_{c|}^{k}+2\,{}'T_{i[j}^{l}U_{n|}^{i}U_{c|k]}U_{b|}^{j}\right)U_{n|l}\\&={}'T_{ji}^{l}U_{b|}^{j}U_{c|}^{i}U_{n|l}.\end{aligned}$$

Hence, this congruence is normal by Theorem 4. ■

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