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THE CENTRALIZER OF $\mathfrak{sl}(0, n)$ IN THE GENERALIZED WITT LIE SUPERALGEBRA OVER FIELDS OF PRIME CHARACTERISTIC

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Abstract. This paper considers centralizers of the Lie superalgebra $\mathfrak{sl}(0, n)$ over prime characteristic fields. Using homological methods, the centralizers of the even and odd parts of $\mathfrak{sl}(0, n)$ in the generalized Witt Lie superalgebra are calculated and a summary of their structural properties is provided.

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1. Introduction

The Lie superalgebra (see [1]) is a generalization of Lie algebras and an important part of the Lie theory. It not only occupies a very important position in mathematics, but also has general applications in modern physics. In theoretical physics, the Lie superalgebra is a Lie algebra with Z_2-graded which can be used to study supersymmetric phenomena (see [2],[9]). In addition, the deformation theory of the universal enveloping algebra in the Lie superalgebra makes a significant contribution to the study of quantum supergroup theory in physics.

At present, the study of non-modular Lie superalgebra has achieved quite systematic results (see [11]). However, there is still a lot of research space for modular Lie superalgebras (see [14],[15]). We know that centralizers play an important role in the study of group structure (see [4],[5]) and the Lie superalgebra is a linearized object of the Lie supergroup, so it also has centralizers. In the literature [6],[8],[13], the centralizers of $\mathfrak{gl}(0, 3)$, $\mathfrak{gl}(m, n)$, $\mathfrak{sl}(0, 3)$ in the generalized Witt Lie superalgebras are studied respectively.

Since the Lie superalgebra can be represented by an adjoint representation, it can be considered a natural module of its subalgebra. Therefore, the centralizer

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can be equivalently regarded as a zero cohomology group (see [10],[12]). This paper is inspired by references (see [6],[7]) and we use homological methods to discuss the centralizers of Lie superalgebras in the generalized Witt Lie superalgebra W. We believe that our findings contribute significantly to the understanding of non-modular Lie superalgebras and have implications for the study of supersymmetric phenomena in modern physics.

2. Preliminary

Let $\mathbb{F}$ be a field of prime characteristic, $\mathrm{Y}_0 = \{1, 2, \ldots, m\}$, $\mathrm{Y}_1 = \{1, 2, \ldots, n\}$,

$$\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k\rangle \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

$\mathbb{B} = \bigcup_{k=0}^{n} \mathbb{B}_k$. The structure of the generalized Witt Lie superalgebra over a prime characteristic field is as follows (see [3]):

$$W = w \oplus \omega,$$

where

$$\begin{aligned} w &= \bigoplus_{i=0}^{\pi+n} w_i = \bigoplus_{i=0}^{\pi+n} span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^u D_k \mid |\alpha| + |u| = i, \alpha \in \mathbb{A}(m, n, \underline{t}), u \in \mathbb{B}, k \in \mathrm{Y}_0 \right\}, \\ \omega &= \bigoplus_{j=0}^{\pi+n} \omega_j = \bigoplus_{j=0}^{\pi+n} span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^u d_b \mid |\alpha| + |u| = j, \alpha \in \mathbb{A}(m, n, \underline{t}), u \in \mathbb{B}, b \in \mathrm{Y}_1 \right\}. \end{aligned}$$

The canonical basis of the Lie superalgebra $\mathfrak{sl}(0, n)$ is $\{\xi_q d_v, \xi_i d_i - \xi_{i+1} d_{i+1}\}$, where $q, v \in \mathrm{Y}_1$, $q \neq v$, $i = 1, \ldots, n-1$.

Definition 2.1. In a Lie algebra L and the L-module A if

$$\ldots \to C^{n-1}(L, A) \xrightarrow{\delta_{n-1}} C^n(L, A) \xrightarrow{\delta_n} C^{n+1}(L, A) \to \ldots$$

(where $n \in \mathbb{N}$, the angle code increases as the arrow progresses.) satisfies:

$$\delta_n \delta_{n-1} = 0 \ (\forall n \in \mathbb{N}); \ \delta_0(a)(x) = x.a \ (\forall x \in L, \ \forall a \in A),$$

where the L-module homomorphism δ_n $(\forall n \in \mathbb{N})$ is the coboundary operator, then this chain is said to be a cochain.

Definition 2.2. Let L be a Lie superalgebra, A be an L-module,

$$H^n(L, A) = Ker\delta_n / (\mathrm{Im}\delta_{n-1})$$

is called an n-dimensional cohomology of L with the coefficient in its module A.

According to the definition, the following lemmas are clearly stated.

Lemma 2.3. *For all L-module A, the centralizers of L in A is*

$$C_A(L) = H^0(L, A) = \{a \in A|\ x \cdot a = 0,\ x \in L\}.$$

Lemma 2.4. *Suppose that A is an L-module and A_1, A_2,..., A_k are submodules of A such that $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_k$. Then*

$$H^n(L, A) = \bigoplus_{i=1}^{k}(L, A_i),\ n \in \mathbb{N}.$$

3. The centralizer $C_W(\mathfrak{sl}(0, n))$

Firstly, we consider $C_w(\mathfrak{sl}(0, n))$.

Proposition 3.1. *The centralizers of $\mathfrak{sl}(0, n)$ in w_0 is*

$$C_{w_0}(sl(0, n)) = H^0(sl(0, n), w_0) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid |\alpha| = 1,\ k \in Y_0\right\}.$$

Proof. According to $w_0 = span_{\mathbb{F}}\left\{X^{(\alpha)}\xi^u D_k \mid |\alpha| + |u| = 1,\ k \in Y_0\right\}$, suppose that

$$f = \sum_{k,l \in Y_0} c_{lk} x_l D_{_k} + \sum_{k \in Y_0}\sum_{j \in Y_1} g_{jk}\xi_j D_k.$$

If $q \neq v$, then

$$[\xi_q d_v, f] = \sum_{k \in Y_0} g_{vk}\xi_q D_k = 0.$$

If $q = v$, then

$$[\xi_1 d_1 - \xi_2 d_2, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{2k}\xi_2 D_k = 0,$$

$$[\xi_1 d_1 - \xi_3 d_3, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{3k}\xi_3 D_k = 0.$$

Similarly,

$$[\xi_1 d_1 - \xi_n d_n, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{nk}\xi_n D_k = 0.$$

Therefore,

$$g_{11} = g_{12} = \cdots = g_{1m} = g_{21} = g_{22} = \cdots = g_{2m} = \cdots = g_{n1} = \cdots = g_{nm} = 0,$$

as desired. $\square$

Proposition 3.2. *The centralizers of $\mathfrak{sl}(0, n)$ in w_1 is*

$$C_{w_1}(sl(0, n)) = H^0(sl(0, n), w_1) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid |\alpha| = 2,\ k \in Y_0\right\}.$$

Proof. Suppose that

$$f=\sum_{k,l,p\in Y_0} c_{lpk}x_lx_pD_k+\sum_{k,l\in Y_0}\sum_{j\in Y_1} g_{ilk}\xi_jx_lD_k+\sum_{k\in Y_0}\sum_{i,j\in Y_1} h_{ijk}\xi_i\xi_jD_k.$$

It follows from $q\neq v$ that $[\xi_qd_v,f]=0$. If $q=v$, then $[\xi_1d_1-\xi_td_t,f]=0,\ t=1,\ldots,n$. By calculating the coefficients, the conclusion is proved. □

Proposition 3.3. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* w_i *is*

$$C_{w_i}(sl(0,n))=H^0(sl(0,n),w_i)=span_{\mathbb{F}}\left\{x^{(\alpha)}D_k\mid |\alpha|=i+1,\ k\in Y_0\right\}.$$

Proof. According to induction, drawing conclusions is straightforward. □

From Proposition 3.3 and Lemma 2.4, it follows that:

Theorem 3.4. *The centralizers of Lie superalgebras in* w *is*

$$C_w(sl(0,n))=H^0(sl(0,n),w)=span_{\mathbb{F}}\left\{x^{(\alpha)}D_k\mid \alpha\in A(m,t),\ k\in Y_0\right\}.$$

Next, we will investigate $C_\omega(sl(0,n))$.

Proposition 3.5. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* ω_0 *is*

$$C_{\omega_0}(sl(0,n))=H^0(sl(0,n),\omega_0)=span_{\mathbb{F}}\left\{\xi_bd_b\mid b\in Y_1\right\}.$$

Proof. According to $\omega_0=span_{\mathbb{F}}\left\{X^{(\alpha)}\xi^ud_b\mid |\alpha|+|u|=1,\ b\in Y_1\right\}$, suppose that

$$f=\sum_{b\in Y_1}\sum_{l\in Y_0} c_{lb}x_ld_b+\sum_{b,j\in Y_1} g_{jb}\xi_jd_b.$$

If $q\neq v$, then

$$[\xi_qd_v,f]=\sum_{b\in Y_1} g_{vb}\xi_qd_b-\sum_{j\in Y_1} g_{jq}\xi_jd_v+\sum_{l\in Y_0} c_{lq}x_ld_v=0.$$

If $q=v$, then

$$\begin{aligned}[][\xi_1d_1-\xi_2d_2,f] &= \sum_{b\in Y_1} g_{1b}\xi_1d_b-\sum_{b\in Y_1} g_{2b}\xi_2d_b+\sum_{j\in Y_1} g_{j2}\xi_jd_2\\ &- \sum_{j\in Y_1} g_{j1}\xi_jd_1+\sum_{l\in Y_0} c_{l2}x_ld_2-\sum_{l\in Y_0} c_{l1}x_ld_1=0,\end{aligned}$$

$$\begin{aligned}[][\xi_1d_1-\xi_3d_3,f] &= \sum_{b\in Y_1} g_{1b}\xi_1d_b-\sum_{b\in Y_1} g_{3b}\xi_3d_b+\sum_{j\in Y_1} g_{j3}\xi_jd_3\\ &- \sum_{j\in Y_1} g_{j1}\xi_jd_1+\sum_{l\in Y_0} c_{l3}x_ld_3-\sum_{l\in Y_0} c_{l1}x_ld_1=0.\end{aligned}$$

Similarly,

$$\begin{aligned}[\xi_1 d_1 - \xi_n d_n, f] &= \sum_{b\in Y_1} g_{1b}\xi_1 d_b - \sum_{b\in Y_1} g_{nb}\xi_n d_b + \sum_{j\in Y_1} g_{jn}\xi_j d_n \\ &\quad - \sum_{j\in Y_1} g_{j1}\xi_j d_1 + \sum_{l\in Y_0} c_{nl} x_l d_n - \sum_{l\in Y_0} c_{l1} x_l d_1 = 0.\end{aligned}$$

Therefore, $c_{ij} = 0$, $g_{ij} = 0$, $i,\ j \in Y_1$ if $i \neq j$, as desired. □

Proposition 3.6. *The centralizers of* $\mathfrak{sl}(0, n)$ *in* ω_1 *is*

$$C_{\omega_1}(sl(0,n)) = H^0(sl(0,n), \omega_1) = span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^{u-\langle b\rangle} \xi_b d_b \mid |\alpha| + |u - \langle b\rangle| = 1,\ b \in Y_1 \right\}.$$

Proof. Suppose that

$$f = \sum_{b\in Y_1} \sum_{l,p\in Y_0} c_{lpb} x_l x_p d_b + \sum_{b,j\in Y_1} \sum_{l\in Y_0} g_{jlb} \xi_j x_l d_b + \sum_{b,i,j\in Y_1} h_{ijb} \xi_i \xi_j d_b.$$

It follows from $q \neq v$ that $[\xi_q d_v, f] = 0$. If $q = v$, then $[\xi_1 d_1 - \xi_t d_t, f] = 0,\ t = 1, \ldots, n$. By calculating the coefficients, the conclusion is proved. □

Proposition 3.7. *The centralizers of* $\mathfrak{sl}(0, n)$ *in* ω_i *is*

$$\begin{aligned} C_{\omega_i}(sl(0,n)) &= H^0(sl(0,n), \omega_i) \\ &= span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^{u-\langle b\rangle} \xi_b d_b \mid |\alpha| + |u - \langle b\rangle| = i+1,\ b \in Y_1 \right\}. \end{aligned}$$

Proof. According to induction, drawing conclusions is straightforward. □

From Proposition 3.7 and Lemma 2.4, it follows that:

Theorem 3.8. *The centralizers of* $\mathfrak{sl}(0, n)$ *in* ω *is*

$$\begin{aligned} C_{\omega}(sl(0,n)) &= H^0(sl(0,n), \omega) \\ &= span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^{u-\langle b\rangle} \xi_b d_b \mid 0 \le |\alpha| + |u - \langle b\rangle| \le \pi + n - 1,\ b \in Y_1 \right\}. \end{aligned}$$

From Theorem 3.4, Theorem 3.8 and Lemma 2.4, it follows that:

Theorem 3.9. *The centralizers of* $\mathfrak{sl}(0, n)$ *in* W *is*

$$\begin{aligned} C_W(sl(0,n)) = H^0(sl(0,n), W) &= H^0(sl(0,n), w) \oplus H^0(sl(0,n), \omega) \\ &= span_{\mathbb{F}} \left\{ x^{(\alpha)} D_k, x^{(\alpha)} \xi^{u-\langle b\rangle} \xi_b d_b \right\}, \\ \text{where } \alpha \in A(m,t),\ & 0 \le |\alpha| + |u| \le \pi + n - 1,\ k \in Y_0,\ b \in Y_1. \end{aligned}$$

The conclusion gives the centralizers of $\mathfrak{sl}(0, n)$ in the generalized Witt Lie superalgebra W. This contributes to the study of the structure of the Lie superalgebra $\mathfrak{sl}(0, n)$.

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