Ikonion Journal of Mathematics 7(1) (2025) 15-33

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

https://doi.org/10.54286/ikjm.1494576

ISSN: 2687-6531

On Generalized Ricci-Recurrent Trans-Sasakian Indefinite Finsler Manifolds

Ayşe Funda Sağlamer[1], Aslı Kalkan Altıntaş[2]

Keywords: *Trans-Sasakian manifold, $\alpha-$Sasakian manifold, $\beta-$Kenmotsu manifold, Pseudo-Finsler metric, Indefinite Finsler manifold, Ricci-recurrent manifold*

Abstract — In this paper generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds are studied.These structures are established on the vector subbundles $(M^0)^h$ and $(M^0)^v$, where M is an $(2n+1)$ dimensional C^∞ manifold, $M^0=(M^0)^h\oplus(M^0)^v$ is a non-empty open submanifold of TM. F^* is the fundamental Finsler function and $F^{2n+1}=(M,M^0,F^*)$ is an indefinite Finsler manifold. We use the Sasaki Finsler metric $G=G^H+G^V=g_{ij}^{F^*}dx^i\otimes dx^j+g_{ij}^{F^*}\,\delta y^i\otimes\delta y^j$ .It is also provided that If $((M^0)^h,\phi^H,\xi^H,\eta^H,G^H)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are one of the $(\varepsilon)-\ \alpha-$Sasakian, $(\varepsilon)-$Sasakian, $(\varepsilon)-$ β-Kenmotsu and $(\varepsilon)-$Kenmotsu manifolds, which are generalized Ricci-recurrent with cyclic Ricci tensor and non-zero $A^H(\xi^H)$, $A^V(\xi^V)$ everywhere, then they are Einstein and Ricci symmetric manifolds, where α,β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

Subject Classification (2020): 53C25, 53C6, 53B40, 53B30

1. Introduction

Oubina introduced the idea of trans-Sasakian manifold of classification (α,β). Indefinite Sasakian manifold is a notable category of indefinite trans-Sasakian manifold for $\alpha=1,\beta=0$. Also, indefinite cosymplectic manifold is the other category of indefinite trans-Sasakian manifold for $\alpha=0,\ \beta=0$. Indefinite Kenmotsu manifold is given with $\alpha=0,\ \beta=1$. Siddiqi, M. Danish, Aliya N. Siddiqui, and Bahadır, Oğuzhan studied the trans-Sasakian manifolds with a quarter-symmetric nonmetric connection [1]. Prasad, R., Gautam, U. K., Prakash, J. and Rai, A. K. studied $(\varepsilon)-$Lorentzian trans-Sasakian manifolds [2]. Prasad, R and Prakash, J. studied On Generalized Ricci- Recurrent Indefinite Trans-Sasakian Manifolds [3]. Prasad, R, Kishor, S. and Srivastava, V. studied On generalized Ricci-recurrent (k,μ) -contact metric manifolds [4].

The papers interested in contact structures with Riemannian metric or pseudo-Riemannian metric but in this paper, we are also related to the contact structures with pseudo-Finsler metric.

After Finsler published his thesis about curves and surfaces, a lot of articles are dedicated to Finsler geometry, see references [5, 6, 7, 8, 9, 10], but the theory of indefinite Finsler manifold has been

[1]aysefunda.saglamer@dpu.edu.tr (Corresponding Author); [2]asli.kalkan.altintas@ogr.dpu.edu.tr

[1]Department of Mathematics, Faculty of Art and Science, Dumlupınar University, Kütahya, TURKEY

[2]Department of Mathematics, Institute of Postgraduation Education, Dumlupınar University, Kütahya, TURKEY

Article History: Received: 2.6.2024 — Accepted: 26.11.2024— Published:

investigated by few researchers [11, 12, 13, 14]. We also make reference to the reader to the recent monograph for detailed information in this field.

Hence, our aim is to present generalized Ricci- recurrent trans-Sasakian indefinite Finsler manifolds and to obtain the formulas for α −Sasakian and β −Kenmotsu indefinite Finsler manifolds. The paper is organized as follows: after introduction and background, we give some preliminaries about indefinite Finsler manifolds. Then, we deal with the trans-Sasakian indefinite Finsler manifolds. Finally, it is shown that if the trans-Sasakian indefinite Finsler manifolds, $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are one of the $(\varepsilon)-$ $\alpha-$Sasakian, $(\varepsilon)-$Sasakian, $(\varepsilon)-$ β-Kenmotsu and (ε) −Kenmotsu manifolds, which are generalized Ricci- recurrent with cyclic Ricci tensor and non-zero $A^H(\xi^H)$, $A^V(\xi^V)$ everywhere, then they are Einstein and Ricci symmetric manifolds, where α, β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

2. Preliminaries

2.1. Indefinite Finsler Manifolds

Let M be a real $(2n+1)-$ dimensional smooth manifold and TM be the tangent bundle of M. A coordinate system in M can be stated with $\{(U,\varphi): x^1, \dots, x^{2n+1}\}$, where U is an open subset of M; for any $x \in U$, $\varphi: U \to \mathbb{R}^{2n+1}$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, \dots, x^{2n+1})$. On M, denote by π the canonical projection of TM and by T_xM the fibre, at $x \in M$, i.e., $T_xM = \pi^{-1}(x)$. Through the coordinate system $\{(U,\varphi): x^i\}$ in M, we can describe a new coordinate system $\{(U^*,\Phi); x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1}\}$ or shortly $\{(U^*,\Phi): x^i, y^i\}$ in TM, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^{4n+2}$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^{2n+1}$, and $\Phi(y_x) = (x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$ for any $x \in U$ and $y_x \in T_xM$. Let M^0 be a non-empty open submanifold of TM such that $\pi(M^0) = M$ and $\theta(M) \cap M^0 = \emptyset$, where θ is the zero section of TM. Assume that $M_x^0 = T_xM \cap M^0$ is a positive conic set, for any $k > 0$ and $y \in M_x^0$. we have $ky \in M_x^0$. Obviously, the largest M^0 holding the above circumstances is $TM \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold.

We now consider a smooth function $F : M^0 \to (0,\infty)$ and take $F^* = F^2$, $F^*: M^0 \to R$, where M^0 is as above. Moreover, suppose that for any coordinate system $\{(U^0, \Phi^0); x^i, y^i\}$ in M^0, the following conditions are fulfilled:

($F1^*$) F^* is positively homogenous of degree two regarding $(y^1, \dots, y^{2n+1})$, we get, for all $k > 0$ and $(x,y) \in \Phi^0(U^0)$,

$$F^*(x^1, \dots, x^{2n+1}, ky^1, \dots, ky^{2n+1}) = k^2 F^*(x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$$

($F2^*$) At all point $(x,y) \in \Phi^0(U^0)$,

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y_i \partial y_j}(x,y), \quad i,j \in \{1,2,\dots,2n+1\}$$

are the components of a quadratic form on $\mathbb{R}^{2n+1}$ with $(2n+1)-q$ positive eigenvalues and q negative eigenvalues $(0 < q < 2n+1)$. In this state $F^{2n+1} = (M, M^0, F^*)$ is called indefinite Finsler manifolds with index q. Particularly, if chosing $q = 1$, we get Lorentzian indefinite Finsler manifolds [12]. Consider the structure of $F^{2n+1} = (M, M^0, F^*)$ indefinite Finsler manifold with index q. Then the tangent mapping $\pi_*: TM^0 \to TM$ of the submersion $\pi: M^0 \to M$ and define the vector bundle $(TM^0)^V = ker\pi_*$. As locally, $\pi^i\ (x,y) = x^i$, we obtain $\pi_*^i\left(\frac{\partial}{\partial x^j}\right) = \delta_j^i\ and\ \pi_*^i\left(\frac{\partial}{\partial y^j}\right) = 0$, on the coordinate neighborhood $U^0 \subset M^0$. Thus, $\left\{\frac{\partial}{\partial y^i}\right\}$ is a basis of $\Gamma\left((TM^0)^V{}_{|_{U^0}}\right)$. We call $(TM^0)^V$ the vertical vector bundle of F^{2n+1}. Locally, on a coordinate neighborhood $U^0 \subset M^0$. we have $X^V = X^i(x,y)\frac{\partial}{\partial y^i}$, where X^i smooth functions on U^0. After we denote by $(T^*M^0)^V$ the dual vector bundle of $(TM^0)^V$. Thus a Finsler 1-form is smooth section of $(T^*M^0)^V$. Assume $\{\delta y^1, \dots, \delta y^{2n+1}\}$ is a dual basis to $\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{2n+1}}\right\}$, i.e., $\delta y^i\left(\frac{\partial}{\partial y^j}\right) = \delta_j^i$. Then each for $w \in (T^*M^0)^V, w^V = w^i(x,y)\,\delta y^i$, where $w^i(x,y) = w\left(\frac{\partial}{\partial y^i}\right)$ [12].

The complementary distribution $(TM^0)^H$ to $(TM^0)^V$ in TM^0 is said a horizontal distribution (non linear connection) on M^0. Thus we can write

$$TM^0 = (TM^0)^H \oplus (TM^0)^V$$

The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^{2n+1}}\right\}$ is a basis in $\Gamma((TM^0)^H)$. Then

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$$

Let X be a vector field on M^0. Then locally we get

$$X = X^i \frac{\delta}{\delta x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$$

Clearly, for $\tilde{X}^i(x,y) = 0$, we obtain the subbundle of $(M^0)^h \subset M^0$ and for $X^i(x,y) = 0$, we obtain the subbundle of $(M^0)^v \subset M^0$. Suppose $\{dx^1, \dots, dx^{2n+1}\}$ is a dual basis to $\left\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^{2n+1}}\right\}$, i.e., $dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta_j^i$. Then each $w \in \Gamma(T^*M^0)^H$ is locally written as $w^H = \widetilde{w}_i(x,y)dx^i$, where $\widetilde{w}_i = w_i - N_i^j w_j$. Thus we can write

$$\delta y^i = dy^i + N_j^i(x,y)dx^j$$

Consider a 1-form w , then

$w = \widetilde{w}_i(x,y)dx^i + w_i(x,y)\ \delta y^i$.

Also, $w^H(X^V) = 0, w^V(X^H) = 0$, where $w = w^H + w^V$ [12].

Definition 2.1. A Finsler connection is a linear connection $\nabla = F\Gamma$ with the property that the horizontal linear space $\left(T_{(x,y)}M^0\right)^H$, $(x,y) \in M^0$ of the distribution N is parallel with respect to ∇.

Similarly, a Finsler connection is called linear connection $\nabla = F\Gamma$ with the vertical linear space $\left(T_{(x,y)}M^0\right)^V$, $(x,y) \in M^0$ of the distribution N parallel relative to ∇.

Necessary and sufficient condition for linear connection ∇ on M^0 to be Finsler connection is

$$(\nabla_X^V Y^H) \quad = 0, (\nabla_X^H Y^V) \quad = 0$$

$$\nabla_X Y = \nabla_X^H Y^H + \nabla_X^V Y^V$$

for each $X, Y \in T_{(x,y)}M^0$.

$$\nabla_X w = \nabla_X^H w^H + \nabla_X^V w^V$$

for all $w \in T^*_{(x,y)} \quad M^0$ [6].

Let ∇ be a Finsler connection and the curvature of this connection is given with the below equation.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R^H(X^H, Y^H)Z^H + R^V(X^V, Y^V)Z^V$$

where

$$R^H(X^H, Y^H)Z^H = \nabla_{X^H} \nabla_{Y^H} Z^H - \nabla_{Y^H} \nabla_{X^H} Z^H - \nabla_{[X^H, Y^H]} Z^H$$

$$R^V(X^V, Y^V)Z^V = \nabla_{X^V} \nabla_{Y^V} Z^V - \nabla_{Y^V} \nabla_{X^V} Z^V - \nabla_{[X^V, Y^V]} Z^V$$

$X, Y, Z \in T_{(x,y)}M^0$ [5].

2.2. Almost Contact Pseudo-Metric Finsler Structures

Consider tensor field ϕ, 1-form η and vector field ξ given as below:

$$\phi = \phi^H + \phi^V = \phi_i^j(x,y) \frac{\delta}{\delta x_i} \otimes dx^j + \widetilde{\phi_i^j}(x,y) \frac{\partial}{\partial y^i} \otimes \delta y^j \tag{2.1}$$

$$\eta = \eta^H + \eta^V = \eta_i(x,y) dx^i + \widetilde{\eta_i}(x,y) \delta y^i \tag{2.2}$$

$$\xi = \xi^H + \xi^V = \xi^i(x,y) \frac{\delta}{\delta x_i} + \widetilde{\xi^i}(x,y) \frac{\partial}{\partial y^i} \tag{2.3}$$

Then, we can write the following statements.

$$(\phi^H)^2 X^H = -X^H + \eta^H(X^H)\, \xi^H, (\phi^V)^2 X^V = -X^V + \eta^V(X^V)\, \xi^V \tag{2.4}$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1 \tag{2.5}$$

$$\phi^H(\xi^H) = \phi^V(\xi^V) = 0 \tag{2.6}$$

$$\eta^H \circ \phi^H = \eta^V \circ \phi^V = 0 \tag{2.7}$$

$$rank(\phi^H) = rank(\phi^V) = 2n \tag{2.8}$$

Thus, (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are called the almost contact Finsler structures on vector bundles $(M^0)^h$ and $(M^0)^v$,respectively, where $M^0 = (M^0)^h \oplus (M^0)^v$. Also, we call that $\left((M^0)^h, \phi^H, \xi^H, \eta^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V)$ are almost contact Finsler manifolds [15].

Let $F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. Then, we define

$$g^{F^*}: \Gamma(TM^0)^V \times \Gamma(TM^0)^V \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)(x,y).$$

Obviously, g^{F^*} is a symmetric Finsler tensor field. g^{F^*} is called the pseudo-Finsler metric of F^{2n+1}. Thus, g^{F^*} is thought to be a pseudo-Riemannian metric on $(TM^0)^V$.

Similarly, the metric for horizontal distrubituon is defined as following:

$$g^{F^*}:\Gamma(TM^0)^H \times \Gamma(TM^0)^H \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)(x,y)$$

[12].

Also,

$$G:\Gamma(TM^0) \quad \times \Gamma(TM^0) \quad \to \mathfrak{F}(M^0)$$

$$G(X,Y) = G^H(X,Y) + G^V(X,Y).$$

is defined. Obviously, G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on M^0 with index $2q$. Then, G is called Sasaki Finsler metric on M^0. Then, G can be defined as below.

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*}\ \delta y^i \ \otimes \delta y^j$$

[12].

Definition 2.2. Suppose that (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are almost contact structures on horizontal and vertical Finsler vector bundles $(M^0)^h$ and $(M^0)^v$. If the G^H and G^V satisfy the following conditions,

$$G^H(\phi X^H, \phi Y^H) = G^H(X^H, Y^H) - \varepsilon\eta^H(X^H)\eta^H(Y^H)$$

$$G^V(\phi X^V, \phi Y^V) = G^V(X^V, Y^V) - \varepsilon\eta^V(X^V)\eta^V(Y^V)$$

$$\eta^H(X^H) = \varepsilon G^H(X^H, \xi^H), \eta^V(X^V) = \varepsilon G^V(X^V, \xi^V)$$

3. Trans-Sasakian Indefinite Finsler Manifolds

We introduce trans-Sasakian indefinite Finsler manifolds in our main results.The almost contact pseudo-metric Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are said to be trans-Sasakian indefinite Finsler manifolds if and only if the following conditions are hold.

$$(\nabla_X^H \phi^H)Y^H = \frac{\alpha}{2}\{G^H(X^H, Y^H)\xi^H - \varepsilon\eta^H(Y^H)X^H\} + \frac{\beta}{2}\{\varepsilon G^H(\phi X^H, Y^H)\xi^H - \eta^H(Y^H)\phi X^H\} \tag{3.1}$$

$$(\nabla_X^V \phi^V)Y^V = \frac{\alpha}{2}\{G^V(X^V, Y^V)\xi^V - \varepsilon\eta^V(Y^V)X^V\} + \frac{\beta}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\}, \tag{3.2}$$

where α and β are smooth functions on $(M^0)^h$ and $(M^0)^v$ then we say such a structure the trans-Sasakian pseudo-metric Finsler structure of type (α, β).

$$(\nabla_X^H \xi^H) = -\varepsilon\frac{\alpha}{2}\ \phi X^H + \frac{\beta}{2}(X^H - \eta^H(X^H)\xi^H) \tag{3.3}$$

$$(\nabla_X^V \xi^V) = -\varepsilon\frac{\alpha}{2}\ \phi X^V + \frac{\beta}{2}(X^V - \eta^V(X^V)\xi^V) \tag{3.4}$$

$$(\nabla_X^H \eta^H)(Y^H) = \frac{\alpha}{2}G^H(X^H, \phi Y^H) + \varepsilon\frac{\beta}{2}G^H(\phi X^H, \phi Y^H) \tag{3.5}$$

$$(\nabla_X^V \eta^V)(Y^V) = \frac{\alpha}{2}G^V(X^V, \phi Y^V) + \varepsilon\frac{\beta}{2}G^V(\phi X^V, \phi Y^V) \tag{3.6}$$

$$R^H(X^H,Y^H)\xi^H = \frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\}$$
$$+\frac{\varepsilon}{2}\{Y^H(\alpha)\phi X^H - X^H(\alpha)\phi Y^H\}$$
$$+\frac{1}{2}\{X^H(\beta)Y^H - Y^H(\beta)X^H + Y^H(\beta)\eta^H(X^H)\xi^H$$
$$- X^H(\beta)\eta^H(Y^H)\xi^H\} \quad (3.7)$$

$$R^V(X^V,Y^V)\xi^V = \frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\}$$
$$+\frac{\varepsilon}{2}\{Y^V(\alpha)\phi X^V - X^V(\alpha)\phi Y^V\}$$
$$+\frac{1}{2}\{X^V(\beta)Y^V - Y^V(\beta)X^V + Y^V(\beta)\eta^V(X^V)\xi^V -$$
$$X^V(\beta)\eta^V(Y^V)\xi^V\} \quad (3.8)$$

$$R^H(\xi^H,X^H)Y^H$$
$$= \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^H(X^H,Y^H)\xi^H - \eta^H(Y^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \varepsilon G^H(\phi X^H,Y^H)\xi^H\}$$
$$+\frac{\varepsilon}{2}\{G^H(\phi Y^H,X^H)\nabla(\alpha) + Y^H(\alpha)\phi X^H\}$$
$$+\frac{1}{2}\{Y^H(\beta)X^H - Y^H(\beta)\eta^H(X^H)\xi^H - G^H(X^H,Y^H)\nabla(\beta)$$
$$+\varepsilon\,\eta^H(X^H)\eta^H(Y^H)\nabla(\beta)\} \quad (3.9)$$

$$R^V(\xi^V,X^V)Y^V$$
$$= \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^V(X^V,Y^V)\xi^V - \eta^V(Y^V)X^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \varepsilon G^V(\phi X^V,Y^V)\xi^V\}$$
$$+\frac{\varepsilon}{2}\{G^V(\phi Y^V,X^V)\nabla(\alpha) + Y^V(\alpha)\phi X^V\}$$
$$+\frac{1}{2}\{Y^V(\beta)X^V - Y^V(\beta)\eta^V(X^V)\xi^V - G^V(X^V,Y^V)\nabla(\beta)$$
$$+\varepsilon\,\eta^V(X^V)\eta^V(Y^V)\nabla(\beta)\} \quad (3.10)$$

$$S^H(X^H,\xi^H) = \left(\frac{n(\alpha^2-\beta^2)_-\xi^H(\beta)}{2}\right)\eta^H(X^H) - \frac{\varepsilon}{2}\phi X^H(\alpha) + \frac{(1-2n)}{2}X^H(\beta), \quad (3.11)$$

$$S^V(X^V,\xi^V) = \left(\frac{n(\alpha^2-\beta^2)_-\xi^H(\beta)}{2}\right)\eta^V(X^V) - \frac{\varepsilon}{2}\phi X^V(\alpha) + \frac{(1-2n)}{2}X^V(\beta), \quad (3.12)$$

$$S^H(\xi^H,\xi^H) = \frac{n(\alpha^2-\beta^2{}_-2\xi^H(\beta))}{2} = S^V(\xi^V,\xi^V) \quad (3.13)$$

If α,β =constant, then the getting α,β =constant from (3.1) and (3.2) we get

$$R^H(X^H,Y^H)\xi^H = \frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\} \quad (3.14)$$

$$R^V(X^V,Y^V)\xi^V = \frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\} \quad (3.15)$$

$$R^H(\xi^H, X^H)Y^H = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^H(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \varepsilon G^H(\phi X^H, Y^H)\xi^H\} \quad (3.16)$$

$$R^V(\xi^V, X^V)Y^V = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^V(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \varepsilon G^V(\phi X^V, Y^V)\xi^V\} \quad (3.17)$$

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^H(Y^H, Z^H)\eta^H(X^H) - G^H(X^H, Z^H)\eta^H(Y^H)\} + \frac{\alpha\beta}{2}\{\eta^H(X^H)G(\phi Y^H, Z^H) - \eta^H(Y^H)G^H(\phi X^H, Z^H)\} \quad (3.18)$$

$$\eta^V(R^V(X^V, Y^V)Z^V) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^V(Y^V, Z^V)\eta^V(X^V) - G^V(X^V, Z^V)\eta^V(Y^V)\} + \frac{\alpha\beta}{2}\{\eta^V(X^V)G(\phi Y^V, Z^V) - \eta^V(Y^V)G^V(\phi X^V, Z^V)\} \quad (3.19)$$

$$\eta^H(R^H(X^H, Y^H)\xi^H) = 0, \eta^V(R^V(X^V, Y^V)\xi^V) = 0 \quad (3.20)$$

$$S^H(X^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}\eta^H(X^H), \quad S^V(X^V, \xi^V) = n\frac{(\alpha^2-\beta^2)}{2}\eta^V(X^V) \quad (3.21)$$

$$S^H(\xi^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}, \quad S^V(\xi^V, \xi^V) = n\frac{(\alpha^2-\beta^2)}{2} \quad (3.22)$$

$$QX^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^H, \; QX^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^V, \; Q\xi^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^H, \; Q\xi^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^V \quad (3.23)$$

Example 3.1. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$can be stated with $\{(U,\varphi): x^1, x^2, x^3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, x^2, x^3)$. On $\mathbb{R}^3$, denote by π the canonical projection of $T\mathbb{R}^3$ and by T_xM the fibre, at $x \in \mathbb{R}^3$, i.e., $T_x\mathbb{R}^3 = \pi^{-1}(x)$. Through the coordinate system $\{(U,\varphi): x^i\}$ in $\mathbb{R}^3$, we can describe a new coordinate system $\{(U^*,\Phi); x^1, x^2, x^3; y^1, y^2, y^3\}$ or shortly $\{(U^*,\Phi): x^i, y^i\}$ in $T\mathbb{R}^3$, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^6$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^3$, and $\Phi(y_x) = (x^1, x^2, x^3; y^1, y^2, y^3)$ for any $x \in U$ and $y_x \in T_x\mathbb{R}^3$. Let $(\mathbb{R}^3)^0$ be a non-empty open submanifold of $T\mathbb{R}^3$ such that $\pi((\mathbb{R}^3)^0) = \mathbb{R}^3$ and $\theta(\mathbb{R}^3) \cap (\mathbb{R}^3)^0 = \emptyset$, where θ is the zero section of $T\mathbb{R}^3$. Assume that $(\mathbb{R}^3)^0_x = T_x\mathbb{R}^3 \cap (\mathbb{R}^3)^0$ is a positive conic set, for any $k > 0$ and $y \in (\mathbb{R}^3)^0_x$. we have $ky \in (\mathbb{R}^3)^0_x$. Obviously, the largest $(\mathbb{R}^3)^0$ holding the above circumstances is $T\mathbb{R}^3 \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X^V_1(x,y)\frac{\partial}{\partial y^1} + X^V_2(x,y)\frac{\partial}{\partial y^2} + X^V_3(x,y)\frac{\partial}{\partial y^3}$, $X^H = X^H_1(x,y)\frac{\delta}{\delta x^1} + X^H_2(x,y)\frac{\delta}{\delta x^2} + X^H_3(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any $X \in T(\mathbb{R}^3)^0$, $X = X^H_i(x,y)\frac{\delta}{\delta x^i} + X^V_i(x,y)\frac{\partial}{\partial y^i}$ (i =1, 2, 3). Consider a 1-form η , $\eta = \eta^H + \eta^V = \eta_i{}^H(x,y)dx^i + \eta_i{}^V(x,y)\delta y^i$ (i =1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on $(\mathbb{R}^3)^0$. Then, G is called Sasaki Finsler metric on $(\mathbb{R}^3)^0$. Then, G can be defined as below:

$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i$ (i=1, 2, 3).

The vector fields

$E_1^H = x_3 \frac{\delta}{\delta x^1}$, $E_2^H = x_3 \frac{\delta}{\delta x^2}$, $E_3^H = x_3 \frac{\delta}{\delta x^3} = \xi^H$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G be the Sasaki Finsler pseudo-metric given by

$G^H(E_1^H, \xi^H) = G^H(E_1^H, E_2^H) = G^H(E_2^H, \xi^H) = 0$

$G^H(E_1^H, E_1^H) = G^H(E_2^H, E_2^H) = 1$, $G^H(\xi^H, \xi^H) = \varepsilon = -1$.

Let η^H be the 1-form derscribed by

$\eta^H(Z^H) = -G^H(Z^H, \xi^H) = -G^H(z_1 E_1^H + z_2 E_2^H + z_3 \xi^H, \xi^H) = z_3$, $\forall Z^H \in (T(\mathbb{R}^3)^0)^H$.

Consider ϕ^H the (1, 1) tensör field stated by

$\phi^H(E_1^H) = -E_2^H$, $\phi^H(E_2^H) = E_1^H$, $\phi^H(\xi^H) = 0$.

Then using the linearity of ϕ^H, we have

$Z^H = z_1 E_1^H + z_2 E_2^H + z_3 \xi^H$, $W^H = w_1 E_1^H + w_2 E_2^H + w_3 \xi^H$

$\phi^H(Z^H) = \phi^H(z_1 E_1^H + z_2 E_2^H + z_3 \xi^H) = z_1 \phi^H(E_1^H) + z_2 \phi^H(E_2^H) + z_3 \phi^H(\xi^H)$

$\phi^H(Z^H) = -z_1 E_2^H + z_2 E_1^H$

$\phi^H(W^H) = \phi^H(w E_1^H + w E_2^H + w_3 \xi^H) = w_1 \phi^H(E_1^H) + w_2 \phi^H(E_2^H) + w_3 \phi^H(\xi^H)$

$\phi^H(W^H) = -w_1 E_2^H + w_2 E_1^H$

$(\phi^H)^2(Z^H) = -z_2 E_2^H - z_1 E_1^H = - Z + \eta^H(Z^H)\xi^H$

Thus we get

$G^H(\phi^H(Z^H), \phi^H(W^H)) = G^H(Z^H, W^H) + \eta^H(Z^H)\,\eta^H(W^H)$

$\forall Z^H \in (T(\mathbb{R}^3)^0)^H$ and $\forall W^H \in (T(\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$[E_1^H, E_2^H] = 0$, $[E_1^H, \xi^H] = -E_1^H$, $[E_2^H, \xi^H] = -E_2^H$.

The connection ∇ of the pseudo-metric G^H is given by

$$2G^H(\nabla_{X^H} Y^H, Z^H) = X^H G^H(Y^H, Z^H) + Y^H G^H(X^H, Z^H) - Z^H G^H(X^H, Y^H)$$
$$-G^H(X^H, [Y^H, Z^H]) - G^H(Y^H, [X^H, Z^H]) + G^H(Z^H, [X^H, Y^H]) \quad (3.24)$$

Which is known as Koszul's formula. Using this formula, we have

$$2G^H(\nabla_{E_1^H} \xi^H, E_1^H) = -G^H(E_1^H, [\xi^H, E_1^H]) - G^H(\xi^H, [E_1^H, E_1^H]) + G^H(E_1^H, [E_1^H, \xi^H])$$
$$= 2G^H(-E_1^H, E_1^H).$$

Thus,

$\nabla_{E_1^H} \xi^H = -E_1^H$, $\nabla_{\xi^H} E_1^H = 0$.

Again by using Koszul's formula we obtain

$$2G^H(\nabla_{E_2^H} \xi^H, E_2^H) = -G^H(E_2^H, [\xi^H, E_2^H]) - G^H(\xi^H, [E_2^H, E_2^H]) + G^H(E_2^H, [E_2^H, \xi^H])$$

$= 2G^H(-E_2^H, E_2^H)$.

Thus,

$\nabla_{E_2^H}\xi^H = -E_2^H$, $\nabla_{\xi^H}E_2^H = 0$.

Also by using Koszul's formula we obtain

$2G^H\left(\nabla_{E_1^H} E_2^H, \xi^H\right) = G^H(E_1^H,[\xi^H, E_2^H]) + G^H(\xi^H,[E_1^H, E_2^H]) - G^H(E_2^H,[E_1^H,\xi^H]) = 0.$

Thus,

$\nabla_{E_1^H} E_2^H = 0$, $\nabla_{E_2^H} E_1^H = 0$

Similarly we get

$2G^H\left(\nabla_{E_1^H} E_1^H, \xi^H\right) = -G^H(E_1^H,[E_1^H,\xi^H]) + G^H(\xi^H,[E_1^H, E_1^H]) - G^H(E_1^H,[E_1^H,\xi^H])$

$= 2G^H(E_1^H, E_1^H) = -2G^H(\xi^H, \xi^H)$.

Thus,

$$\nabla_{E_1^H} E_1^H = -\xi^H .$$

(3.24) further yields

$\nabla_{E_2^H} E_2^H = -\xi^H$, $\nabla_{\xi^H} E_1^H = 0$, $\nabla_{\xi^H} E_2^H = 0$, $\nabla_{E_2^H} E_1^H = 0$.

If we use the equations we found

$(\nabla_X^H\xi^H) = x_1\nabla_{E_1^H}\xi^H + x_2\nabla_{E_2^H}\xi^H = (-x_1) E_1^H - (x_2) E_2^H$,

$\forall X^H \in (T(\mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ satisfy (3.3) for $\alpha = 0$, $\beta = -2$, $\varepsilon = -1$.

With the help of the above results it can be verified that

$$R^H(E_1^H, E_2^H) E_2^H = E_1^H, \quad R^H(\xi^H, E_2^H) E_2^H = \xi^H, \quad R^H(E_1^H, \xi^H)\xi^H = -E_1^H$$
$$R^H(E_2^H, \xi^H)\xi^H = -E_2^H, \quad R^H(E_2^H, E_1^H) E_1^H = E_2^H, \quad R^H(\xi^H, E_1^H) E_1^H = \xi^H$$

$S^H(\xi^H, \xi^H) = G^H(R^H(E_1^H, \xi^H)\xi^H, E_1^H) + G^H(R^H(E_2^H, \xi^H)\xi^H, E_2^H)$

$= G^H(-E_1^H, E_1^H) + G^H(-E_2^H, E_2^H) = -2$

$S^H(\xi^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2} = -2$

4. Generalized Ricci-Recurrent Trans Sasakian Indefinite Finsler Manifolds

Definition 4.1. Trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are said to be recurrent if $\forall X^H, Y^H, Z^H, W^H \in (TM^0)^H$ and $\forall X^V, Y^V, Z^V, W^V \in (TM^0)^V$

$$\left(\nabla_{X^H}^H R^H\right)(Y^H, Z^H)W^H = A^H(X^H)R^H(Y^H, Z^H)W^H$$

and

$$\left(\nabla_{X^V}^V R^V\right)(Y^V, Z^V)W^V = A^V(X^V)R^V(Y^V, Z^V)W^V,$$

where A^H is 1-form on $(M^0)^h$ such that $A^H(X^H) = G^H(X^H, (A^*)^H)$ and $(A^*)^H$ is called associated vector field to the 1-form A^H (A^V is 1-form on $(M^0)^v$ such that $A^V(X^V) = G^V(X^V, (A^*)^V)$ and $(A^*)^V$ is called associated vector field to the 1-form A^V).

If A^H and A^V vanishes identically on $(M^0)^h$ and $(M^0)^v$, the recurrent manifold reduces to locally symmetric manifold due to Cartan, i.e. $\nabla^H R^H = 0$ and $\nabla^V R^V = 0$.

Definition 4.2. Trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are said to be Ricci-recurrent if $\forall\, X^H, Y^H, Z^H, W^H \in (TM^0)^H$ and $\forall\, X^V, Y^V, Z^V, W^V \in (TM^0)^V$

$$\left(\nabla^H_{X^H} S^H\right)(Y^H, Z^H) = A^H(X^H) S^H(Y^H, Z^H) \text{ and } \left(\nabla^V_{X^V} S^V\right)(Y^V, Z^V) = A^V(X^V) S^V(Y^V, Z^V),$$

where A^H is 1-form on $(M^0)^h$ and A^V is 1-form on $(M^0)^v$. If A^H and A^V vanishes identically on $(M^0)^h$ and $(M^0)^v$, the Ricci-recurrent manifold becomes a Ricci- symmetric manifold, i.e. $\nabla^H S^H = 0$ and $\nabla^V S^V = 0$.

It is well known that an Einstein manifold is a Ricci-symmetric manifold.

The non-flat trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are called generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds if their Ricci tensors S^H and S^V, satisfy the conditions

$$\left(\nabla^H_{X^H} S^H\right)(Y^H, Z^H) = A^H(X^H) S^H(Y^H, Z^H) + B^H(X^H) G^H(Y^H, Z^H) \tag{4.1}$$

$$\left(\nabla^V_{X^V} S^V\right)(Y^V, Z^V) = A^V(X^V) S^V(Y^V, Z^V) + B^V(X^V) G^V(Y^V, Z^V) \tag{4.2}$$

where A^H, B^H are 1-forms on $(M^0)^h$ and A^V, B^V are 1-forms on $(M^0)^v$, $\forall\, X^H, Y^H, Z^H, W^H \in (TM^0)^H$ and $\forall\, X^V, Y^V, Z^V, W^V \in (TM^0)^V$. In particular, if 1-form B^H vanishes identically, then $(M^0)^h$ reduces to well known Ricci-recurrent trans-Sasakian indefinite Finsler manifold (if 1-form B^V vanishes identically, then $(M^0)^v$ reduces to Ricci-recurrent trans-Sasakian indefinite Finsler manifold [3].

Theorem 4.1. Let $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ be generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds. Then, 1-forms A^H, A^V, B^H, B^V are related as

$$B^H(X^H) = -\,\varepsilon\, n \frac{(\alpha^2-\beta^2)}{2} A^H(X^H),\; B^H(\xi^H) = -\,\varepsilon\, n \frac{(\alpha^2-\beta^2)}{2} A^H(\xi^H)$$

$$B^V(X^V) = -\,\varepsilon\, n \frac{(\alpha^2-\beta^2)}{2} A^V(X^V),\; B^V(\xi^V) = -\,\varepsilon\, n \frac{(\alpha^2-\beta^2)}{2} A^V(\xi^V)$$

Proof: We have

$$\left(\nabla^H_{X^H} S^H\right)(Y^H, Z^H) = X^H S^H(Y^H, Z^H) - S^H\left(\nabla^H_{X^H} Y^H, Z^H\right) - S^H\left(Y^H, \nabla^H_{X^H} Z^H\right) \tag{4.3}$$

$$\left(\nabla^V_{X^V} S^V\right)(Y^V, Z^V) = X^V S^V(Y^V, Z^V) - S^V\left(\nabla^V_{X^V} Y^V, Z^V\right) - S^V\left(Y^V, \nabla^V_{X^V} Z^V\right) \tag{4.4}$$

From (4.1) and (4.3), we get

$$A^H(X^H) S^H(Y^H, Z^H) + B^H(X^H) G^H(Y^H, Z^H) = X^H S^H(Y^H, Z^H) - S^H\left(\nabla^H_{X^H} Y^H, Z^H\right) - S^H\left(Y^H, \nabla^H_{X^H} Z^H\right)$$

Putting $Y^H = Z^H = \xi^H$ in above equation, we obtain

$$A^H(X^H) S^H(\xi^H, \xi^H) + B^H(X^H) G^H(\xi^H, \xi^H) = X^H S^H(\xi^H, \xi^H) - 2 S^H\left(\nabla^H_{X^H} \xi^H, \xi^H\right), \tag{4.5}$$

from equations (3.3), (3.21) and (4.5), we get

$$\varepsilon B^H(X^H) + n \frac{(\alpha^2-\beta^2)}{2} A^H(X^H) = -2 S^H\left(-\varepsilon \frac{\alpha}{2}\, \phi X^H + \frac{\beta}{2}(X^H - \eta^H(X^H)\xi^H), \xi^H\right),$$

$$\varepsilon B^H(X^H) + n \frac{(\alpha^2-\beta^2)}{2} A^H(X^H) = \varepsilon \alpha\, S^H(\phi X^H, \xi^H) - \beta S^H(X^H, \xi^H) + \beta\, \eta^H(X^H) S^H(\xi^H, \xi^H),$$

from (3.21) and (3.22), we obtain

$$\varepsilon B^H(X^H) + n\frac{(\alpha^2-\beta^2)}{2}A^H(X^H) = 0$$

$$B^H(X^H) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(X^H). \tag{4.6}$$

Putting $X^H = \xi^H$ in equation (4.6), we obtain

$$B^H(\xi^H) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(\xi^H) \tag{4.7}$$

Similarly from (4.2) and (4.4), we get

$$A^V(X^V)S^V(Y^V,Z^V) + B^V(X^V)G^V(Y^V,Z^V) = X^V S^V(Y^V,Z^V) - S^V\left(\nabla^V_{X^V}Y^V,Z^V\right) - S^V\left(Y^V,\nabla^V_{X^V}Z^V\right).$$

Putting $Y^V = Z^V = \xi^V$ in above equation, we obtain

$$A^V(X^V)S^V(\xi^V,\xi^V) + B^V(X^V)G^V(\xi^V,\xi^V) = X^V S^V(\xi^V,\xi^V) - 2S^V\left(\nabla^V_{X^V}\xi^V,\xi^V\right), \tag{4.8}$$

from equations (3.3), (3.22) and (4.8), we get

$$\varepsilon B^V(X^H) + n\frac{(\alpha^2-\beta^2)}{2}A^V(X^V) = \varepsilon\alpha\, S^V(\phi X^V,\xi^V) - \beta S^V(X^V,\xi^V) + \eta^V(X^V)S^V(\xi^V,\xi^V) = 0$$

and from (3.21) and (3.22), we obtain

$$B^V(X^V) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^V(X^V). \tag{4.9}$$

Putting $X^V = \xi^V$ in equation (4.9), we get

$$B^V(\xi^V) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^V(\xi^V). \tag{4.10}$$

Let $(A^*)^H$ and $(B^*)^H$ be the associated vector fields of A^H and B^H,respectively, so

$$A^H(X^H) = G^H(X^H,(A^*)^H) \text{ and } B^H(X^H) = G^H(X^H,(B^*)^H).$$

From (4.6), we get

$$G^H(X^H,(B^*)^H) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(X^H) = G^H\left(X^H, -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}(A^*)^H\right)$$

$$(B^*)^H = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}(A^*)^H\ . \tag{4.11}$$

For $\alpha-$ Sasakian indefinite Finsler manifold, the equations (4.6), (4.7) and (4.11) becomes

$$B^H(X^H) = -\,\varepsilon\, n\frac{(\alpha^2)}{2}A^H(X^H),\quad B^H(\xi^H) = -\,\varepsilon\, n\frac{(\alpha^2)}{2}A^H(\xi^H),\quad (B^*)^H = -\,\varepsilon\, n\frac{(\alpha^2)}{2}(A^*)^H.$$

For $(\varepsilon)-$ Sasakian manifold, the equation (4.6) becomes

$$B^H(X^H) = -\,\varepsilon\, n\frac{1}{2}A^H(X^H)$$

which implies

$$(B^*)^H = -\,\varepsilon\, n\frac{1}{2}(A^*)^H.$$

Let $(A^*)^V$ and $(B^*)^V$ be the associated vector fields of A^V and B^V,respectively, so

$$A^V(X^V) = G^V(X^V,(A^*)^V) \text{ and } B^V(X^V) = G^V(X^V,(B^*)^V),$$

From (4.9), we obtain

$$G^V(X^V,(B^*)^V) = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^V(X^V) = G^V\left(X^V, -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}(A^*)^V\right)$$

$$(B^*)^V = -\,\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}(A^*)^V \tag{4.12}$$

For $\alpha -$ Sasakian indefinite Finsler manifold, the equations (4.9), (4.10) and (4.12) becomes

$$B^V(X^V) = -\,\varepsilon\, n\frac{(\alpha^2)}{2}A^V(X^V), \quad B^V(\xi^V) = -\,\varepsilon\, n\frac{(\alpha^2)}{2}A^V(\xi^V), \quad (B^*)^V = -\,\varepsilon\, n\frac{(\alpha^2)}{2}(A^*)^V.$$

For $(\varepsilon) -$ Sasakian manifold, the equation (4.9) becomes

$$B^V(X^V) = -\,\varepsilon\, n\frac{1}{2}A^V(X^V), \tag{4.13}$$

which implies

$$(B^*)^V = -\,\varepsilon\, n\frac{1}{2}(A^*)^V.$$

Hence we have the following Lemmas:

Lemma 4.1. In a generalized Ricci-recurrent (ε)**-**Sasakian indefinite Finsler manifold or $(\varepsilon) - \alpha -$ Sasakian indefinite Finsler manifold $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$, $(B^*)^H$ and $(A^*)^H$ have same or opposite directions if ε is -1 and 1 respectively.

Lemma 4.2. In a generalized Ricci-recurrent $(\varepsilon) -$ Sasakian indefinite Finsler manifold or $(\varepsilon) - \alpha -$ Sasakian indefinite Finsler manifold $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, $(B^*)^V$ and $(A^*)^V$ have same or opposite directions if ε is -1 and 1 respectively.

For $(\varepsilon) -$Kenmotsu indefinite Finsler manifold $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$, the equation (4.6) becomes

$$B^H(X^H) = \varepsilon\, n\frac{1}{2}A^H(X^H), \tag{4.14}$$

which implies

$$(B^*)^H = \varepsilon\, n\frac{1}{2}(A^*)^H\ . \tag{4.15}$$

For $(\varepsilon) -$ β-Kenmotsu indefinite Finsler manifold $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$, the equation (4.6) becomes

$$B^H(X^H) = \varepsilon\, n\frac{(\beta^2)}{2}A^H(X^H), \tag{4.16}$$

which implies

$$(B^*)^H = \varepsilon\, n\frac{\beta^2}{2}(A^*)^H\ . \tag{4.17}$$

For $(\varepsilon) -$Kenmotsu indefinite Finsler manifold $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the equation (4.9) becomes

$$B^V(X^V) = \ \varepsilon\, n\frac{(1)}{2}A^V(X^V), \tag{4.18}$$

For $(\varepsilon) -$ β-Kenmotsu indefinite Finsler manifold $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the equation (4.9) becomes

$$B^V(X^V) = \varepsilon\, n\frac{(\beta^2)}{2}A^V(X^V), \tag{4.19}$$

which implies

$$(B^*)^V = \varepsilon\, n\frac{\beta^2}{2}(A^*)^V\ . \tag{4.20}$$

Hence we have the following Lemmas:

Lemma 4.3. In a generalized Ricci-recurrent $(\varepsilon)-$Kenmotsu or $(\varepsilon)-\beta-$Kenmotsu indefinite Finsler manifold $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$, $(B^*)^H$ and $(A^*)^H$ have same or opposite directions if ε is 1 and -1 respectively.

Lemma 4.4. In a generalized Ricci-recurrent $(\varepsilon)-$Kenmotsu or $(\varepsilon)-\beta-$Kenmotsu indefinite Finsler manifold $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$, $(B^*)^V$ and $(A^*)^V$ have same or opposite directions if ε is 1 and -1 respectively.

5. Generalized Ricci-Recurrent Trans Sasakian Indefinite Finsler Manifolds with Cyclic Ricci Tensor

Indefinite Finsler manifolds $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are said to admit cyclic Ricci tensor, if these manifolds are Ricci-recurrent [3]:

$$\left(\nabla^H_{X^H}S^H\right)(Y^H,Z^H)+\left(\nabla^H_{Y^H}S^H\right)(Z^H,X^H)+\left(\nabla^H_{Z^H}S^H\right)(X^H,Y^H)=0 \tag{5.1}$$

$$\left(\nabla^V_{X^V}S^V\right)(Y^V,Z^V)+\left(\nabla^V_{Y^V}S^V\right)(Z^V,X^V)+\left(\nabla^V_{Z^V}S^V\right)(X^V,Y^V)=0. \tag{5.2}$$

Theorem 5.1. Let $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ be generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds with cyclic Ricci tensor, the Ricci tensors satisfy

$$A^H(\xi^H)S^H(X^H,Y^H)=\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(\xi^H)G^H(X^H,Y^H) \tag{5.3}$$

$$A^V(\xi^V)S^V(X^V,Y^V)=\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^V(\xi^V)G^V(X^V,Y^V) \tag{5.4}$$

Proof: Suppose that $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds with cyclic Ricci tensor. Then in view of (4.1), (5.1), we get

$$A^H(X^H)S^H(Y^H,Z^H)+A^H(Y^H)S^H(Z^H,X^H)+A^H(Z^H)S^H(X^H,Y^H)+B^H(X^H)G^H(Y^H,Z^H)+B^H(Y^H)G^H(Z^H,X^H)+B^H(Z^H)G^H(X^H,Y^H)=0 \tag{5.5}$$

Putting $Z^H=\xi^H$ in equation (5.5), we obtain

$$A^H(X^H)S^H(Y^H,\xi^H)+A^H(Y^H)S^H(\xi^H,X^H)+A^H(\xi^H)S^H(X^H,Y^H)+B^H(X^H)G^H(Y^H,\xi^H)+B^H(Y^H)G^H(\xi^H,X^H)+B^H(\xi^H)G^H(X^H,Y^H)=0$$

or

$$-A^H(\xi^H)S^H(X^H,Y^H)=n\frac{(\alpha^2-\beta^2)}{2}A^H(X^H)\eta^H(Y^H)+n\frac{(\alpha^2-\beta^2)}{2}A^H(Y^H)\eta^H(X^H)-\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(\xi^H)G^H(X^H,Y^H)-n\frac{(\alpha^2-\beta^2)}{2}A^H(X^H)\eta^H(Y^H)-n\frac{(\alpha^2-\beta^2)}{2}A^H(Y^H)\eta^H(X^H)=-\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

or

$$A^H(\xi^H)S^H(X^H,Y^H)=\varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

Let $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ be generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds with cyclic Ricci tensor.Then in view of (4.2), (5.2), we get

$$A^V(X^V)S^V(Y^V,Z^V)+A^V(Y^V)S^V(Z^V,X^V)+A^V(Z^V)S^V(X^V,Y^V)+B^V(X^V)G^V(Y^V,Z^V)+B^V(Y^V)G^V(Z^V,X^V)+B^V(Z^V)G^V(X^V,Y^V)=0 \tag{5.6}$$

Putting $Z^V=\xi^V$ in equation (5.6), we obtain

$$A^V(X^V)S^V(Y^V,\xi^V) + A^V(Y^V)S^V(\xi^V,X^V) + A^V(\xi^V)S^V(X^V,Y^V) + B^V(X^V)G^V(Y^V,\xi^V) + B^V(Y^V)G^V(\xi^V,X^V) + B^V(\xi^V)G^V(X^V,Y^V) = 0$$

or

$$A^V(\xi^V)S^V(X^V,Y^V) = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}A^V(\xi^V)G^V(X^V,Y^V)$$

Corollary 5.1. For the generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifold with cyclic Ricci tensor $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$, we have the following statements:

1. If $\alpha \neq 0$, $\beta = 0$, α,β are constant functions defined on $(M^0)^h$, $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a $(\varepsilon)-\alpha-$ Sasakian indefinite Finsler manifold, then from equation (5.3)

$$A^H(\xi^H)S^H(X^H,Y^H) = \varepsilon\, n\frac{(\alpha^2)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

$$S^H(X^H,Y^H) = \varepsilon\, n\tfrac{(\alpha^2)}{2}G^H(X^H,Y^H), \text{ if } A^H(\xi^H)\neq 0.$$

2. If $\alpha = 1$, $\beta = 0$, α,β are constant functions defined on $(M^0)^h$, $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a $(\varepsilon)-$ Sasakian indefinite Finsler manifold, then from equation (5.3)

$$A^H(\xi^H)S^H(X^H,Y^H) = \varepsilon\, n\frac{(1)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

$$S^H(X^H,Y^H) = \varepsilon\, n\tfrac{(1)}{2}G^H(X^H,Y^H), \text{ if } A^H(\xi^H)\neq 0.$$

3. If $\alpha = 0$, $\beta \neq 0$, α,β are constant functions defined on $(M^0)^h$, $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a $(\varepsilon)-$ β-Kenmotsu indefinite Finsler manifold, then from equation (5.3)

$$A^H(\xi^H)S^H(X^H,Y^H) = -\varepsilon\, n\frac{(\beta^2)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

$$S^H(X^H,Y^H) = -\varepsilon\, n\tfrac{(\beta^2)}{2}G^H(X^H,Y^H), \text{ if } A^H(\xi^H)\neq 0.$$

4. If $\alpha = 0$, $\beta = 1$, α,β are constant functions defined on $(M^0)^h$, $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a $(\varepsilon)-$Kenmotsu indefinite Finsler manifold, then from equation (5.3)

$$A^H(\xi^H)S^H(X^H,Y^H) = -\varepsilon\, n\frac{(1)}{2}A^H(\xi^H)G^H(X^H,Y^H)$$

$$S^H(X^H,Y^H) = -\varepsilon\, n\tfrac{(1)}{2}G^H(X^H,Y^H), \text{ if } A^H(\xi^H)\neq 0.$$

Theorem 5.2. Let $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ be a generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds with cyclic Ricci tensor. If $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is one of $(\varepsilon)-\ \alpha-$Sasakian, $(\varepsilon)-$ Sasakian, $(\varepsilon)-$ β-Kenmotsu and $(\varepsilon)-$Kenmotsu manifolds with non-zero $A^H(\xi^H)$ everywhere, then $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is Einstein and Ricci symmetric.

If $\alpha = 0$, $\beta = 0$, α,β are constant functions defined on $(M^0)^h$, $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ is a $(\varepsilon)-$ cosymplectic indefinite Finsler manifold, then from equation (5.3)

$$A^H(\xi^H)S^H(X^H,Y^H) = 0$$

$S^H(X^H,Y^H) = 0$, if $A^H(\xi^H)\neq 0$.

Lemma 5.1: If $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ be a generalized Ricci-recurrent $(\varepsilon)-$ cosymplectic indefinite Finsler manifolds with cyclic Ricci tensor and non-zero $A^H(\xi^H)$ everywhere, then $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ is Ricci flat.

Corollary 5.2. For the generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifold with cyclic Ricci tensor $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, we have the following statements:

5. If $\alpha \neq 0$, $\beta = 0$, α, β are constant functions defined on $(M^0)^v$, $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$is a $(\varepsilon) - \alpha -$ Sasakian indefinite Finsler manifold, then from equation (5.4)

$$A^V(\xi^V)S^V(X^V, Y^V) = \varepsilon\, n\frac{(\alpha^2)}{2}A^V(\xi^V)G^V(X^V, Y^V)$$

$$S^V(X^V, Y^V) = \varepsilon\, n\tfrac{(\alpha^2)}{2}G^V(X^V, Y^V), \text{ if } A^V(\xi^V) \neq 0.$$

6. If $\alpha = 1$, $\beta = 0$, α, β are constant functions defined on $(M^0)^v$, $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is a $(\varepsilon) -$ Sasakian indefinite Finsler manifold, then from equation (5.4)

$$A^V(\xi^V)S^V(X^V, Y^V) = \varepsilon\, n\frac{(1)}{2}A^V(\xi^V)G^V(X^V, Y^V)$$

$$S^V(X^V, Y^V) = \varepsilon\, n\tfrac{(1)}{2}G^V(X^V, Y^V), \text{ if } A^V(\xi^V) \neq 0.$$

7. If $\alpha = 0$, $\beta \neq 0$, α, β are constant functions defined on $(M^0)^v$, $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is a $(\varepsilon) - \beta$-Kenmotsu indefinite Finsler manifold, then from equation (5.4)

$$A^V(\xi^V)S^V(X^V, Y^V) = -\varepsilon\, n\frac{(\beta^2)}{2}A^V(\xi^V)G^V(X^V, Y^V)$$

$$S^V(X^V, Y^V) = -\varepsilon\, n\tfrac{(\beta^2)}{2}G^H\left(X^V, Y^V\right), \text{ if } A^V(\xi^V) \neq 0.$$

8. If $\alpha = 0$, $\beta = 1$, α, β are constant functions defined on $(M^0)^v$, $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$is a $(\varepsilon)-$Kenmotsu indefinite Finsler manifold, then from equation (5.4)

$$A^V(\xi^V)S^V(X^V, Y^V) = -\varepsilon\, n\frac{(1)}{2}A^V(\xi^V)G^V(X^V, Y^V)$$

$$S^V(X^V, Y^V) = -\varepsilon\, n\tfrac{(1)}{2}G^H\left(X^V, Y^V\right), \text{ if } A^V(\xi^V) \neq 0.$$

Theorem 5.3. Let $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ be a generalized Ricci-recurrent trans-Sasakian indefinite Finsler manifolds with cyclic Ricci tensor. If $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is one of $(\varepsilon) - \alpha -$ Sasakian, $(\varepsilon) -$ Sasakian, $(\varepsilon) -$ β-Kenmotsu and $(\varepsilon)-$Kenmotsu manifolds with non-zero $A^H(\xi^H)$ everywhere, then $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is Einstein and Ricci symmetric.

If $\alpha = 0$, $\beta = 0$, α, β are constant functions defined on $(M^0)^v$, $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is a $(\varepsilon) -$ cosymplectic indefinite Finsler manifold, then from equation (5.4)

$$A^V(\xi^V)S^V(X^V, Y^V) = 0$$

$S^V(X^V, Y^V) = 0$, if $A^V(\xi^V) \neq 0$.

Lemma 5.2. If $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ be a generalized Ricci-recurrent $(\varepsilon) -$ cosymplectic indefinite Finsler manifolds with cyclic Ricci tensor and non-zero $A^V(\xi^V)$ everywhere, then$((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ is Ricci flat.

Example 5.1. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$can be stated with $\{(U,\varphi): x_1, x_2, x_3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x_1, x_2, x_3)$. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X_1^V(x,y)\frac{\partial}{\partial y^1} + X_2^V(x,y)\frac{\partial}{\partial y^2} + X_3^V(x,y)\frac{\partial}{\partial y^3}$, $X^H = X_1^H(x,y)\frac{\delta}{\delta x^1} + X_2^H(x,y)\frac{\delta}{\delta x^2} + X_3^H(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$ and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any X$\in$ $T(\mathbb{R}^3)^0$, $X = X_i^H(x,y)\frac{\delta}{\delta x^i} + X_i^V(x,y)\frac{\partial}{\partial y^i}$ (i =1, 2, 3). Consider a η, 1-form, $\eta = \eta^H + \eta^V = \eta_i{}^H(x,y)dx^i + \eta_i{}^V(x,y)\delta y^i$ (i =1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

The vector fields

$$E_1^H = \frac{e^{x_1}}{{x_3}^2}\frac{\delta}{\delta x^1} \ , \ E_2^H = \frac{e^{x_2}}{{x_3}^2}\frac{\delta}{\delta x^2} \ , \ E_3^H = -\frac{\varepsilon}{2}\frac{\delta}{\delta x^3} = \xi^H$$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G be the Sasaki Finsler pseudo-metric given by

$$G^H(E_1^H, \xi^H) = G^H(E_1^H, E_2^H) = G^H(E_2^H, \xi^H) = 0$$

$$G^H(E_1^H, E_1^H) = \varepsilon_1, G^H(E_2^H, E_2^H) = \varepsilon_2 , G^H(\xi^H, \xi^H) = \varepsilon .$$

Let η^H be the 1-form derscribed by

$$\eta^H(Z^H) = \varepsilon\ G^H(Z^H,\ \xi^H) = \varepsilon\, G^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H,\ \xi^H) = z_3 ,\ \forall\, Z^H \in (T(\mathbb{R}^3)^0)^H.$$

Consider the (1, 1) tensör field ϕ^H stated by

$$\phi^H(E_1^H) = E_2^H\ , \phi^H(E_2^H) = -E_1^H\ ,\ \phi^H(\xi^H) = 0.$$

Then using the linearity of ϕ^H, we have

$$Z^H = z_1E_1^H + z_2E_2^H + z_3\,\xi^H\ , W^H = w_1E_1^H + w_2E_2^H + w_3\,\xi^H$$

$$\phi^H(Z^H) = \phi^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H) = z_1\phi^H(E_1^H) + z_2\phi^H(E_2^H) + z_3\,\phi^H(\xi^H)$$

$$\phi^H(Z^H) = z_1E_2^H - z_2E_1^H$$

$$\phi^H(W^H) = \phi^H(wE_1^H + wE_2^H + w_3\,\xi^H) = w_1\phi^H(E_1^H) + w_2\phi^H(E_2^H) + w_3\,\phi^H(\xi^H)$$

$$\phi^H(W^H) = w_1E_2^H - w_2E_1^H$$

$$(\phi^H)^2(Z^H) = -z_2E_2^H - z_1E_1^H = -Z + \eta^H(Z^H)\xi^H$$

Thus we get

$$G^H\big(\phi^H(Z^H), \phi^H(W^H)\big) = G^H(Z^H, W^H) - \varepsilon\,\eta^H(Z^H)\,\eta^H(W^H)$$

$\forall\ Z^H \in (T(\mathbb{R}^3)^0)^H$ and $\forall\ W^H \in (T(\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$$[E_1^H, E_2^H] = 0\ ,\ [E_1^H, \xi^H] = -\frac{\varepsilon}{x_3}E_1^H\ ,\ [E_2^H, \xi^H] = -\frac{\varepsilon}{x_3}\,E_2^H.$$

Using Koszul’s formula, we have

$$2G^H\left(\nabla_{E_1^H}\xi^H,\ E_1^H\right) = -G^H(E_1^H, [\xi^H,\ E_1^H]) - G^H(\xi^H, [E_1^H,\ E_1^H]) + G^H(E_1^H, [E_1^H, \xi^H])$$

$$= 2G^H(-\frac{\varepsilon}{x_3}E_1^H,\ E_1^H).$$

Thus,

$\nabla_{E_1^H}\xi^H = -\frac{\varepsilon}{x_3}E_1^H$, $\nabla_{\xi^H}E_1^H = 0$.

Again by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_2^H}\xi^H,\ E_2^H\right) = -G^H(E_2^H,[\xi^H,\ E_2^H\ ]\ ) - G^H(\xi^H,[\ E_2^H,\ E_2^H]) + G^H(E_2^H\ ,[\ E_2^H,\xi^H])$$

$$= 2G^H(-\frac{\varepsilon}{x_3}\ E_2^H,\ E_2^H).$$

Thus,

$\nabla_{E_2^H}\xi^H = -\frac{\varepsilon}{x_3}\ E_2^H$, $\nabla_{\xi^H}E_2^H = 0$.

Also by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_1^H}E_2^H,\ \xi^H\right) = G^H(E_1^H,[\xi^H,\ E_2^H\ ]\ ) + G^H(\xi^H,[\ E_1^H,\ E_2^H]) - G^H(E_2^H\ ,[\ E_1^H,\xi^H]) = 0.$$

Thus,

$\nabla_{E_1^H}E_2^H = 0$, $\nabla_{E_2^H}E_1^H = 0$

Similarly we get

$$2G^H\left(\nabla_{E_1^H}E_1^H,\ E_2^H\right) = -G^H(E_1^H,[E_1^H,\ E_2^H\ ]\ ) + G^H(\ E_2^H,[\ E_1^H,\ E_1^H]) - G^H(E_1^H\ ,[\ E_1^H,\ E_2^H])$$

$$= 2G^H\left(-\frac{\varepsilon}{x_3}E_1^H\ ,\ E_2^H\right) = 0.$$

Thus,

$$\nabla_{E_1^H}E_1^H = -\frac{\varepsilon}{x_3}E_1^H$$

(3.17) further yields

$$\nabla_{E_2^H}E_2^H = -\frac{\varepsilon}{x_3}\ E_2^H.$$

If we use the equations we found

$(\nabla_X^H\xi^H) = x_1\nabla_{E_1^H}\xi^H + x_2\nabla_{E_2^H}\xi^H = (-x_1)\frac{\varepsilon}{x_3}E_1^H - (x_2)\frac{\varepsilon}{x_3}E_2^H$,

$\forall\ X^H \in (T(\mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left((\mathbb{R}^3)^0)^h\ ,\phi^H,\xi^H,\eta^H,G^H\right)$ satisfy (3.3) for $\alpha = 0$, $\beta = -\frac{2\varepsilon}{x_3}$.

With the help of the above results it can be verified that

$$R^H(\ E_1^H,\ \xi^H)\ \xi^H = -\frac{1}{2{x_3}^2}\ E_1^H$$

$$R^H(\ E_2^H,\ \xi^H)\ \xi^H = -\frac{1}{2{x_3}^2}\ E_2^H,$$

$$R^H(\ E_1^H,\ E_2^H)\ E_2^H = 0,\ R^H(\ E_1^H,\ E_1^H)\ E_1^H = 0$$

$$\varepsilon_3\ G^H(R^H(\ E_1^H,\ \xi^H)\ \xi^H,\ E_2^H) + \varepsilon_1\ G^H(R^H(\ E_1^H,\ E_1^H)\ E_1^H,\ E_2^H) + \varepsilon_2\ G^H(R^H(\ E_1^H,\ E_2^H)\ E_2^H,\ E_2^H) = S^H(\ E_1^H,\ E_2^H) = 0,$$

$$\varepsilon_3\ G^H(R^H(\ E_1^H,\ \xi^H)\ \xi^H,\ \xi^H) + \varepsilon_1\ G^H(R^H(\ E_1^H,\ E_1^H)\ E_1^H,\ \xi^H) + \varepsilon_2\ G^H(R^H(\ E_1^H,\ E_2^H)\ E_2^H,\ \xi^H) =$$

$$S^H(\ E_1^H,\ \xi^H) = 0,$$

$$\varepsilon_3\, G^H(R^H(E_2^H, \xi^H)\,\xi^H, \xi^H) + \varepsilon_1\, G^H(R^H(E_2^H, E_1^H)\,E_1^H, \xi^H) + \varepsilon_2\, G^H(R^H(E_2^H, E_2^H)\,E_2^H, \xi^H) =$$

$$S^H(E_2^H, \xi^H) = 0,$$

$$S^H(\xi^H, \xi^H) = \varepsilon_1\, G^H(R^H(E_1^H, \xi^H)\,\xi^H, E_1^H) + \varepsilon_2 G^H(R^H(E_2^H, \xi^H)\,\xi^H, E_2^H)$$

$$= -\frac{1}{{x_3}^2}$$

$$S^H(E_1^H, E_1^H) = S^H(E_2^H, E_2^H) = -\frac{1}{{x_3}^2}$$

$$S^H(E_1^H, E_1^H) + S^H(E_2^H, E_2^H) + S^H(\xi^H, \xi^H) = -\frac{3}{{x_3}^2}.$$

$$S^H(E_1^H, E_2^H) = S^H(E_1^H, \xi^H) = S^H(E_2^H, \xi^H) = 0$$

$$(\nabla^H_{X^H} R^H)(E_1^H, \xi^H)\,\xi^H = -\frac{\varepsilon}{2{x_3}^3}\mathrm{a}_3\, E_1^H = \frac{\varepsilon}{x_3}\mathrm{a}_3\, R^H(E_1^H, \xi^H)\,\xi^H = A^H(X^H) R^H(E_1^H, \xi^H)\,\xi^H$$

$$(\nabla^H_{X^H} R^H)(E_2^H, \xi^H)\,\xi^H = -\frac{\varepsilon}{2{x_3}^3}\mathrm{a}_3\, E_2^H = \frac{\varepsilon}{x_3}\mathrm{a}_3\, R^H(E_2^H, \xi^H)\,\xi^H = A^H(X^H) R^H(E_2^H, \xi^H)\,\xi^H$$

$$A^H(X^H) = \frac{\varepsilon}{x_3}\mathrm{a}_3$$

for any $X^{\mathrm{H}} \in (T(\mathbb{R}^3)^0)^{\mathrm{H}}$ $(X^H = \mathrm{a}_1 \frac{\delta}{\delta x^1} + \mathrm{a}_2 \frac{\delta}{\delta x^2} + \mathrm{a}_3 \frac{\delta}{\delta x^3})$.

$$(\nabla^H_{X^H} S^H)(E_1^H, E_2^H) = (\nabla^H_{X^H} S^H)(E_1^H, \xi^H) = (\nabla^H_{X^H} S^H)(E_2^H, \xi^H) = 0$$

This implies that there exist a Ricci- recurrent trans- Sasakian ((ε) −Kenmotsu) indefinite Finsler manifold of dimension 3. Therefore, we have

$$S^H(E_{\mathrm{i}}^H, E_{\mathrm{i}}^H) = -\frac{1}{{x_3}^2}\, \varepsilon_{\mathrm{i}} G^H(E_{\mathrm{i}}^H, E_{\mathrm{i}}^H)$$

for i= 1, 2, 3, and $\alpha = 0$, $\beta = -\frac{2\varepsilon}{x_3}$. Hence, M is an Einstein manifold.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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