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Research Article

ON ρ −STATISTICAL CONVERGENCE OF SEQUENCES OF FUZZY NUMBERS

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Abstract: *In this study, we present the concepts of ρ −statistically convergence for sequences of fuzzy numbers as well as strong $\left(w_\rho(F)\right)$ summability and ρ −Cauchy statistically convergence for sequences of fuzzy numbers. We also provide several results concerning these concepts.*

Keywords: *Cesàro summability, Statistical convergence, Strongly ρ −Cesàro summability.*

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1. Introduction

Fast [1] gave short description of statistical convergence 1951. Schoenberg [2] investigated statistical convergence as a summability method and outlined several fundamental properties associated with it. This concept has been applied by many researchers under different names to measurement theory, locally convex spaces, summability theory, Banach spaces, trigonometric series in Fourier analysis and theory of fuzzy set ([3],[4],[5],[6]). The concept of statistical convergence depend on the density subsets of the set $\mathbb{N}$. The natural density of a subset K of $\mathbb{N}$ is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n: k \in K\}|$, if the limit exists, where the vertical bars indicate number of the elements in $\{k \le n: k \in K\}$.

If x is a sequence such that satisfies feature P for all k apart from a set of naturally density zero, then we say that x_k satifies P for "almost all k" and we shortened this by "$a.a.k$."

Fuzzy set theory, which is a very valuable logic with accuracy, was first introduced by Zadeh [7] in 1965. The applications of this theory span various fields, including fuzzy topological spaces, fuzzy measurements, fuzzy mathematical programming, and fuzzy logic. The concept of fuzzy number sequence is first encountered in Matloka's paper [8].

Matloka [8] defined the concept of bounded and convergent sequences of fuzzy numbers and studied their some properties. Since then, many studies on sequences of fuzzy numbers have been made and studies on this subject are still ongoing ([9], [10], [11], [12], [13], [14]).

A fuzzy number is fuzzy set $Z: \mathbb{R} \to [0,1]$ with the following properties:

$i)$ Z is normal, that is, there exists an $z_0 \in \mathbb{R}$ such that $Z(z_0) = 1$;

$ii)$ Z is fuzzy convex, that is, for $z, t \in \mathbb{R}$ and $0 \le \lambda \le 1, Z(\lambda z + (1-\lambda)t) \ge \min[Z(z), Z(t)]$;

$iii)$ Z is upper semicontinuous;

$iv)$ $suppZ = cl\{z \in \mathbb{R}: Z(z) > 0\}$, or denoted by $[Z]^0$, is compact.

The definition α −level set $[Z]^\alpha$ of a fuzzy number is determined by

$$[Z]^{\alpha} = \begin{cases} \{z \in \mathbb{R}: Z(z) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ suppZ, & \text{if } \alpha = 0. \end{cases}$$

It is evident that Z is a fuzzy number is necessary and sufficient for $[Z]^{\alpha}$ is a closed interval for each $\alpha \in [0,1]$ and $[Z]^1 \neq \emptyset$. The set of all fuzzy number sequences will be denoted as $L(\mathbb{R})$. The distance between two fuzzy numbers Z and T, we use the metric

$$d(Z,T) = \sup_{0 \leq \alpha \leq 1} d_H([Z]^{\alpha}, [T]^{\alpha})$$

Let $Z^{\alpha} = \left[\underline{Z}^{\alpha}, \overline{Z}^{\alpha}\right]$ and $T^{\alpha} = \left[\underline{T}^{\alpha}, \overline{T}^{\alpha}\right]$. Then, the Hausdorff metric is characterized by

$$d_H([Z]^{\alpha}, [T]^{\alpha}) = \max\left\{\left|\underline{Z}^{\alpha} - \underline{T}^{\alpha}\right|, \left|\overline{Z}^{\alpha} - \overline{T}^{\alpha}\right|\right\}.$$

It is known that d is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), d)$ is a complete metric space.

Nuray and Savaş [15] defined the concept of statistical convergence for sequences of fuzzy numbers. A sequence $Z = (Z_k)$ of fuzzy numbers is a function $Z: \mathbb{N} \to L(\mathbb{R})$. Let $Z = (Z_k)$ be a sequence of fuzzy numbers. Then the sequence $Z = (Z_k)$ fuzzy numbers, is called statistically convergent to the fuzzy number Z_0 if for each $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0.$$

The set of all fuzzy number sequences demonstrating statistically convergent will be denoted as $S(F)$.

Çakallı [16] defined the concept of $\rho -$statistically convergence. Subsequently, many authors have done a great deal of work on $\rho -$ statistical convergence([17],[18],[19],[20],[21],[22]). The aim of this paper is to extend the investigation conducted by Çakallı [16].

2. Main Results

In this section, we present the concepts of $\rho -$statistically convergence for sequences of fuzzy numbers, strong $\left(w_{\rho}(F)\right)$ summability for sequences of fuzzy numbers and $\rho -$Cauchy statistically convergence for sequences of fuzzy numbers. We also provide several results pertaining to these concepts.

Definition 2.1. Let (Z_k) be a fuzzy number sequence, the sequence $Z = (Z_k)$ is is called $\rho -$statistically convergent to the fuzzy number Z_0 if

$$\lim_{n \to \infty} \frac{1}{\rho_n} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\rho = (\rho_n)$ is a non-decreasing sequence for each $n \in \mathbb{Z}^{+}$ tending to ∞ such that $\limsup_{n} \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta Z_n = Z_{n+1} - Z_n$ for each $n \in \mathbb{Z}^{+}$.

In this case, either $S_{\rho}(F) - \lim Z_k = Z_0$ or $Z_k \to Z_0 \left(S_{\rho}(F)\right)$ is used as a notation. The set of all fuzzy number sequences demonstrating ρ-statistical convergence will be denoted as $S_{\rho}(F)$. If for each

$n \in \mathbb{N}$ $\rho = (\rho_n) = n$, the concept of being $\rho -$statistically convergent is equivalent to being statistically convergent.

Definition 2.2. Let (Z_k) be a fuzzy number sequence, the sequence $Z = (Z_k)$ is is called strong $\rho -$convergent (or $\left(w_\rho(F)\right) -$convergent) to Z_0 if

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} d(Z_k, Z_0) = 0.$$

In this case, either $\left(w_\rho(F)\right) - \lim Z_k = Z_0$ or $\lim_{k\to\infty} Z_k \to Z_0 \left(w_\rho(F)\right)$ is used as a notation. The set of all fuzzy number sequences demonstrating strong $\rho -$convergent will be denoted as $\left(w_\rho(F)\right)$.

Theorem 2.1 Let (Z_k) and (T_k) be two fuzzy numbers sequences, $\rho = (\rho_n)$ is a non-decreasing sequence for each $n \in \mathbb{Z}^+$ tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta Z_n = Z_{n+1} - Z_n$ for each $n \in \mathbb{Z}^+$. Then

(i) $Z_k \to Z_0 \left(S_\rho(F)\right)$ and $c \in \mathbb{C}$ implies $(cZ_k) \to cZ_0 \left(S_\rho(F)\right)$,

(ii) $Z_k \to Z_0 \left(S_\rho(F)\right)$ and $T_k \to T_0 \left(S_\rho(F)\right)$ implies $(Z_k + T_k) \to (Z_0 + T_0) \left(S_\rho(F)\right)$.

Proof. (i) For $c = 0$, the proof is clear. Let $c \neq 0$, the inequality leads to the proof

$$\frac{1}{\rho_n} |\{k \leq n : d(cZ_k, cZ_0) \geq \varepsilon\}| \leq \frac{1}{\rho_n} \left|\left\{k \leq n : d(Z_k, Z_0) \geq \frac{\varepsilon}{c}\right\}\right|.$$

(ii) Let $Z_k \to Z_0 \left(S_\rho(F)\right)$ and $T_k \to T_0 \left(S_\rho(F)\right)$, we can write

$$\frac{1}{\rho_n} |\{k \leq n : d(Z_k + T_k, Z_0 + T_0) \geq \varepsilon\}|$$

$$\leq \frac{1}{\rho_n} \left|\left\{k \leq n : d(Z_k, Z_0) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\rho_n} \left|\left\{k \leq n : d(T_k, T_0) \geq \frac{\varepsilon}{2}\right\}\right|$$

for each $\varepsilon > 0$ and thus if $Z_k \to Z_0 \left(S_\rho(F)\right)$ and $T_k \to T_0 \left(S_\rho(F)\right)$ then $(Z_k + T_k) \to (Z_0 + T_0) \left(S_\rho(F)\right)$.

Definition 2.3 Let (Z_k) be a fuzzy number sequence, the sequence $Z = (Z_k)$ is called $S_\rho(F) -$Cauchy sequence if there exists a subsequence $\left(Z_{k'(n)}\right)$ of Z such that $k'(n) \leq n$ for every n, $\lim_{n\to\infty} Z_{k'(n)} = Z_0$ and for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\left\{k \leq n : d\left(Z_k, Z_{k'(n)}\right) \geq \varepsilon\right\}\right| = 0,$$

where $\rho = (\rho_n)$ is non-decreasing sequence for each $n \in \mathbb{Z}^+$ tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta Z_n = Z_{n+1} - Z_n$ for each $n \in \mathbb{Z}^+$.

Theorem 2.2. The subsequent statements are mutually equivalent:

(i) (Z_k) is a ρ −statistical convergence,

(ii) (Z_k) is a ρ −Cauchy statistical convergence,

(iii) (Z_k) is a sequence of fuzzy numbers for which there is a ρ −statistically convergent sequence of fuzzy numbers T such that $Z_k = T_k\ a.a.k$.

Theorem 2.3 Let (Z_k) be a fuzzy number sequence, the sequence $Z = (Z_k)$ is $S_\rho(F)$ −convergent a necessary and sufficient condition is that (Z_k) is an $S_\rho(F)$ −Cauchy sequence.

Proof. Let's consider Z_k is an S_ρ −Cauchy sequence. For each $\varepsilon > 0$, we can say

$$\frac{1}{\rho_n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}| \le \frac{1}{\rho_n}\left|\left\{k \le n: d\left(Z_k, Z_{k'(n)}\right) \ge \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\rho_n}\left|\left\{k \le n: d\left(Z_{k'(n)}, Z_0\right) \ge \frac{\varepsilon}{2}\right\}\right|.$$

Hence, we get $Z_k \to Z_0\left(S_\rho(F)\right)$.

The proof to the contrary is obvious.

Theorem 2.4 Let (Z_k) be a fuzzy number sequence, $\rho = (\rho_n)$ is non-decreasing sequence for each $n \in \mathbb{Z}^+$ tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta Z_n = Z_{n+1} - Z_n$ for each $n \in \mathbb{Z}^+$. If for each $n \in \mathbb{N}$, $\liminf\left(\frac{\rho_n}{n}\right) \ge 1$, then $S(F) \subset S_\rho(F)$.

Proof. Let's consider $Z_k \to Z_0(S(F))$, the following inequality leads to the proof, for every $\varepsilon > 0$

$$\frac{1}{n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}| = \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}| \ge \frac{1}{\rho_n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}|.$$

Theorem 2.5. Let (Z_k) be a fuzzy number sequence, $\rho = (\rho_n)$ and $\varrho = (\varrho_n)$ be two sequences such that $\rho_n \le \varrho_n$ for all $n \in \mathbb{N}$. If $\liminf\left(\frac{\rho_n}{\varrho_n}\right) > 0$, then $S_\rho(F) \subset S_\varrho(F)$.

Proof. Suppose that $Z_k \to Z_0\left(S_\rho(F)\right)$, the following inequality leads to the proof, for every $\varepsilon > 0$

$$\frac{1}{\varrho_n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}| \le \frac{\rho_n}{\varrho_n}\frac{1}{\rho_n}|\{k \le n: d(Z_k, Z_0) \ge \varepsilon\}|.$$

Corollary 2.1 Let (Z_k) be a fuzzy number sequence, $\rho = (\rho_n)$ and $\varrho = (\varrho_n)$ be two sequences such that $\rho_n \le \varrho_n$ for all $n \in \mathbb{N}$. If $\liminf\left(\frac{\rho_n}{\varrho_n}\right) > 0$, then $S(F) \subset S_\rho(F) \subset S_\varrho(F)$.

Theorem 2.6. If $(Z_k) \to Z_0\left(w_\rho(F)\right)$, then $(Z_k) \to Z_0\left(S_\rho(F)\right)$.

Proof. Suppose that $(Z_k) \to Z_0\left(w_\rho(F)\right)$, for $\varepsilon > 0$, we can write

$$\frac{1}{\rho_n}\sum_{k=1}^{n} d(Z_k,Z_0) = \frac{1}{\rho_n}\left(\sum_{\substack{k=1\\ d(Z_k,Z_0)\geq\varepsilon}}^{n} d(Z_k,Z_0) + \sum_{\substack{k=1\\ d(Z_k,Z_0)<\varepsilon}}^{n} d(Z_k,Z_0)\right)$$
$$\geq \frac{1}{\rho_n}\sum_{\substack{k=1\\ d(Z_k,Z_0)\geq\varepsilon}}^{n} d(Z_k,Z_0)$$
$$\geq \varepsilon\frac{1}{\rho_n}|\{k\leq n: d(Z_k,Z_0)\geq\varepsilon\}|.$$

If we take the limit for $n\to\infty$, we have

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n: d(Z_k,Z_0)\geq\varepsilon\}| = 0.$$

Thus, the desired outcome is obtained.

Corollary 2.2 Let (Z_k) be a fuzzy number sequence. If $(Z_k)\to Z_0(n\to\infty)$, then $(Z_k)\to Z_0\left(S_\rho(F)\right)$.

The opposite of the Theorem 2.6 and Corollary 2.2 aren't true, mostly. For example, let the $Z=(Z_k)$ sequence be as follows:

$$Z_k(z) = \begin{cases} \left.\begin{matrix} \frac{k}{k+2}z+\frac{2-2k}{k+2}, & z\in\left[\frac{2k-2}{k},3\right] \\ -\frac{k}{k+2}z+\frac{4k+2}{k+2}, & z\in\left[3,\frac{4k+2}{k}\right] \\ 0, & \text{otherwise}\end{matrix}\right\}, & \text{if} k=n^3 \ (n=1,2,\dots) \\ \left.\begin{matrix} z-2, & z\in[2,3] \\ -z+4, & z\in[3,4] \\ 0, & \text{otherwise}\end{matrix}\right\} := Z_0 & \text{if} k\neq n^3 \end{cases}$$

If we choose $(\rho_n)=n$,

$$\frac{1}{\rho_n}|\{k\leq n: d(Z_k,Z_0)\geq\varepsilon\}| = \frac{\sqrt[3]{n}}{n}\to 0(n\to\infty).$$

Moreover,

$$\frac{1}{\rho_n}\sum_{k=1}^{n} d(Z_k,Z_0) = \infty\ (n\to\infty)$$

so, Z_k is not $\left(w_\rho(F)\right)$ convergent Z_0.

Ethical statement

The author declares that this document does not require ethics committee approval or any special permission. Our study does not cause any harm to the environment.

Conflict of interest

The author declares no potential conflicts of interest related to this article's research, authorship, and publication.

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