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An application of recurrence relations to central force fields

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Abstract
By generalizing the relationships between auxiliary variables that appear in the works of Roy et al. (1972) and Roy & Moran (1973), we investigate the dynamics of non-interacting particles under the action of a central force field whose force function is a solution to a certain differential equation. The method used by Roy et al. (1972), Broucke (1971), and later by Sitarski (1979), was utilized in such a way that the obtained recursive equations can be used to describe the motion of a particle under such a field. The class of such fields includes both gravitational and non-gravitational force fields. Several numerical and historically essential examples and detailed discussions of various cases are given.

Key words: celestial mechanics – methods: numerical – methods: analytical – history and philosophy of astronomy

1 Introduction

The dynamics of particles under the action of a force directed towards or from a fixed center is one of the oldest and most interesting topics. During the 1970s, a series of articles (Roy et al. 1972; Roy & Moran 1973; Moran et al. 1973; Black 1973) about the numerical integration, using high order expansions in time, of the motion for the two-body problem and the motion of two bodies perturbed by some disturbing force were published. Broucke (1971) used their method to study the solutions to the N-body problem. Sitarski (1979) used a similar method for the dynamics of comets having high eccentric orbits by considering this problem as a special case of the N-body problem. Saad et al. (2008) presented a new symbolic algorithm for the dynamics of comets. Alghamdi & Alshaery (2020) studied circular restricted gravitational three-body problem using similar methods. Hadjifotinou (2000) used the recurrent relations for the integration of motion of finitely many satellites orbiting around a planet. All these articles are based on the techniques and methods developed by Steffensen (1956, 1957) (e.g. Roy 2020, Ch. 4.13; Valtonen & Karttunen 2006, Ch. 2.11; Broucke 1971). Roy et al. (1972) proved that integration by recurrent equations is an efficient and accurate method for the present settings. Since then it has been shown by many researchers (Black 1973; Broucke 1971; Hadjifotinou 2000) that the obtained reliable results are satisfactory in speed, precision, and accuracy for the applications.

The present work is motivated by these papers and focuses on systems of recursive equations obtained from second-order equations modeling the dynamics of particles. In our approach, we further generalize this method to a certain class, called type-I, of central force fields by using the relationships between auxiliary variables that appear in the works of Roy et al. (1972) and Roy & Moran (1973). Most of the force fields appearing in the related literature are particular cases of type-I central force fields. After generalizing the method used by Roy et al. (1972), Broucke (1971), and later by Sitarski (1979), we use systems of recursive equations to study the motion of a particle under such a field. Concrete examples of type-I central force fields include both gravitational and non-gravitational force fields, and hence, they can be used in certain problems appearing in the dynamics of space flight.

The structure of the paper is as follows. In §2, we introduce the notions of type-I and type-II central force fields. In §3 we prove several statements that show that the introduced notions are well-defined. In §4, we study certain properties of the auxiliary variables needed in the numerical solutions. In §5, we focus on the dynamics of non-interacting particles under the action of type-I central force fields. In §6 we present some results in the case of type-II fields.

2 Preliminaries

We prefer the mathematical approach given in Hirsch & Smale (1974) because it is closer to the works of Smale (1967) in celestial mechanics. Throughout, f denotes a real valued analytic function defined on the open interval $(0, \infty)$. We denote by $\mathbf{r}$ the position of a particle in a space with respect to the standard non-rotating rectangular coordinate system. We denote the Euclidean norm of $\mathbf{r}$ by r, that is $r = \|\mathbf{r}\|$. Further, we assume everywhere in the present work that r is sufficiently many times differentiable, and, is non-vanishing in the sense that zero does belong to the closure of the set $\{t\colon r(t) \neq 0\}$. The unit vector $\hat{\mathbf{r}}$ in the direction of $\mathbf{r}$ is defined by the condition $\mathbf{r} = r\hat{\mathbf{r}}$.

Following Hirsch & Smale (1974, Ch. 2.4), by a *central force field* induced by the function f we mean a field $F_f = F$ defined everywhere over the space, except possibly at the origin, satisfying

$$F(\mathbf{r}) = -f(r)\mathbf{r}$$

whenever $r \neq 0$, see also Remark 1 below. In this case, $\|F(\mathbf{r})\| = |f(r)|r$.

Many analytical and local statistical properties of F_f are induced by those of f. The transformation that maps such an analytic function f to its induced central force field F_f is linear, i.e. $F_{kf_1+f_2} = kF_{f_1} + F_{f_2}$ for any scalar $k \in \mathbb{R}$. It further follows that if P_n is the nth polynomial approximation of f around $r_0 > 0$ then the truncation error in the induced

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central force field of P_n is given by a high dimensional version of Taylor's theorem, see Lemma 2 below. Similarly, if P is the nth Lagrange interpolating polynomial associated with f and $n+1$ distinct arguments $r_0, r_1, \ldots, r_n$ in $(0, \infty)$ then an error bound for F_f and F_P is given by a high dimensional version of Lagrange error formula, see Lemma 3.

Definition 1. We say that a central force field F induced by the function f is type-I if there exists some analytic function q on $(0, \infty)$ such that f is a solution of the differential equation $f' + qf = 0$. Similarly, we say that F is type-II if there exists some analytic functions p and q on the same interval $(0, \infty)$ such that the function f is a solution of the differential equation $f'' + pf' + qf = 0$.

One can show that if F is type-I, it is also type-II. However, F can be type-II but not type-I, for example $F(\mathbf{r}) = \sin(r)\mathbf{r}$ is as such. It is also clear that F is type-I (or type-II) if and only if kF for $k \in \mathbb{R}$ is type-I (or type-II, respectively). Further, if F is induced by some f and is type-I then the central vector fields induced by rf and f/r, respectively, are also type-I, see Lemma 1 below.

The problem that we are interested in is the approximations of the solutions of the differential equation

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -f(r)\mathbf{r} \tag{1}$$

in which the force field appearing on the right side is either type-I or type-II.

Remark 1. In the present article, $f(r)$ denotes the analytical function inducing F_f in the non-normalized equation Eq. (1). Since

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -f(r)\mathbf{r} = -f(r)r\hat{\mathbf{r}}$$

the function $f(r)r$ can be regarded as the force directed towards or from a fixed center. It follows from Lemma 1 given below that being type-I, as given Definition 1, is invariant under normalization with respect to any choice of the Euclidean norm. In general, the force depends not only on r but also other magnitudes such as time, velocity, etc. (Danby 1988, Ch. 4; Whittaker & McCrae 1988, Ch. IV), however, we follow the conventions given in Hirsch & Smale (1974, Ch. 2.4).

Example 1. Let $f(r) = \frac{K}{r^3}$ for $r > 0$ where K is a positive constant. The central force field F induced by f is type-I with $q(r) = 3/r$. We note that $q(r)$ is independent of K. The Eq. (1) can be regarded as the equations of relative motion of one of the bodies about another body in a two body system where the constant K depends on the choice of units (Musielak & Quarles 2017, Ch. 2; Beutler 2005, Ch. 3; Heggie 2005; Valtonen & Karttunen 2006, Ch. 3; Roy et al. 1972). Indeed, if $\mathbf{r}$ denotes the radius vector from a body P_1 of mass M to a body P of mass m then the equations of relative motion for the body P about the body P_1 is given by Eq. (1) with $f(r) = \frac{K}{r^3}$ where $K = G(M+m)$ and G is the constant of gravitation.

Example 2. Let $f(r) = k^2$ for some constant k. In this case, Eq. (1) models a high dimensional version of a simple harmonic oscillator (Hirsch & Smale 1974, Ch. 2.4). The corresponding central force field is type-I. In dimension 1, Eq. (1) is an equation for the undriven undamped oscillator.

Example 3. If $f(r) = \mu/r^4$ then Eq. (1) can be rewritten as

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{\mu}{r^3}\hat{\mathbf{r}}$$

whose solutions are known to produce Cotes' spirals (Danby 1988, Ch. 4.7; Whittaker & McCrae 1988, Ch. IV). The resulting central force field is type-I with $q(r) = 4/r$.

Consider an analytic function $u(t)$ defined over a domain containing zero. We denote the jth coefficient of the expansion of $u(t)$ around $t = 0$ by $\widehat{[u]}_j$ for $j \geq 0$. One advantage of the bracket notation $\widehat{[\cdot]}$ is that it allows one to express coefficients appearing in the expansions of composite functions in a more compact way. For instance, let w be a positive non-vanishing analytic function of t. Then $\widehat{\left[1/\sqrt{w}\right]}_j$ represents the jth coefficient in the expansion of $1/\sqrt{w}$ around zero.

By a recursive transform of a differential-algebraic system (abbreviated as DAS), we mean a system of recursive equations (abbreviated as SRE) consisting of those obtained from the given DAS. In general, the SRE of a DAS is not unique because it may further contain auxiliary recursive equations which are needed to solve the system, see for instance §4.

Remark 2. In Broucke (1971), it was stated that for the problem of 10 bodies revolving around the Sun, the first 25 terms of the expansions were used. In general, the choices of step sizes and the number of terms in expansions vary and depend on the problem itself (for instance Valtonen & Karttunen 2006, Ch. 2.11). Black (1973) reported that Myachin & Sizova (1972) used a fixed step size and a varying number of terms. Further applications related to regular polygons and star clusters were given in Black (1973). Sitarski (1979, Sec. 5) stated that, in the case of comets having high eccentric orbits, 12 terms in the expansions produced the desired results. A detailed discussion related to the size of integration steps can also be found in Sitarski (1979). It is further reported in Sitarski (1979) that Mercurial terms in the equation of the comet's motion affect the step size. In the present paper, we use similar numbers. However, the step size of Example 11 given below is chosen relatively smaller due to the nature of the problem.

3 The Well Definedness of Type

Let $\Omega = C^\omega(0, \infty)$ denote the linear space of all real valued and real analytic functions on the interval $(0, \infty)$. The following lemma states that the solution space of the equation $f' + qf = 0$ is invariant under multiplications by r and by $1/r$.

Lemma 1. A function $f \in \Omega$ satisfies $f' + q_1 f = 0$ for some $q_1 \in \Omega$ if and only if the function $g(r) = \frac{f(r)}{r}$ satisfies $g' + q_2 g = 0$ for some $q_2 \in \Omega$.

Proof. Given q_1 let us put

$$q_2(r) = q_1(r) + \frac{1}{r}$$

for every r. Because r is non-vanishing, $q_2 \in \Omega$. It follows that

$$\begin{aligned} g'(r) + q_2(r)g(r) &= \frac{f'(r)r - f(r)}{r^2} + \left(q_1(r) + \frac{1}{r}\right)\frac{f(r)}{r} \\ &= \frac{f(r)}{r}\left(-q_1(r) - \frac{1}{r} + q_1(r) + \frac{1}{r}\right) \\ &= 0 \end{aligned}$$

for every such r. Conversely, suppose that $g' + q_2 g = 0$. If we let

$$q_1(r) = q_2(r) - \frac{1}{r}$$

for every r and $f(r) = rg(r)$ then it follows that

$$\begin{aligned} f'(r) + q_1(r)f(r) &= (rg(r))' + \left(q_2(r) - \frac{1}{r}\right) rg(r) \\ &= g'(r)r + g(r) + rq_2(r)g(r) - g(r) \\ &= -rq_2(r)g(r) + rq_2(r)g(r) \\ &= 0 \end{aligned}$$

for every such r. □

The following lemma shows that the nomenclatures type-I and type-II, as given in Definition 1, are well-defined.

Proposition 1. The type of a central force field does not depend on the functions that are inducing it.

Proof. Let $f, g \in \Omega$ induce F. Without loss of generality, for every $\mathbf{r}$ we have $f(r)\mathbf{r} = g(r)\mathbf{r}$, and hence, $(f(r) - g(r))\mathbf{r} = 0$ where $\mathbf{r}$ is nonzero. Hence, $f(r) = g(r)$ for every r. Since $\mathbf{r}$ runs through the space freely except the origin, $f = g$ on the interval $(0, \infty)$. Since both f and g are analytic, if at least one of $f(0)$ or $g(0)$ is finite then the other is also finite, and their values at zero are equal to each other. Hence, $f' + qf = 0$ if and only $g' + qg = 0$. The other case is similar. □

In the following lemma, $P_n(r)$ denotes the nth approximating polynomial for $f \in \Omega$ about $r_0 > 0$.

Lemma 2. Denote by F_f and F_{P_n} be the central force fields induced by f and P_n, respectively. Then there exists $\xi(r)$ depending on r such that

$$\|(F_f - F_{P_n})(\mathbf{r})\| = \frac{|f^{n+1}(\xi(r))|}{(n+1)!}|(r - r_0)^{n+1}|r$$

for every $r > 0$.

Example 4. This example shows that F_{P_n} may not be type-I even though F_f is type-I. Let $f(r) = K/r^3$ as in Example 1. Then

$$P_1(r) = \frac{K}{r_0^3} - \frac{3K}{r_0^4}(r - r_0)$$

and there exists no $q \in \Omega$ such that $P_1' + qP_1 = 0$.

Let P be the nth Lagrange interpolating polynomial associated with f and $n+1$ distinct arguments $r_0, r_1, \ldots, r_n$ in $(0, \infty)$ with $r_i \in [R_0, R_1]$ for $R_0, R_1 > 0$.

Lemma 3. Denote by F_f and F_P be the central force fields induced by f and P, respectively. Then there exists $\xi(r)$ between R_0 and R_1

$$\|(F_f - F_P)(\mathbf{r})\| = \frac{|f^{n+1}(\xi(r))|}{(n+1)!}|(r - r_0)(r - r_1)\cdots(r - r_n)|r$$

for every $r \in [R_0, R_1]$.

4 The auxiliary variables w, s, σ and υ

This section is devoted to studying properties of three variables appearing in Black (1973); Moran et al. (1973); Roy & Moran (1973); Roy et al. (1972), and an additional auxiliary variable. Differential and algebraic relations between these variables allow us to obtain approximations to the solution of Eq. (1). Further, since some of these relations are independent of f, the corresponding SRE can be precomputed and can be used whenever needed, see Example 7.

Let $\mathbf{r}$ be as given in §2. Similar to Roy et al. (1972), we write

$$w = \frac{1}{r^2}, \quad s = \mathbf{r} \cdot \dot{\mathbf{r}}, \quad \sigma = ws, \quad \upsilon = \sqrt{w} = \frac{1}{r} \tag{2}$$

where s is defined in terms of the classical dot product. The auxiliary variables w, s and σ were used in Black (1973); Moran et al. (1973); Roy & Moran (1973); Roy et al. (1972). In the present paper, we further use υ since it simplifies the recursive formulae.

The following remark reflects the central idea of the present work.

Remark 3. In Roy et al. (1972), there is another auxiliary variable, called u, defined by

$$u = \frac{1}{r^3} \,,$$

(Roy et al. 1972, Eq.6), where $1/r^3$ appears in the equations (Roy et al. 1972, Eq.1) of relative motion. In the present work, we do not use u because if a central force field is type-I then the analytic function inducing that field satisfies a differential equation. This allows us not to use u but to use the differential equation instead.

Remark 4. Broucke (1971) studied the case of the general N-body problem. He introduced and used auxiliary variables $s_i = 1/r_i^3$ for each of the N planets revolving around the Sun and $s_{ij} = r_{ij}^{-3}$ for each pair of the planets. Sitarski (1979) used a similar auxiliary variable in the case of the dynamics of the comet.

The following lemma is needed in Proposition 2.

Lemma 4. We have

$$\begin{aligned} \widehat{[s^2]}_j = &\sum_{j_1=0}^{j} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j-j_1} (j_3+1)(j_2+1) \\ &\times \left[\widehat{[\mathbf{r}]}_{j_2+1} \cdot \widehat{[\mathbf{r}]}_{j_1-j_2}\right] \left[\widehat{[\mathbf{r}]}_{j_3+1} \cdot \widehat{[\mathbf{r}]}_{j-j_1-j_3}\right] \end{aligned}$$

and

$$\begin{aligned} \widehat{[s^3]}_j = &\sum_{j_2=0}^{j} \sum_{j_1=0}^{j_2} \sum_{j_3=0}^{j_2} \sum_{j_4=0}^{j_2-j_1} \sum_{j_5=0}^{j-j_1} (j_3+1)(j_4+1)(j_5+1) \\ &\times \Bigg[\left[\widehat{[\mathbf{r}]}_{j_3+1} \cdot \widehat{[\mathbf{r}]}_{j_2-j_3}\right] \left[\widehat{[\mathbf{r}]}_{j_4+1} \cdot \widehat{[\mathbf{r}]}_{j_1-j_2-j_4}\right] \\ &\left[\widehat{[\mathbf{r}]}_{j_5+1} \cdot \widehat{[\mathbf{r}]}_{j-j_1-j_5}\right] \Bigg] \end{aligned}$$

for $j \geq 0$.

Proof. Since $s = \mathbf{r} \cdot \dot{\mathbf{r}}$ we have

$$\widehat{[s]}_j = \sum_{i=0}^{j} (i+1)\widehat{[\mathbf{r}]}_{i+1} \cdot \widehat{[\mathbf{r}]}_{j-i}$$

for $j \geq 0$. As s^2 and s^3 denote the second and third powers of

s, respectively, we have

$$\widehat{[s^2]}_j = \sum_{j_1=0}^{j} \widehat{[s]}_{j_1} \widehat{[s]}_{j-j_1}$$

and

$$\widehat{[s^3]}_j = \sum_{j_2=0}^{j} \sum_{j_1=0}^{j_2} \widehat{[s]}_{j_1} \widehat{[s]}_{j_2-j_1} \widehat{[s]}_{j-j_2} .$$

for $j \geq 0$. The results follow from these. □

The following lemma is needed in the sequel.

Lemma 5. If $p \geq 1$ is independent of time then

$$\frac{\mathrm{d}}{\mathrm{d}t} w^{p/2} = -pw^{\frac{p+2}{2}} s = -pw^{\frac{p}{2}} \sigma \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} w^{-p/2} = pw^{\frac{2-p}{2}} s.$$

Proof. Let $p \geq 1$. Since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r^p}\right) = -p\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r^{p+2}},$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} w^{p/2} = -pw^{\frac{p+2}{2}} s = -pw^{\frac{p}{2}} \sigma .$$

Similarly, it follows from

$$\frac{\mathrm{d}r^p}{\mathrm{d}t} = pr^{p-2}\mathbf{r}\cdot\dot{\mathbf{r}}$$

that

$$\frac{\mathrm{d}}{\mathrm{d}t} w^{-p/2} = pw^{\frac{2-p}{2}} s$$

. □

Lemma 6. In the above setting, we have $(\dot{r})^2 = s^2 w$ and

$$\frac{\ddot{r}}{\dot{r}} = \frac{\dot{s}}{s} - sw.$$

Proof. We have

$$(\dot{r})^2 = (\frac{s}{r})^2 = s^2 w$$

and

$$\frac{\ddot{r}}{\dot{r}} = \frac{\dot{s}r - s\dot{r}}{sr} = \frac{\dot{s}}{s} - sw$$

. □

5 Results Related Type-I Central Force Fields

Consider Eq. (1) together with $f' + qf = 0$ over the interval $(0, \infty)$. If we put $\gamma = f(r)$ then we have

$$\dot{\gamma} = f'(r)\dot{r} = -q(r)f(r)sv.$$

The differential algebraic system (DAS) corresponding to Eq. (1) can be written as

$$\begin{aligned} \dot{\gamma} &= -q(r)\gamma sv, \\ \dot{v} &= -v\sigma, \\ s &= \mathbf{r}\cdot\dot{\mathbf{r}}, \\ \sigma &= ws, \\ w &= v^2, \\ \ddot{\mathbf{r}} &= -\gamma\mathbf{r}. \end{aligned} \tag{3}$$

We note that $q(r)$ appears only in the first equation above. Further, in the above system, there will be no advantage if we replace w with v, and vice versa. Indeed, since $v = \sqrt{w}$, it is possible to rewrite all equations containing v in terms of w. In that case, one has to deal with the time derivative of $\sqrt{w}$. On the other hand, if one changes every occurrence of w to v^2 then the equation $\sigma = v^2 s$ contains v^2. However, the equation $w = v^2$ is already listed in Eq. (3).

Example 5. If $f(r) = K/r^3$ for some positive constant K, see Example 1, then

$$f'(r)\dot{r} = \frac{-3Ks}{r^5} = -3f(r)sw = -3\gamma sw$$

which agrees with

$$q(r) = 3\sqrt{w} = 3/r ,$$

in terms of r. Thus, the first equation in Eq. (3) simplifies to $\dot{\gamma} = -3\gamma\sigma$, which is same as the first equation of the system (Roy et al. 1972, Eq. 7). A similar discussion for the case of two body problem is also given in Beutler (2005, Ch. 7.4.3), and in Roy (2020, Ch. 4.13).

Example 6. If $f(r) = K/r^p$ for some constants $p \geq 1$ and $K > 0$ then one can show that $q(r) = p/r$. Hence, in view of Example 5, the first equation of in Eq. (3) becomes

$$\dot{\gamma} = -p\gamma\sigma = -p\gamma sw .$$

The corresponding central force field generally appears in applications (Danby 1988, Prob. 4.11.6). In the case of

$$f(r) = \frac{K_1}{r^{p_1}} + \frac{K_2}{r^{p_2}}$$

with $p_1, p_2 \geq 1$ and $K_1, K_2 > 0$ one obtains

$$q(r) = \frac{K_2 p_2 r^{p_1} + K_1 p_1 r^{p_2}}{r\left(K_2 r^{p_1} + K_1 r^{p_2}\right)}$$

which is analytic over $(0, \infty)$. It follows that the first equation in Eq. (3) simplifies to

$$\dot{\gamma} = -\frac{(K_1 p_1 \nu^{p_1} + K_2 p_2 \nu^{p_2})}{K_1 \nu^{p_1} + K_2 \nu^{p_2}} \gamma s \nu^2$$

Example 7. Let $f(r) = k^2$ for some nonzero constant k, see Example 2. It follows from $f' + qf = 0$ that q is zero everywhere. Therefore, the Eq. (3) further simplifies, by Lemma 5, to

$$\begin{aligned} \dot{v} &= -v\sigma, \\ s &= \mathbf{r}\cdot\dot{\mathbf{r}}, \\ \sigma &= ws, \\ w &= v^2, \\ \ddot{\mathbf{r}} &= -\gamma\mathbf{r} \end{aligned}$$

though the DAS formed by w, s, σ and v is independent of the computation of $\mathbf{r}$ by recursions. This example suggests that one can precompute the SRE of this independent DAS for various initial values. We informally remark that one can even design special-purpose hardware for the numerical computations of these types of independent subsystems (Heggie 2005, Sec. 3.3).

Theorem 1. The SRE of Eq. (3) is given by

$$\begin{aligned}
\widehat{[\gamma]}_{j+1} &= \frac{-1}{j+1}\sum_{j_3=0}^{j}\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\widehat{[q(r)]}_{j_1}\widehat{[\gamma]}_{j_2-j_1}\widehat{[s]}_{j_3-j_2}\widehat{[\upsilon]}_{j-j_3},\\
\widehat{[\upsilon]}_{j+1} &= \frac{-1}{j+1}\sum_{j_1=0}^{j}\widehat{[\upsilon]}_{j_1}\widehat{[\sigma]}_{j-j_1},\\
\widehat{[s]}_{j} &= \sum_{j_1=0}^{j}(j-j_1+1)\widehat{[\mathbf{r}]}_{j_1}\cdot\widehat{[\mathbf{r}]}_{j-j_1+1},\\
\widehat{[\sigma]}_{j} &= \sum_{j_1=0}^{j}\widehat{[w]}_{j}\widehat{[s]}_{j-j_1},\\
\widehat{[w]}_{j} &= \sum_{j_1=0}^{j}\widehat{[\upsilon]}_{j}\widehat{[\upsilon]}_{j-j_1},\\
\widehat{[\mathbf{r}]}_{j+2} &= \frac{-1}{(j+1)(j+2)}\sum_{j_1=0}^{j}\widehat{[\gamma]}_{j_1}\widehat{[\mathbf{r}]}_{j-j_1},
\end{aligned} \tag{4}$$

for $j\geq 0$ where the initial values $\widehat{[w]}_0, \widehat{[s]}_0, \widehat{[\sigma]}_0, \widehat{[\upsilon]}_0, \widehat{[\mathbf{r}]}_0$, and $\widehat{[\mathbf{r}]}_1$ of recursions are obtained from the initial values of Eq. (3).

Remark 5. In general, initial values of Eq. (1) consists of the data $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$. By using Eq. (2), we determine the initial values of all variables appearing in the SRE given in Theorem 1. We note that this approach has the advantage that the SRE given in Theorem 1 does not change even if the initial time value $t_0 = 0$ is changed to t_1 with $t_0 \neq t_1$. In that case, $\widehat{[\cdot]}$ evaluates the coefficients with respect to time at t_1.

Remark 6. In the views of §4 and Theorem 1, the scalar magnitudes $\gamma, \upsilon, s, \sigma, w$, the vector $\mathbf{r} = (x, y)$ and the function q are all transformed into sequences $\widehat{[\gamma]}_j, \widehat{[\upsilon]}_j, \widehat{[s]}_j, \widehat{[\sigma]}_j, \widehat{[w]}_j, \widehat{[x]}_j, \widehat{[y]}_j$ and $\widehat{[q]}_j$ for $j\geq 0$, respectively. Let $x(0) = X_0$, $y(0) = Y_0$, $x'(0) = X_1$, $y'(0) = Y_1$ be the initial values of the problem. We put $\widehat{[x]}_0 = X_0$, $\widehat{[x]}_1 = X_1$, $\widehat{[y]}_0 = Y_0$, and $\widehat{[Y]}_0 = Y_1$. The corresponding SRE can be iterated for any $j\geq 0$ and the number of terms N of the expansion provides a stopping condition for the iteration procedure. Once a stopping condition is satisfied then $\widehat{[x]}_j$ and $\widehat{[y]}_j$ are evaluated for $j = 0, 1, \ldots, N$. On the other hand, not all variables are needed to be evaluated up to order N because of the difference of degrees appearing in the SRE given in Theorem 1.

Proposition 2. In order to compute $\widehat{[\mathbf{r}]}_\ell$ for some $\ell\geq 4$ it suffices to compute $\widehat{[\gamma]}_j$ for $j = 0, 1, \ldots, \ell-2$; $\widehat{[\upsilon]}_j$ for $j = 0, 1, \ldots, \ell-3$; $\widehat{[s]}_j$ for $j = 0, 1, \ldots, \ell-3$; $\widehat{[\sigma]}_j$ for $j = 0, 1, \ldots, \ell-4$; and $\widehat{[q]}_j$ for $j = 0, 1, \ldots, \ell-3$.

Proof. By using the equation of $\widehat{[\mathbf{r}]}$ given in Theorem 1, we see that $\widehat{[\mathbf{r}]}_\ell$ depends on $\widehat{[\gamma]}_{\ell-2}$. Similarly, $\widehat{[\gamma]}_{\ell-2}$ depends on $\widehat{[\upsilon]}_{\ell-3}$, $\widehat{[s]}_{\ell-3}$ and $\widehat{[q]}_{\ell-3}$. Consider the magnitude $\widehat{[\sigma]}_{\ell-3}$ together with the only equation

$$\widehat{[\upsilon]}_{j+1} = \frac{-1}{j+1}\sum_{j_1=0}^{j}\widehat{[\upsilon]}_{j_1}\widehat{[\sigma]}_{j-j_1},$$

appearing in the SRE of Theorem 1 containing σ on its right side. Because the maximum value of the index $j-j_1$ is j, $\widehat{[\sigma]}_{\ell-3}$ produces $\widehat{[\upsilon]}_{\ell-2}$, which is not needed from the previous step. Hence, by structural induction it suffices to compute $\widehat{[\sigma]}_j$ for $j = 0, 1, \ldots, \ell-4$. □

Remark 7. Suppose that an explicit formula, in terms r, for q is available. In the SRE given in Theorem 1, one requires to compute $\widehat{[q(r)]}_j$ with respect to time. Suppose further that the values of components of $\frac{\mathrm{d}^i\mathbf{r}}{\mathrm{d}t^i}$ for $i = 0, 1, \ldots, j$ at $t = 0$ are known. Then one may use McKiernan's formula (McKiernan 1956),

$$\widehat{[q(r)]}_j = \frac{1}{j!}\left[\sum_{j_1=1}^{j}\frac{\mathrm{d}^{j_1}}{\mathrm{d}x^{j_1}}[q(x)]_{x=r(t)} \times \sum_{j_2=0}^{j_1}\frac{(-1)^{j_1-j_2}}{j_2!(j_1-j_2)!}[r(t)]^{j_1-j_2}\frac{\mathrm{d}^j}{\mathrm{d}t^j}[r(t)]^{j_2}\right]_{t=0}$$

for the jth component of the recursive transform of the composite function $q(r)$. When the summations appearing on the right side of this formula expanded, the order of derivatives of q and r go up to exactly j. More details about this formula can be found in McKiernan (1956).

Example 8. The SRE corresponding to the high dimensional harmonic oscillator of Example 7, is given by

$$\widehat{[\mathbf{r}]}_{j+2} = \frac{-k^2}{(j+1)(j+2)}\widehat{[\mathbf{r}]}_j,$$

for $j\geq 0$ together with the independent SRE formed by w, s, σ and υ.

Example 9. If $f(r) = \mu/r^4$ then $q(r) = 4/r$, see Example 3. Depending on the initial values and μ one obtains Cotes' spirals of various types (Danby 1988, Ch. 4.7; Whittaker & McCrae 1988, Ch. IV). In this case, $q = 4\upsilon$ and the initial value of γ of the corresponding SRE is μ/r_0^4 where r_0 is the norm of the initial position. Now we give a concrete example. Let $\mu = 7$ with initial data $x(0) = -40$, $y(0) = -40$, $x'(0) = 0.067246$, $y'(0) = 0$ with step size $h = 0.1$ on the interval $[0, 1500]$. For the number of terms of the expansions concerning the time, we use 12. The value $x'(0) = 0.067246$ is chosen for the purpose of obtaining an epispiral as shown in Figure 1. The solution curve sensitively depends on $x'(0)$ in the sense that small changes in $x'(0)$ cause drastic changes in the solution. The second and third graphs given in Figure 1 illustrate respectively the error vectors and their magnitudes obtained from the standard solver of Mathematica and the SRE of Theorem 1. The norm of the error becomes larger as the solution curve approaches the center of the field.

In the following example, e denotes the base of Napier's logarithm, not the eccentricity. Instead of $\exp(f(x))$ we write $e^{f(x)}$.

Example 10. Let $p_1\geq 1$ and $p_2 > 1$ be integers. If $f(r) = e^{\frac{p_1}{p_2-1}r^{1-p_2}}$ then $q(r) = p_1/r^{p_2} = p_1\upsilon^{p_2}$. Hence, the

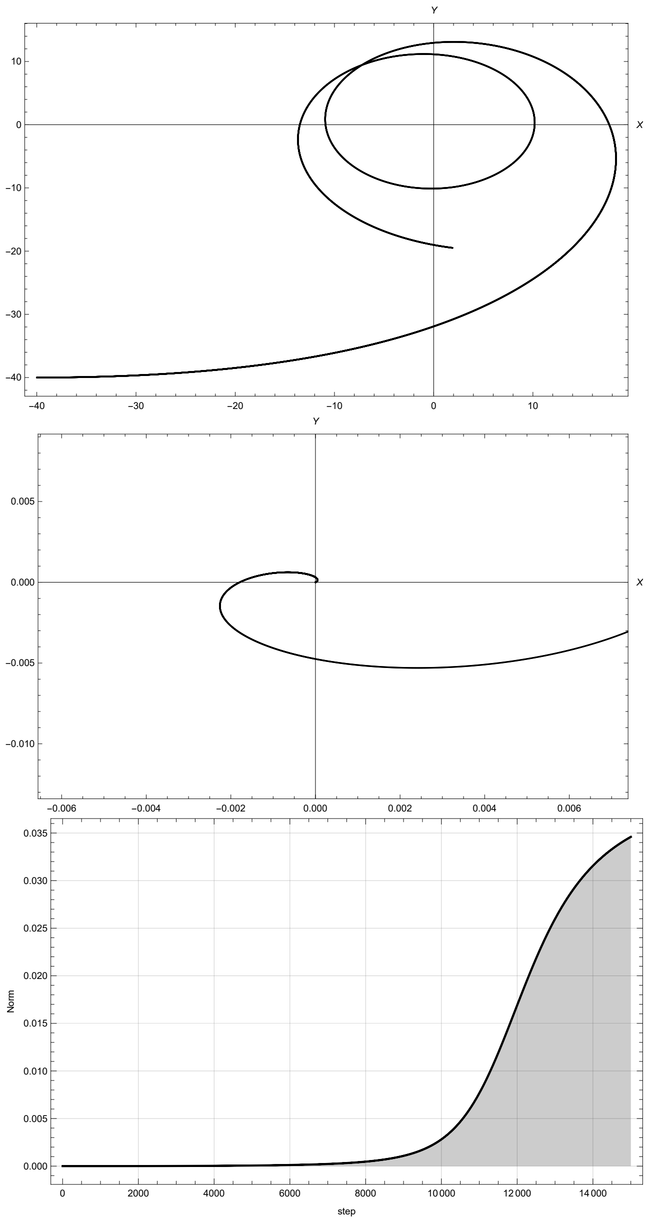

Figure 1. (Top Panel) A numerical solution for Cotes' spirals given in Example 9 (Middle Panel) The error vectors with respect to the standard Mathematica solver (Bottom Panel) The norm of the error vectors showing that the global error is less than 0.035 in the given range

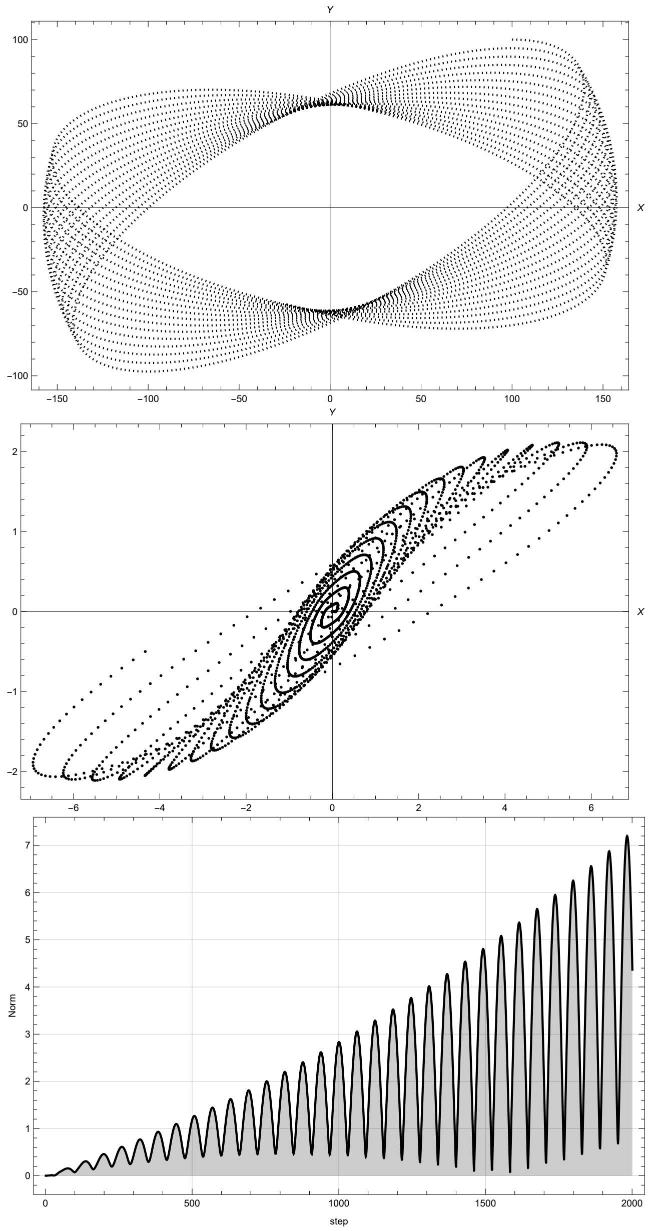

Figure 2. (Top Panel) A numerical solution in the case $f(r) = e^{\frac{p_1}{p_2-1}r^{1-p_2}}$, see Example 10 (Middle Panel) The error vectors with respect to the standard Mathematica solver (Bottom Panel) The norm of the error vectors showing a quasi-periodic global error

corresponding DAS is

$$
\begin{aligned}
\dot{\gamma} &= -p_1 \upsilon^{p_2} \gamma s \upsilon, \\
\dot{\upsilon} &= -\upsilon \sigma, \\
s &= \mathbf{r} \cdot \dot{\mathbf{r}}, \\
\sigma &= w s, \\
w &= \upsilon^2, \\
\ddot{\mathbf{r}} &= -\gamma \mathbf{r}.
\end{aligned}
$$

It follows that for the SRE given in Eq. (4) of Theorem 1, it suffices to compute

$$
\begin{aligned}
\widehat{[q(r)]}_j &= p_1 \widehat{[\upsilon^{p_2}]}_j \\
&= p_1 \sum_{j_{p_2-1}=0}^{j} \sum_{j_{p_2-2}=0}^{j_{p_2-1}} \cdots \sum_{j_1=0}^{j_2} \prod_{m=1}^{p_2} \widehat{[\upsilon]}_{j_m - j_{m-1}}
\end{aligned}
$$

with $j_0 = 0$ and $j_{p_2} = j$. Let us give a concrete example. We choose $p_1 = 8$ and $p_2 = 2$ so that $f(r) = e^{8/r}$. Consider the initial data $x(0) = 100$, $y(0) = 100$, $x'(0) = 100$, $y'(0) = 0$ with step size $h = 0.05$ on the interval $[0, 100]$. For the number of terms of the expansions, we use 13. The solution curve is given Figure 2 together with the comparison of the standard

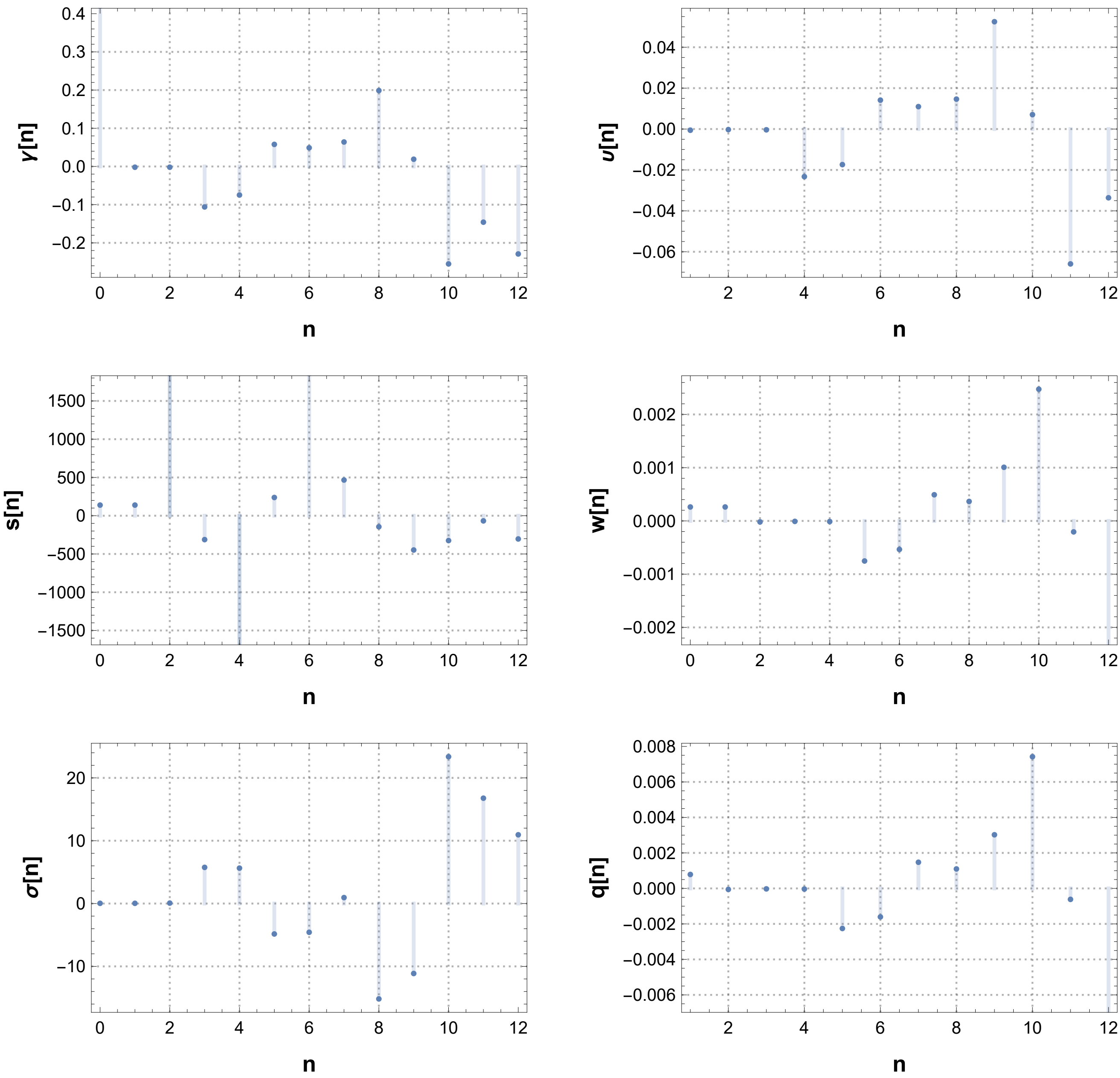

Figure 3. Plots of the auxiliary variables at the initial location of the numerical example given in Example 10

solver of Mathematica and the SRE of Theorem 1. The third graph illustrates the fact that the norm of the error changes in a quasi-periodic manner becoming smaller and larger as time increases. In the views of Remark 6 and Proposition 2, Figure 3 illustrates the changes in the auxiliary variables.

Example 11. Let $Q(r) = a_{-1} + a_0 \ln r + \frac{a_1}{r} + \cdots + \frac{a_n}{r^n}$ for some real constants $a_{-1}, a_0, \ldots, a_n$. Then $f(r) = e^{Q(r)}$ is a solution of $f' + qf = 0$ with $q(r) = -Q'(r)$. Thus, if we set $f(r) = e^{Q(r)}$ then Eq. (1) becomes

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -e^{a_{-1}+a_0 \ln r + \frac{a_1}{r} + \cdots + \frac{a_n}{r^n}}\mathbf{r}$$

whose DAS contains

$$\begin{aligned}\dot\gamma &= -q(r)\gamma s\upsilon \\ &= (\frac{a_0}{r} - \frac{a_1}{r^2} - \cdots - \frac{n a_n}{r^{n+1}})\gamma s\upsilon\end{aligned}$$

as one of its equations, see Eq. (3). Since $\upsilon = 1/r$, we can write

$$\dot\gamma = (a_0\upsilon - a_1\upsilon^2 - \cdots - n a_n \upsilon^{n+1})\gamma s\upsilon,$$

so that $\widehat{[\upsilon]}_j$ for $j = 0, 1, 2, \ldots \ell$ determines $\widehat{[\upsilon^k]}_\ell$ for $k = 1, \ldots, n+1$. We consider the case $Q(r) = 1 + \ln r - \frac{5}{r} - \frac{50}{r^2}$ so that

$$\dot\gamma = (\upsilon + 5\upsilon^2 + 100\upsilon^3)\gamma s\upsilon$$

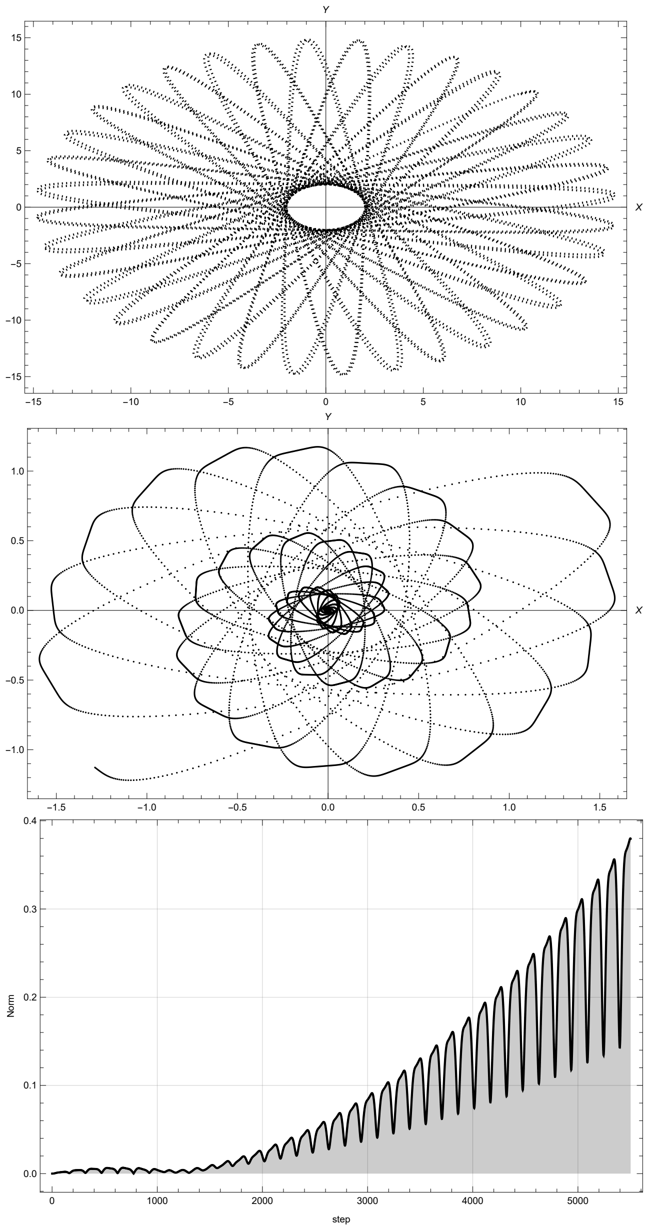

Figure 4. (Top Panel) Solution of the numerical example given in Example 11 (Middle Panel) The error vectors with respect to the standard Mathematica solver (Bottom Panel) The norm of the error vectors showing a small nonlinear and accumulation of error

in terms of υ, γ and s only. Consider the initial data $x(0) = 10$, $y(0) = 10$, $x'(0) = 20$, $y'(0) = 10$ with step size $h = 0.005$ on the interval $[0, 55]$. For the number of terms of the expansions, we use 12. The solution curve is given Figure 4 together with the comparison of the standard solver of Mathematica and the SRE of Theorem 1. Contrary to the case of Example 10, the third graph of Figure 4 implies a nonlinear and increasing accumulation of error which is consistent with the second graph. In this case, $h = 0.005$ is relatively smaller than the step sizes of the previous examples and the accumulated error is much smaller, see also Remark 2.

6 A Partial Result Related Type-II Central Force Fields

Consider Eq. (1) together with $f'' + pf' + qf = 0$ over the interval $(0, \infty)$. If we put

$$\begin{aligned} \gamma &= f(r), \\ \kappa &= \dot\gamma = f'(r)\dot r = f'(r)s\sqrt{w} \end{aligned}$$

then it follows that

$$\dot\kappa = f''(r)(\dot r)^2 + f'(r)\ddot r.$$

Since $f''+pf'+qf = 0$ over the interval $(0, \infty)$ and $r \in (0, \infty)$, we have

$$f''(r) = -p(r)f'(r) - q(r)f(r)\ .$$

Hence,

$$\begin{aligned} \dot\kappa &= \left[-p(r)f'(r) - q(r)f(r)\right](\dot r)^2 + f'(r)\ddot r \\ &= -p(r)f'(r)(\dot r)^2 - q(r)f(r)(\dot r)^2 + f'(r)\ddot r \end{aligned}$$

which contains the term $f'(r)\ddot r$.

Lemma 7. In the above setting, if $\dot r \neq 0$ then

$$f'(r)\ddot r = \kappa\frac{\dot s}{s} - \kappa s w = \kappa\frac{\dot s}{s} - s^3\upsilon^4$$

.

Proof.

$$\begin{aligned} f'(r)\ddot r &= f'(r)\frac{\mathrm{d}s\sqrt{w}}{\mathrm{d}t} \\ &= f'(r)\dot s\sqrt{w} - f'(r)\frac{s^2}{r^3} \\ &= \frac{\kappa r}{s}\dot s\sqrt{w} - \frac{\kappa r}{s}\frac{s^2}{r^3} \\ &= \kappa\frac{\dot s}{s} - \kappa s w \end{aligned}$$

because

$$f'(r) = \frac{\kappa}{\dot r} = \frac{\kappa r}{s}$$

whenever $\dot r \neq 0$. Furthermore, this implies that $s = \mathbf{r}\cdot\dot{\mathbf{r}} \neq 0$ because $\mathbf{r}$ is assumed to be away from the origin. We finally note that one can also multiply both sides of $\ddot r = -s^2\upsilon^3 + \dot s\upsilon$ by $f'(r)$ to obtain the same result. ☐

It follows from Lemma 7 that

$$\begin{aligned} \dot\kappa &= -p(r)f'(r)(\dot r)^2 - q(r)f(r)(\dot r)^2 + f'(r)\ddot r \\ &= -p(r)\kappa s\upsilon - q(r)\gamma s^2 w + \kappa\frac{\dot s}{s} - \kappa s w. \end{aligned}$$

because $\kappa = f'(r)\dot r$ and $f(r)(\dot r)^2 = \gamma s^2 w$. The corresponding DAS can be written as

$$\begin{aligned} \dot\gamma &= \kappa, \\ \dot\kappa &= -p(r)\kappa s\upsilon - q(r)\gamma s^2 w + \kappa\frac{\dot s}{s} - \kappa s w, \\ \dot\upsilon &= -\upsilon\sigma, \\ s &= \mathbf{r}\cdot\dot{\mathbf{r}}, \\ \sigma &= ws, \\ w &= \upsilon^2, \\ \ddot{\mathbf{r}} &= -\gamma\mathbf{r} \end{aligned} \tag{5}$$

in which $\dot\kappa$ can also be expressed in such a way that it contains a term containing s^3, see Lemma 7. Unlike the reduction in the central force fields of type-I, in the type-II case we further need $\widehat{[s^2]}_j$ or $\widehat{[s^3]}_j$ for $j \geq 0$, see Lemma 4.

Theorem 2. In the SRE of Eq. (5), the recursive transform of the second equation can be found from

$$
\begin{aligned}
\widehat{[p(r)\kappa s v]}_j &= \sum_{j_3=0}^{j}\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2} \widehat{[p(r)]}_{j_1}\widehat{[\kappa]}_{j_2-j_1}\widehat{[s]}_{j_3-j_2}\widehat{[v]}_{j-j_3},\\
\widehat{[q(r)\gamma s^2 w]}_j &= \sum_{j_3=0}^{j}\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2} \widehat{[q(r)]}_{j_1}\widehat{[\gamma]}_{j_2-j_1}\widehat{[s^2]}_{j_3-j_2}\widehat{[w]}_{j-j_3},\\
\widehat{\left[\kappa\frac{\dot{s}}{s}\right]}_j &= \sum_{j_1=0}^{j}\widehat{[\kappa]}_{j_1}\widehat{\left[\frac{\dot{s}}{s}\right]}_{j-j_1},\\
\widehat{[\kappa s w]}_j &= \sum_{j_2=0}^{j}\sum_{j_1=0}^{j_2}\widehat{[\kappa]}_{j_1}\widehat{[s]}_{j_2-j_1}\widehat{[w]}_{j-j_2},\\
\widehat{[\kappa]}_{j+1} &= -\widehat{[p(r)\kappa s v]}_j - \widehat{[q(r)\gamma s^2 w]}_j + \widehat{\left[\kappa\frac{\dot{s}}{s}\right]}_j - \widehat{[\kappa s w]}_j
\end{aligned}
$$

provided that the coefficients of $\dot{s}/s$ in the expansion with respect to time are known.

7 Conclusion

In the present paper, by using auxiliary variables w, s, σ and v and the equation $f' + qf = 0$, we generalized one of the numerical methods used by Roy et al. (1972), Broucke (1971), and Sitarski (1979) to determine dynamics a particle under the action of a central force field induced by f. The current method is more comprehensive in its applicability, as it can be used to study a particle's motion under both gravitational and non-gravitational central forces. We remark that Roy (2020) applied this numerical method to perturbation problems, while the current approach remains untested. In the present settings, what we mean by a perturbation problem is that the particle's trajectory is disturbed by a term added to the second derivative. In the gravitational case, this term is usually called the perturbing acceleration.

The case of N-body problem, in which multiple interacting bodies affect each other's motion was discussed in Broucke (1971). The present method might be useful in investigating the dynamics of a particle in a compound field with N different centers. We further remark that the recent advancements in the theory of integral equations and decomposition methods may have further applications in the present settings.

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