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Binomial Transforms of the Third-Order Jacobsthal and Modified Third-Order Jacobsthal Polynomials

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Abstract

In this study, we define the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials. Further, the generating functions, Binet formulas and summation of these binomial transforms are found by recurrence relations. Also, we establish the relations between these transforms by deriving new formulas. Finally, the Vajda, d'Ocagne, Catalan and Cassini formulas for these transforms are obtained.

1. Introduction

The study of number sequences has been the subject of several studies published in recent decades. Algebraic properties, generating function, Binet's formula and some well-known identities have been studied in this research topic.
In 2013, Cook and Bacon [1] introduced the notion of third-order Jacobsthal numbers $\{J_n^{(3)}\}_{n\geq\mathbb{N}}$ as an extension to the famous properties of the Jacobsthal sequence. The recurrence relation of this number is $J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}$ for $n \geq 0$, where $J_0^{(3)} = 0$ and $J_1^{(3)} = J_2^{(3)} = 1$. A new study on the modified third-order Jacobsthal numbers $K_{n+2}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 6J_n^{(3)}$ was published in 2020 by Morales [2]. The recurrence relation of this number is $K_{n+3}^{(3)} = K_{n+2}^{(3)} + K_{n+1}^{(3)} + 2K_n^{(3)}$ for $n \geq 0$, where $K_0^{(3)} = 3$, $K_1^{(3)} = 1$ and $K_2^{(3)} = 3$. In addition, Soykan et. al. in [3] studied the binomial transforms of the generalized third-order Jacobsthal numbers.
Some generalizations of third-order Jacobsthal numbers can be obtained in various ways (see, e.g., [4–6]). A natural extension is to consider for $x \in \mathbb{C}$ sequences of third-order Jacobsthal and modified third-order Jacobsthal polynomials $\{J_n^{(3)}(x)\}_{n\geq\mathbb{N}}$ and $\{K_n^{(3)}(x)\}_{n\geq\mathbb{N}}$, respectively. Third-order Jacobsthal and modified third-order Jacobsthal polynomials are defined by the recurrence relations

$$\begin{aligned} &J_{n+3}^{(3)}(x) = (x-1)J_{n+2}^{(3)}(x) + (x-1)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x), \\ &J_0^{(3)}(x) = 0,\ J_1^{(3)}(x) = 1,\ J_2^{(3)}(x) = x-1 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &K_{n+3}^{(3)}(x) = (x-1)K_{n+2}^{(3)}(x) + (x-1)K_{n+1}^{(3)}(x) + xK_n^{(3)}(x), \\ &K_0^{(3)}(x) = 3,\ K_1^{(3)}(x) = x-1,\ K_2^{(3)}(x) = x^2-1, \end{aligned} \tag{1.2}$$

respectively. For more information, see [7].
On the other hand, some matrix-based transforms can be introduced for a given sequence. The binomial transform is one such transform and there are also other transforms such as rising and falling binomial transforms (see, e.g., [8]). Also, there is an interesting study on watermarking and the binomial transform. In [9], Falcón and Plaza studied the binomial transforms of the k-Fibonacci sequences. In [10], Prodinger gave some information about binomial transform. In [11], a novel Binomial transform based fragile watermarking technique has been proposed for color image authentication. In [12], Yilmaz defined and studied the binomial transforms of the Balancing and

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Lucas-Balancing polynomials. In [13], Özkoç and Gündüz studied the binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion. In [14], Yilmaz and Aktaş studied special transforms of the generalized bivariate Fibonacci and Lucas polynomials. Other examples can be reviewed in [15, 16].
Now we give some preliminaries related our study. Given an integer sequence $\Psi = \{\psi_0, \psi_1, \psi_2, \cdots\}$, the binomial transform $\mathscr{B}$ of the sequence Ψ, $\mathscr{B}(\Psi) = \{\Phi_n\}$, is given by

$$\Phi_n = \sum_{j=0}^{n} \binom{n}{j} \psi_j.$$

Furthermore, in [17], Boyadzhiev studied the following properties of the binomial transform Φ_n:

$$\sum_{j=0}^{n} \binom{n}{j} j\psi_j = n(\Phi_n - \Phi_{n-1})$$

and

$$\sum_{j=1}^{n} \binom{n}{j} \psi_j j^{-1} = \sum_{j=1}^{n} \Phi_j j^{-1}.$$

Motivated essentially by the previous papers, the objective of this study is to apply the binomial transforms to the third-order Jacobsthal $\{J_n^{(3)}(x)\}$ and modified third-order Jacobsthal polynomials $\{K_n^{(3)}(x)\}$ in Eqs. (1.1) and (1.2). Furthermore, the generating functions of binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials are found by recurrence relations. Also, we describe the Vajda and d'Ocagne formulas and the relations between these transforms by deriving new formulas.

2. Binomial Transforms of Third-Order Jacobsthal Polynomials

In this section, we will mainly focus on binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials to get some important results. In fact, as a middle step, we will also present the recurrence relations, generating functions and Binet formulas.

Definition 2.1. *Let $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ be the third-order Jacobsthal and modified third-order Jacobsthal polynomials, respectively. The binomial transforms of these polynomials can be expressed as follows:*

1. *the binomial transform of the third-order Jacobsthal polynomial is*

$$\mathscr{J}_n(x) = \sum_{j=0}^{n} \binom{n}{j} J_j^{(3)}(x),$$

2. *the binomial transform of the modified third-order Jacobsthal polynomial is*

$$\mathscr{K}_n(x) = \sum_{j=0}^{n} \binom{n}{j} K_j^{(3)}(x).$$

Before starting the results, it is useful to say $\binom{n}{j} = 0$ for $j > n$.
The following lemma will be key to the proof of the next theorem.

Lemma 2.2. *For $n \geq 0$, the following equalities hold:*

$$\mathscr{J}_{n+1}(x) - \mathscr{J}_n(x) = \sum_{j=o}^{n} \binom{n}{j} J_{j+1}^{(3)}(x), \tag{2.1}$$

$$\mathscr{K}_{n+1}(x) - \mathscr{K}_n(x) = \sum_{j=o}^{n} \binom{n}{j} K_{j+1}^{(3)}(x). \tag{2.2}$$

Proof. We will only prove Eq. (2.1) since the proof of Eq. (2.2) is analogous. By using Definition 2.1 and the well known binomial equality for $1 \leq j \leq n$

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}, \tag{2.3}$$

we obtain

$$\begin{aligned}
\mathscr{J}_{n+1}(x) &= \sum_{j=1}^{n+1} \binom{n+1}{j} J_j^{(3)}(x) + J_0^{(3)}(x) \\
&= \sum_{j=0}^{n+1} \binom{n}{j} J_j^{(3)}(x) + \sum_{j=1}^{n+1} \binom{n}{j-1} J_j^{(3)}(x) \\
&= \sum_{j=0}^{n} \binom{n}{j} \left(J_j^{(3)}(x) + J_{j+1}^{(3)}(x) \right),
\end{aligned}$$

which is desired result. □

Theorem 2.3. *For $n \geq 0$, we have to*

1. *the recurrence relation of sequences $\{\mathscr{J}_n(x)\}$ is*
$$\mathscr{J}_{n+3}(x) = (x+2)\left[\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right] + (x+1)\mathscr{J}_n(x), \tag{2.4}$$
with initial conditions $\mathscr{J}_0(x) = 0$, $\mathscr{J}_1(x) = 1$ and $\mathscr{J}_2(x) = x+1$.
2. *the recurrence relation of sequences $\{\mathscr{K}_n(x)\}$ is*
$$\mathscr{K}_{n+3}(x) = (x+2)\left[\mathscr{K}_{n+2}(x) - \mathscr{K}_{n+1}(x)\right] + (x+1)\mathscr{K}_n(x), \tag{2.5}$$
with initial conditions $\mathscr{K}_0(x) = 3$, $\mathscr{K}_1(x) = x+2$ and $\mathscr{K}_2(x) = x^2+2x$.

Proof. Similar to the proof of the previous theorem, only the first case (2.4) will be proved. We omit the other cases since the proofs are similar. By considering Definition 2.1 and $J_0^{(3)}(x) = 0$, we obtain

$$\mathscr{J}_{n+3}(x) = \sum_{j=0}^{n+3}\binom{n+3}{j}J_j^{(3)}(x) = \sum_{j=0}^{n+2}\binom{n+3}{j+1}J_{j+1}^{(3)}(x).$$

By taking into account Eq. (2.3), we get

$$\mathscr{J}_{n+3}(x) = \sum_{j=0}^{n+2}\left[\binom{n+1}{j+1} + 2\binom{n+1}{j} + \binom{n+1}{j-1}\right]J_{j+1}^{(3)}(x).$$

By considering recurrence relations of third-order Jacobsthal polynomials

$$J_{j+3}^{(3)}(x) = (x-1)J_{j+2}^{(3)}(x) + (x-1)J_{j+1}^{(3)}(x) + xJ_j^{(3)}(x), \; j \geq 0,$$

and the equality $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$, we obtain

$$\begin{aligned}
\mathscr{J}_{n+3}(x) &= \sum_{j=0}^{n+2}\binom{n+1}{j+1}J_{j+1}^{(3)}(x) + 2\sum_{j=0}^{n+2}\binom{n+1}{j}J_{j+1}^{(3)}(x) + \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j+1}^{(3)}(x) \\
&= \sum_{j=0}^{n+1}\binom{n+1}{j}J_j^{(3)}(x) + 2\sum_{j=0}^{n+1}\binom{n+1}{j}J_{j+1}^{(3)}(x) \\
&\quad + \sum_{j=0}^{n+2}\binom{n+1}{j-1}\left[(x-1)J_j^{(3)}(x) + (x-1)J_{j-1}^{(3)}(x) + xJ_{j-2}^{(3)}(x)\right] \\
&= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) \\
&\quad + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) \\
&\quad - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-2}^{(3)}(x).
\end{aligned}$$

Using Lemma 2.2 and $\binom{n+1}{j-1} = \binom{n}{j-1} + \binom{n}{j-2}$, we have

$$\begin{aligned}
\mathscr{J}_{n+3}(x) &= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) + x\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) - \mathscr{J}_{n+1}(x) \\
&\quad - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n}{j-1}J_{j-2}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n}{j-2}J_{j-2}^{(3)}(x) \\
&= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) + x\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) - \mathscr{J}_{n+1}(x) + (x+1)\mathscr{J}_n(x) \\
&\quad - \sum_{j=0}^{n}\binom{n}{j}\left[J_{j+1}^{(3)}(x) - (x-1)J_j^{(3)}(x) - (x-1)J_{j-1}^{(3)}(x) - xJ_{j-2}^{(3)}(x)\right] \\
&= (x+2)\left[\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right] + (x+1)\mathscr{J}_n(x)
\end{aligned}$$

which completes the proof in this case. □

Remark 2.4. *For $n \geq 0$ and $x = 2$ in Theorem 2.3, we have to*

1. *the recurrence relation of binomial transform for third-order Jacobsthal numbers $J_n^{(3)}$ is*
$$\mathscr{J}_{n+3} = 4\left[\mathscr{J}_{n+2} - \mathscr{J}_{n+1}\right] + 3\mathscr{J}_n,$$
with initial conditions $\mathscr{J}_0 = 0$, $\mathscr{J}_1 = 1$ and $\mathscr{J}_2 = 3$.
2. *the recurrence relation of binomial transform for modified third-order Jacobsthal numbers $K_n^{(3)}$ is*
$$\mathscr{K}_{n+3} = 4\left[\mathscr{K}_{n+2} - \mathscr{K}_{n+1}\right] + 3\mathscr{K}_n,$$
with initial conditions $\mathscr{K}_0 = 3$, $\mathscr{K}_1 = 4$ and $\mathscr{K}_2 = 8$.

Also, the generating functions for third-order Jacobsthal and modified third-order Jacobsthal polynomials play a vital role in determining some important identities of these new polynomial sequences. In the following theorem, we develop the generating functions for the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 2.5. *The generating functions of the binomial transforms for $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ are*

$$g(\mathscr{J}_n(x);\lambda) = \sum_{j=0}^{\infty} \mathscr{J}_j(x)\lambda^j = \frac{\lambda}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3} \tag{2.6}$$

and

$$g(\mathscr{K}_n(x);\lambda) = \sum_{j=0}^{\infty} \mathscr{K}_j(x)\lambda^j = \frac{3-(2x+4)\lambda+(x+2)\lambda^2}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3}. \tag{2.7}$$

Proof. We omit the third-order Jacobsthal case in Eq. (2.6) since the proof is similar. For Eq. (2.7), assume that $g(\mathscr{K}_n(x);\lambda)$ is the generating function of the binomial transform for $\{\mathscr{K}_n(x)\}$. The, we obtain

$$g(\mathscr{K}_n(x);\lambda) = \sum_{j=0}^{\infty} \mathscr{K}_j(x)\lambda^j = \mathscr{K}_0(x)+\mathscr{K}_1(x)\lambda+\mathscr{K}_2(x)\lambda^2+\cdots.$$

Using Theorem 2.3, we have

$$\begin{aligned} g(\mathscr{K}_n(x);\lambda) &= \mathscr{K}_0(x)+\mathscr{K}_1(x)\lambda+\mathscr{K}_2(x)\lambda^2+\sum_{j=3}^{\infty}\left((x+2)\left[\mathscr{K}_{j-1}(x)-\mathscr{K}_{j-2}(x)\right]+(x+1)\mathscr{K}_{j-3}(x)\right)\lambda^j \\ &= \mathscr{K}_0(x)+\left(\mathscr{K}_1(x)-(x+2)\mathscr{K}_0(x)\right)\lambda+\left(\mathscr{K}_2(x)-(x+2)(\mathscr{K}_1(x)-\mathscr{K}_0(x))\right)\lambda^2 \\ &\quad +(x+2)\lambda g(\mathscr{K}_n(x);\lambda)-(x+2)\lambda^2 g(\mathscr{K}_n(x);\lambda)-(x+1)\lambda^3 g(\mathscr{K}_n(x);\lambda). \end{aligned}$$

Now rearrangement the equation implies that

$$g(\mathscr{K}_n(x);\lambda) = \frac{\mathscr{K}_0(x)+\left(\mathscr{K}_1(x)-(x+2)\mathscr{K}_0(x)\right)\lambda+\left(\mathscr{K}_2(x)-(x+2)(\mathscr{K}_1(x)-\mathscr{K}_0(x))\right)\lambda^2}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3},$$

which is equal to $\sum_{j=0}^{\infty}\mathscr{K}_j(x)\lambda^j$ in the theorem. □

Further, we note that $g(\mathscr{J}_n(x);\lambda)$ and $g(\mathscr{K}_n(x);\lambda)$ may be obtained from the generating functions of the third-order Jacobsthal and third-order Jacobsthal polynomials in [7], we have

$$g\left(J_n^{(3)}(x);\lambda\right) = \frac{\lambda}{1-(x-1)\lambda-(x-1)\lambda^2-x\lambda^3}$$

and

$$g\left(K_n^{(3)}(x);\lambda\right) = \frac{3-(x-1)\lambda-(x-1)\lambda^2}{1-(x-1)\lambda-(x-1)\lambda^2-x\lambda^3}.$$

It is seen by using the following result proved by Prodinger in [10]:

$$g(\mathscr{J}_n(x);\lambda) = \frac{1}{1-\lambda}g\left(J_n^{(3)}(x);\frac{\lambda}{1-\lambda}\right)$$

and

$$g(\mathscr{K}_n(x);\lambda) = \frac{1}{1-\lambda}g\left(K_n^{(3)}(x);\frac{\lambda}{1-\lambda}\right).$$

To derive new identities of the binomial transform of third-order Jacobsthal and modified third-order Jacobsthal polynomials, we now present an explicit formula for $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ for $n \geq 0$.

Theorem 2.6. *The Binet formulas of sequences $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ are*

$$\mathscr{J}_n(x) = \frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)} \tag{2.8}$$

and

$$\mathscr{K}_n(x) = (x+1)^n+\omega_1^n+\omega_2^n,$$

where ω_1 and ω_2 are the conjugate roots of the characteristic equation $\lambda^3-(x+2)\lambda^2+(x+2)\lambda-(x+1)=0$.

Proof. (2.8): From Theorem 2.5 and Eq. (2.6), we have

$$g(\mathscr{J}_n(x);\lambda)=\sum_{j=0}^{\infty}\mathscr{J}_j(x)\lambda^j=\frac{\lambda}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3}.$$

Using the partial fraction decomposition, $g(\mathscr{J}_n(x);\lambda)$ can be expressed as

$$g(\mathscr{J}_n(x);\lambda)=\frac{1}{\Phi(x)}\left[\frac{x}{1-(x+1)\lambda}+\frac{\omega_2(x+\omega_1)}{i\sqrt{3}(1-\omega_1\lambda)}-\frac{\omega_1(x+\omega_2)}{i\sqrt{3}(1-\omega_2\lambda)}\right],$$

where $\Phi(x)=x^2+x+1$.
However, note that $\omega_1+\omega_2=1$, $\omega_1-\omega_2=i\sqrt{3}$ and $\omega_1\omega_2=1$. Then, we have

$$\begin{aligned}g(\mathscr{J}_n(x);\lambda)&=\frac{1}{\Phi(x)}\left[\frac{x}{1-(x+1)\lambda}+\frac{\omega_2(x+\omega_1)}{i\sqrt{3}(1-\omega_1\lambda)}-\frac{\omega_1(x+\omega_2)}{i\sqrt{3}(1-\omega_2\lambda)}\right]\\&=\frac{1}{\Phi(x)}\sum_{n=0}^{\infty}\left[x(x+1)^n+\frac{\omega_2(x+\omega_1)\omega_1^n}{i\sqrt{3}}-\frac{\omega_1(x+\omega_2)\omega_2^n}{i\sqrt{3}}\right]x^n\\&=\sum_{n=0}^{\infty}\left[\frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)}\right]x^n.\end{aligned}$$

Thus, by the equality of generating function, we get

$$\mathscr{J}_n(x)=\frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)}.$$

The proof of the binomial transform of modified third-order Jacobsthal polynomials $\mathscr{K}_n(x)$ can be seen by taking Theorem 2.5 and Eq. (2.7). □

3. Some Properties of Binomial Transforms of Third-Order Jacobsthal Polynomials

Now, we give the sums of binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 3.1. *For $n\geq 3$, sums of sequences $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ are*

$$\sum_{j=0}^{n}\mathscr{J}_j(x)=\frac{1}{x}\left[\mathscr{J}_{n+2}(x)-(x+1)(\mathscr{J}_{n+1}(x)-\mathscr{J}_n(x))\right] \tag{3.1}$$

and

$$\sum_{j=0}^{n}\mathscr{K}_j(x)=\frac{1}{x}\left[\mathscr{K}_{n+2}(x)-(x+1)(\mathscr{K}_{n+1}(x)-\mathscr{K}_n(x))+x-1\right]. \tag{3.2}$$

Proof. (3.1): By considering recurrence relation in Eq. (2.4), we have

$$\begin{aligned}\mathscr{J}_3(x)&=(x+2)\left[\mathscr{J}_2(x)-\mathscr{J}_1(x)\right]+(x+1)\mathscr{J}_0(x)\\\mathscr{J}_4(x)&=(x+2)\left[\mathscr{J}_3(x)-\mathscr{J}_2(x)\right]+(x+1)\mathscr{J}_1(x)\\\mathscr{J}_5(x)&=(x+2)\left[\mathscr{J}_4(x)-\mathscr{J}_3(x)\right]+(x+1)\mathscr{J}_2(x)\\&\cdots\\\mathscr{J}_{n-1}(x)&=(x+2)\left[\mathscr{J}_{n-2}(x)-\mathscr{J}_{n-3}(x)\right]+(x+1)\mathscr{J}_{n-4}(x)\\\mathscr{J}_n(x)&=(x+2)\left[\mathscr{J}_{n-1}(x)-\mathscr{J}_{n-2}(x)\right]+(x+1)\mathscr{J}_{n-3}(x).\end{aligned}$$

Adding these equations, we obtain

$$\begin{aligned}\sum_{j=0}^{n}\mathscr{J}_j(x)&=(x+2)\mathscr{J}_{n-1}(x)+(x+1)\sum_{j=0}^{n-3}\mathscr{J}_j(x)\\&=(x+2)\mathscr{J}_{n-1}(x)+(x+1)\sum_{j=0}^{n}\mathscr{J}_j(x)-(x+1)\left[\mathscr{J}_n(x)+\mathscr{J}_{n-1}(x)+\mathscr{J}_{n-2}(x)\right].\end{aligned}$$

Then, using the relation $\mathscr{J}_{n+3}(x)=(x+2)\left[\mathscr{J}_{n+2}(x)-\mathscr{J}_{n+1}(x)\right]+(x+1)\mathscr{J}_n(x)$, we have

$$\sum_{j=0}^{n}\mathscr{J}_j(x)=\frac{1}{x}\left[(x+1)(\mathscr{J}_n(x)+\mathscr{J}_{n-2}(x))-\mathscr{J}_{n-1}(x)\right]=\frac{1}{x}\left[\mathscr{J}_{n+2}(x)-(x+1)(\mathscr{J}_{n+1}(x)-\mathscr{J}_n(x))\right].$$

Similar to (3.1), by considering equation (2.5), Eq. (3.2) into account similar to the proof of (3.1). □

Now, we give the sums of the first n of binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials with even subscripts.

Corollary 3.2. *Sums of sequences $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ with even subscripts are*

$$\sum_{j=0}^{n} \mathscr{J}_{2j}(x) = \frac{1}{x}\left[(x+3)\mathscr{J}_{2n+2}(x) - (x^2+3x+3)\mathscr{J}_{2n+1}(x) + (2x^2+5x+3)\mathscr{J}_{2n}(x) - x\right]$$

and

$$\sum_{j=0}^{n} \mathscr{K}_{2j}(x) = \frac{1}{x}\left[(x+3)\mathscr{K}_{2n+2}(x) - (x^2+3x+3)\mathscr{K}_{2n+1}(x) + (2x^2+5x+3)\mathscr{K}_{2n}(x) - (2x^3+7x^2+12x+15)\right].$$

Proof. The proof can be easily established using [18, Theorem 2.1]. □

Now, we give the combinatorial equalities of the binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 3.3. *For $n \geq 0$, we have the equalities*

$$\sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \left(-\frac{x+2}{x+1}\right)^i \mathscr{J}_{i+j}(x) = (x+1)^{-n}\mathscr{J}_{3n}(x) \tag{3.3}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \left(-\frac{x+2}{x+1}\right)^i \mathscr{K}_{i+j}(x) = (x+1)^{-n}\mathscr{K}_{3n}(x). \tag{3.4}$$

Proof. (3.3): Let λ stand for a root of the characteristic equation of Eq. (2.4). Then, we have $\lambda^3 = (x+2)(\lambda^2-\lambda)+x+1$ and we can write by considering binomial expansion with $x+1 \neq 0$:

$$\begin{aligned}
\left(\frac{\lambda^3}{x+1}\right)^n &= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{\lambda^3}{x+1}-1\right)^i \\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{x+2}{x+1}(\lambda^2-\lambda)\right)^i \\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{x+2}{x+1}\lambda^2\right)^j\left(-\frac{x+2}{x+1}\lambda\right)^{i-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left(-\frac{x+2}{x+1}\right)^i\lambda^{i+j}.
\end{aligned}$$

If we replace to ω_1 and ω_2 by λ and rearrange, then we obtain

$$\begin{aligned}
\frac{\mathscr{J}_{3n}(x)}{(x+1)^n} &= \frac{1}{(x+1)^n}\left[\frac{x(x+1)^{3n}}{x^2+x+1} + \frac{\omega_1^{3n-1}}{(x+\omega_2)(\omega_1-\omega_2)} - \frac{\omega_2^{3n-1}}{(x+\omega_1)(\omega_1-\omega_2)}\right] \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left(-\frac{x+2}{x+1}\right)^i\mathscr{J}_{i+j}(x),
\end{aligned}$$

where ω_1 and ω_2 are the roots of the characteristic equation $\lambda^3-(x+2)\lambda^2+(x+2)\lambda-(x+1)=0$. Finally, Eq. (3.4) can be obtained in a similar way. □

For simplicity of notation, let

$$\begin{aligned}
\mathscr{Z}_n(x) &= \frac{(x+\omega_1)\omega_1^{n-1}-(x+\omega_2)\omega_2^{n-1}}{\omega_1-\omega_2} = \frac{A\omega_1^n - B\omega_2^n}{\omega_1-\omega_2}, \\
\mathscr{W}_n &= \omega_1^n+\omega_2^n = \frac{1}{x^2+x+1}\left[(x+2)\mathscr{Z}_{n+1}(x)-(2x+1)\mathscr{Z}_n(x)\right],
\end{aligned} \tag{3.5}$$

where $A=\omega_2 x+1$ and $B=\omega_1 x+1$.
Further, the Binet formula of the binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials are given by

$$\mathscr{J}_n(x) = \frac{1}{x^2+x+1}\left[x(x+1)^n+\mathscr{Z}_n(x)\right] \tag{3.6}$$

and

$$\mathscr{K}_n(x) = (x+1)^n+\mathscr{W}_n.$$

Note that $\mathscr{Z}_{n+2}(x) = \mathscr{Z}_{n+1}(x) - \mathscr{Z}_{n+2}(x)$, with initial conditions $\mathscr{Z}_0(x) = -x$ and $\mathscr{Z}_1(x) = 1$.
The Vajda's identity for the sequence $\mathscr{Z}_n(x)$ and binomial transform of third-order Jacobsthal polynomials is given in the next theorem.

Theorem 3.4. *Let $n \geq 0$, $p \geq 0$, $q \geq 0$ be integers. Then, we have*

$$\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x) - \mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x) = (x^2+x+1)\mathscr{A}_p\mathscr{A}_q \tag{3.7}$$

and

$$\begin{aligned}&\mathscr{J}_{n+p}(x)\mathscr{J}_{n+q}(x) - \mathscr{J}_n(x)\mathscr{J}_{n+p+q}(x)\\ &\quad= \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_p\mathscr{A}_q - x^2(x+1)^n\left(\mathscr{B}_{n+p}(q)-(x+1)^p\mathscr{B}_n(q)\right)\right],\end{aligned} \tag{3.8}$$

where $\mathscr{A}_n = \frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ and $\mathscr{B}_n(q) = \mathscr{Z}_{n+q}(x)-(x+1)^q\mathscr{Z}_n(x)$.

Proof. (3.7): By using Eq. (3.5), $A=\omega_2 x+1$ and $B=\omega_1 x+1$ and $AB=x^2+x+1$, we have

$$\begin{aligned}&\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x) - \mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x)\\ &\qquad= \frac{1}{(\omega_1-\omega_2)^2}\left[\left(A\omega_1^{n+p}-B\omega_2^{n+p}\right)\left(A\omega_1^{n+q}-B\omega_2^{n+q}\right)-(A\omega_1^n-B\omega_2^n)\left(A\omega_1^{n+p+q}-B\omega_2^{n+p+q}\right)\right]\\ &\qquad= \frac{1}{(\omega_1-\omega_2)^2}\left[AB(\omega_1^p-\omega_2^p)\left(\omega_1^q-\omega_2^q\right)\right]\\ &\qquad= (x^2+x+1)\mathscr{A}_p\mathscr{A}_q,\end{aligned}$$

where $\mathscr{A}_n = \frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ is the n-th companion sequence of $\mathscr{Z}_n(x)$.
(3.8): By formulas (3.5), (3.6) and Eq. (2.4), we get

$$\begin{aligned}&\mathscr{J}_{n+p}(x)\mathscr{J}_{n+q}(x) - \mathscr{J}_n(x)\mathscr{J}_{n+p+q}(x)\\ &= \frac{1}{(x^2+x+1)^2}\left[\left(x(x+1)^{n+p}+\mathscr{Z}_{n+p}(x)\right)\left(x(x+1)^{n+q}+\mathscr{Z}_{n+q}(x)\right)-\left(x(x+1)^n+\mathscr{Z}_n(x)\right)\left(x(x+1)^{n+p+q}+\mathscr{Z}_{n+p+q}(x)\right)\right]\\ &= \frac{1}{(x^2+x+1)^2}\left[\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x)-\mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x)+x^2(x+1)^{n+p}\mathscr{B}_n(q)-x^2(x+1)^n\mathscr{B}_{n+p}(q)\right]\\ &= \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_p\mathscr{A}_q-x^2(x+1)^n\left(\mathscr{B}_{n+p}(q)-(x+1)^p\mathscr{B}_n(q)\right)\right],\end{aligned}$$

where $\mathscr{B}_n(q) = \mathscr{Z}_{n+q}(x)-(x+1)^q\mathscr{Z}_n(x)$. □

It is easily seen that for special values of p and q by Theorem 3.4, we get new identities for binomial transform of the third-order Jacobsthal polynomials:

- Catalan's identity: $q=-p$.
- Cassini's identity: $p=1$, $q=-1$.
- d'Ocagne's identity: $p=1$, $q=m-n$, with $m \geq n$.

Corollary 3.5. *Catalan identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 0$, $p \geq 0$ be integers such that $n \geq p$. Then*

$$\begin{aligned}&\mathscr{J}_{n+p}(x)\mathscr{J}_{n-p}(x) - (\mathscr{J}_n(x))^2\\ &\quad= \frac{1}{(x^2+x+1)^2}\left[-(x^2+x+1)\mathscr{A}_p^2 - x^2(x+1)^n\left(\mathscr{B}_{n+p}(-p)-(x+1)^p\mathscr{B}_n(-p)\right)\right].\end{aligned}$$

Corollary 3.6. *Cassini identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}&\mathscr{J}_{n+1}(x)\mathscr{J}_{n-1}(x) - (\mathscr{J}_n(x))^2\\ &\quad= \frac{1}{(x^2+x+1)^2}\left[-(x^2+x+1) - x^2(x+1)^n\left(\mathscr{B}_{n+1}(-1)-(x+1)\mathscr{B}_n(-1)\right)\right].\end{aligned}$$

Corollary 3.7. *d'Ocagne identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 0$, $m \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned}&\mathscr{J}_{n+1}(x)\mathscr{J}_m(x) - \mathscr{J}_n(x)\mathscr{J}_{m+1}(x)\\ &\quad= \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_{m-n} - x^2(x+1)^n\left(\mathscr{B}_{n+1}(m-n)-(x+1)\mathscr{B}_n(m-n)\right)\right].\end{aligned}$$

4. Conclusion

In this paper, we first define the binomial transforms of third-order Jacobsthal polynomials $\mathscr{J}_n(x)$ and give some identities of this new sequence of polynomials. By taking into account these transforms and its properties, identities of the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal numbers can also be obtained. Furthermore, if we replace $x=2$ in $\mathscr{J}_n(x)$, we obtain the binomial transform of third-order Jacobsthal numbers and if we replace $x=2$ in $\mathscr{K}_n(x)$, we obtain the binomial transform of modified third-order Jacobsthal numbers (in the same sense as Soykan in [3]). Finally, we obtained the generating functions, Binet formulas, summations, and relationships for the binomial transforms of the third-order Jacobsthal and modified third-order Jacobsthal polynomials.

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