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THE CENTRALIZER OF $\mathfrak{sl}(0,n)$ IN THE GENERALIZED WITT LIE SUPERALGEBRA OVER FIELDS OF PRIME CHARACTERISTIC

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Abstract. This paper considers centralizers of the Lie superalgebra $\mathfrak{sl}(0,n)$ over prime characteristic fields. Using homological methods, the centralizers of the even and odd parts of $\mathfrak{sl}(0,n)$ in the generalized Witt Lie superalgebra are calculated and a summary of their structural properties is provided.

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1. Introduction

The Lie superalgebra (see [1]) is a generalization of Lie algebras and an important part of the Lie theory. It not only occupies a very important position in mathematics, but also has general applications in modern physics. In theoretical physics, the Lie superalgebra is a Lie algebra with Z_2-graded which can be used to study supersymmetric phenomena (see [2],[9]). In addition, the deformation theory of the universal enveloping algebra in the Lie superalgebra makes a significant contribution to the study of quantum supergroup theory in physics.

At present, the study of non-modular Lie superalgebra has achieved quite systematic results (see [11]). However, there is still a lot of research space for modular Lie superalgebras (see [14],[15]). We know that centralizers play an important role in the study of group structure (see [4],[5]) and the Lie superalgebra is a linearized object of the Lie supergroup, so it also has centralizers. In the literature [6],[8],[13], the centralizers of $\mathfrak{gl}(0,3)$, $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(0,3)$ in the generalized Witt Lie superalgebras are studied respectively.

Since the Lie superalgebra can be represented by an adjoint representation, it can be considered a natural module of its subalgebra. Therefore, the centralizer can be equivalently regarded as a zero cohomology group (see [10],[12]). This paper is inspired by references (see [6],[7]) and we use homological methods to discuss

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the centralizers of Lie superalgebras in the generalized Witt Lie superalgebra W. We believe that our findings contribute significantly to the understanding of non-modular Lie superalgebras and have implications for the study of supersymmetric phenomena in modern physics.

2. Preliminary

Let $\mathbb{F}$ be a field of prime characteristic, $\mathrm{Y}_0 = \{1, 2, \ldots, m\}$, $\mathrm{Y}_1 = \{1, 2, \ldots, n\}$,

$$\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k \rangle \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

$\mathbb{B} = \bigcup_{k=0}^{n} \mathbb{B}_k$. The structure of the generalized Witt Lie superalgebra over a prime characteristic field is as follows (see [3]):

$$W = w \oplus \omega,$$

where

$$\begin{aligned} w &= \bigoplus_{i=0}^{\pi+n} w_i = \bigoplus_{i=0}^{\pi+n} span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^u D_k \mid |\alpha| + |u| = i, \alpha \in \mathbb{A}(m, n, \underline{t}), u \in \mathbb{B}, k \in \mathrm{Y}_0 \right\}, \\ \omega &= \bigoplus_{j=0}^{\pi+n} \omega_j = \bigoplus_{j=0}^{\pi+n} span_{\mathbb{F}} \left\{ x^{(\alpha)} \xi^u d_b \mid |\alpha| + |u| = j, \alpha \in \mathbb{A}(m, n, \underline{t}), u \in \mathbb{B}, b \in \mathrm{Y}_1 \right\}. \end{aligned}$$

The canonical basis of the Lie superalgebra $\mathfrak{sl}(0, n)$ is $\{\xi_q d_v, \xi_i d_i - \xi_{i+1} d_{i+1}\}$, where $q, v \in \mathrm{Y}_1$, $q \neq v$, $i = 1, \ldots, n-1$.

Definition 2.1. In a Lie algebra L and the L-module A if

$$\ldots \to C^{n-1}(L, A) \xrightarrow{\delta_{n-1}} C^n(L, A) \xrightarrow{\delta_n} C^{n+1}(L, A) \to \ldots$$

(where $n \in \mathbb{N}$, the angle code increases as the arrow progresses.) satisfies:

$$\delta_n \delta_{n-1} = 0 \ (\forall n \in \mathbb{N}); \ \delta_0(a)(x) = x.a \ (\forall x \in L, \ \forall a \in A),$$

where the L-module homomorphism δ_n ($\forall n \in \mathbb{N}$) is the coboundary operator, then this chain is said to be a cochain.

Definition 2.2. Let L be a Lie superalgebra, A be an L-module,

$$H^n(L, A) = Ker\delta_n / (\mathrm{Im}\delta_{n-1})$$

is called an n-dimensional cohomology of L with the coefficient in its module A.

According to the definition, the following lemmas are clearly stated.

Lemma 2.3. *For all L-module A, the centralizers of L in A is*

$$C_A(L) = H^0(L, A) = \{a \in A \mid x \cdot a = 0, \ x \in L\}.$$

Lemma 2.4. *Suppose that A is an L-module and A_1, A_2,..., A_k are submodules of A such that $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_k$. Then*

$$H^n(L,A) = \bigoplus_{i=1}^{k}(L,A_i),\ n \in \mathbb{N}.$$

3. The centralizer $C_W(\mathfrak{sl}(0,n))$

Firstly, we consider $C_w(\mathfrak{sl}(0,n))$.

Proposition 3.1. *The centralizers of $\mathfrak{sl}(0,n)$ in w_0 is*

$$C_{w_0}(sl(0,n)) = H^0(sl(0,n),w_0) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid |\alpha| = 1,\ k \in Y_0\right\}.$$

Proof. According to $w_0 = span_{\mathbb{F}}\left\{X^{(\alpha)}\xi^u D_k \mid |\alpha| + |u| = 1,\ k \in Y_0\right\}$, suppose that

$$f = \sum_{k,l \in Y_0} c_{lk}x_l D_{_k} + \sum_{k \in Y_0}\sum_{j \in Y_1} g_{jk}\xi_j D_k.$$

If $q \neq v$, then

$$[\xi_q d_v, f] = \sum_{k \in Y_0} g_{vk}\xi_q D_k = 0.$$

If $q = v$, then

$$[\xi_1 d_1 - \xi_2 d_2, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{2k}\xi_2 D_k = 0,$$

$$[\xi_1 d_1 - \xi_3 d_3, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{3k}\xi_3 D_k = 0.$$

Similarly,

$$[\xi_1 d_1 - \xi_n d_n, f] = \sum_{k \in Y_0} g_{1k}\xi_1 D_k - \sum_{k \in Y_0} g_{nk}\xi_n D_k = 0.$$

Therefore,

$$g_{11} = g_{12} = \cdots = g_{1m} = g_{21} = g_{22} = \cdots = g_{2m} = \cdots = g_{n1} = \cdots = g_{nm} = 0,$$

as desired. $\square$

Proposition 3.2. *The centralizers of $\mathfrak{sl}(0,n)$ in w_1 is*

$$C_{w_1}(sl(0,n)) = H^0(sl(0,n),w_1) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid |\alpha| = 2,\ k \in Y_0\right\}.$$

Proof. Suppose that

$$f = \sum_{k,l,p \in Y_0} c_{lpk}x_l x_p D_{_k} + \sum_{k,l \in Y_0}\sum_{j \in Y_1} g_{ilk}\xi_j x_l D_k + \sum_{k \in Y_0}\sum_{i,j \in Y_1} h_{ijk}\xi_i\xi_j D_k.$$

It follows from $q \neq v$ that $[\xi_q d_v, f] = 0$. If $q = v$, then $[\xi_1 d_1 - \xi_t d_t, f] = 0,\ t = 1,\ldots,n$. By calculating the coefficients, the conclusion is proved. $\square$

Proposition 3.3. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* w_i *is*

$$C_{w_i}(sl(0,n)) = H^0(sl(0,n), w_i) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid |\alpha| = i+1,\ k \in Y_0\right\}.$$

Proof. According to induction, drawing conclusions is straightforward. □

From Proposition 3.3 and Lemma 2.4, it follows that:

Theorem 3.4. *The centralizers of Lie superalgebras in* w *is*

$$C_w(sl(0,n)) = H^0(sl(0,n), w) = span_{\mathbb{F}}\left\{x^{(\alpha)}D_{_k} \mid \alpha \in A(m,t),\ k \in Y_0\right\}.$$

Next, we will investigate $C_\omega(sl(0,n))$.

Proposition 3.5. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* ω_0 *is*

$$C_{\omega_0}(sl(0,n)) = H^0(sl(0,n), \omega_0) = span_{\mathbb{F}}\left\{\xi_b d_b \mid b \in Y_1\right\}.$$

Proof. According to $\omega_0 = span_{\mathbb{F}}\left\{X^{(\alpha)}\xi^u d_b \mid |\alpha| + |u| = 1,\ b \in Y_1\right\}$, suppose that

$$f = \sum_{b\in Y_1}\sum_{l\in Y_0} c_{lb} x_l d_{_b} + \sum_{b,j\in Y_1} g_{jb}\xi_j d_b.$$

If $q \neq v$, then

$$[\xi_q d_v, f] = \sum_{b\in Y_1} g_{vb}\xi_q d_b - \sum_{j\in Y_1} g_{jq}\xi_j d_v + \sum_{l\in Y_0} c_{lq} x_l d_v = 0.$$

If $q = v$, then

$$\begin{aligned}[\xi_1 d_1 - \xi_2 d_2, f] &= \sum_{b\in Y_1} g_{1b}\xi_1 d_b - \sum_{b\in Y_1} g_{2b}\xi_2 d_b + \sum_{j\in Y_1} g_{j2}\xi_j d_2 \\ &- \sum_{j\in Y_1} g_{j1}\xi_j d_1 + \sum_{l\in Y_0} c_{l2} x_l d_2 - \sum_{l\in Y_0} c_{l1} x_l d_1 = 0,\end{aligned}$$

$$\begin{aligned}[\xi_1 d_1 - \xi_3 d_3, f] &= \sum_{b\in Y_1} g_{1b}\xi_1 d_b - \sum_{b\in Y_1} g_{3b}\xi_3 d_b + \sum_{j\in Y_1} g_{j3}\xi_j d_3 \\ &- \sum_{j\in Y_1} g_{j1}\xi_j d_1 + \sum_{l\in Y_0} c_{l3} x_l d_3 - \sum_{l\in Y_0} c_{l1} x_l d_1 = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}[\xi_1 d_1 - \xi_n d_n, f] &= \sum_{b\in Y_1} g_{1b}\xi_1 d_b - \sum_{b\in Y_1} g_{nb}\xi_n d_b + \sum_{j\in Y_1} g_{jn}\xi_j d_n \\ &- \sum_{j\in Y_1} g_{j1}\xi_j d_1 + \sum_{l\in Y_0} c_{nl} x_l d_n - \sum_{l\in Y_0} c_{l1} x_l d_1 = 0.\end{aligned}$$

Therefore, $c_{ij} = 0$, $g_{ij} = 0$, $i,\ j \in Y_1$ if $i \neq j$, as desired. □

Proposition 3.6. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* ω_1 *is*

$$C_{\omega_1}(sl(0,n)) = H^0(sl(0,n), \omega_1) = span_{\mathbb{F}}\left\{x^{(\alpha)}\xi^{u-\langle b\rangle}\xi_b d_b \mid |\alpha| + |u - \langle b\rangle| = 1,\ b \in Y_1\right\}.$$

Proof. Suppose that

$$f=\sum_{b\in Y_1}\sum_{l,p\in Y_0} c_{lpb}x_lx_pd_b+\sum_{b,j\in Y_1}\sum_{l\in Y_0} g_{jlb}\xi_jx_ld_b+\sum_{b,i,j\in Y_1} h_{ijb}\xi_i\xi_jd_b.$$

It follows from $q\neq v$ that $[\xi_qd_v,f]=0$. If $q=v$, then $[\xi_1d_1-\xi_td_t,f]=0,\ t=1,\dots,n$. By calculating the coefficients, the conclusion is proved. □

Proposition 3.7. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* ω_i *is*

$$C_{\omega_i}(sl(0,n))=H^0(sl(0,n),\omega_i)$$
$$=span_{\mathbb{F}}\left\{x^{(\alpha)}\xi^{u-\langle b\rangle}\xi_bd_b\mid |\alpha|+|u-\langle b\rangle|=i+1,\ b\in Y_1\right\}.$$

Proof. According to induction, drawing conclusions is straightforward. □

From Proposition 3.7 and Lemma 2.4, it follows that:

Theorem 3.8. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* ω *is*

$$C_{\omega}(sl(0,n))=H^0(sl(0,n),\omega)$$
$$=span_{\mathbb{F}}\left\{x^{(\alpha)}\xi^{u-\langle b\rangle}\xi_bd_b\mid 0\le|\alpha|+|u-\langle b\rangle|\le\pi+n-1,\ b\in Y_1\right\}.$$

From Theorem 3.4, Theorem 3.8 and Lemma 2.4, it follows that:

Theorem 3.9. *The centralizers of* $\mathfrak{sl}(0,n)$ *in* W *is*

$$C_W(sl(0,n))=H^0(sl(0,n),W)=H^0(sl(0,n),w)\oplus H^0(sl(0,n),\omega)$$
$$=span_{\mathbb{F}}\left\{x^{(\alpha)}D_{k},x^{(\alpha)}\xi^{u-\langle b\rangle}\xi_bd_b\right\},$$
$$where\ \alpha\in A(m,t),\ 0\le|\alpha|+|u|\le\pi+n-1,\ k\in Y_0,\ b\in Y_1.$$

The conclusion gives the centralizers of $\mathfrak{sl}(0,n)$ in the generalized Witt Lie superalgebra W. This contributes to the study of the structure of the Lie superalgebra $\mathfrak{sl}(0,n)$.

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