Isı Bilimi ve Tekniği Dergisi, 26, 2, 17-22, 2006
J. of Thermal Science and Technology

ISSBN 1300-3615

A SIMPLE CORRELATION EQUATION FOR OPTIMIZATION OF RADIATING RECTANGULAR FINS WITH VARIABLE THERMAL CONDUCTIVITY

Cihat ARSLANTÜRK
Department of Mechanical Engineering, Engineering Faculty, Ataturk University
25240 Erzurum, Turkey, carslan@atauni.edu.tr

(Geliş Tarihi: 20. 03. 2006)

Abstract: Radiating extended surfaces are widely used to enhance heat transfer between primary surface and the environment. The performance of such a surface is significantly affected by variable thermal conductivity, particularly in the case of large temperature differences. The aim of the present work is to evaluate the optimum dimensions of radiating rectangular fins with temperature-dependent thermal conductivity for a fixed fin volume. The nonlinear fin equation is solved by the Adomian decomposition method for obtaining the temperature distribution within such fins. The optimum geometry which maximizes the heat transfer rate from the fins of fixed volume is found by using the data from the solution. Derived condition of the optimality is a simple correlation relation between the fin parameter and thermal conductivity parameter describing the variation of thermal conductivity. The resulting correlation equation is a very suitable tool for optimum design of radiating rectangular fins with temperature-dependent thermal conductivity.
Keywords: Fin, Optimization, Variable thermal conductivity.

DEĞİŞKEN ISI İLETİM KATSAYILI DİKDÖRTGEN KANATLARIN OPTİMİZASYONU İÇİN BİR KORELASYON DENKLEMİ

Özet: Işınımla ısı yayan genişletilmiş yüzeyler geniş kullanım alanına sahiptir. Böyle bir yüzeyin, özellikle büyük sıcaklık farkları söz konusu olduğunda, performansı ısı iletim katsayısının değişken olmasından önemli ölçüde etkilenir. Bu çalışmanın amacı, değişken ısı iletim katsayısına sahip ışınımla ısı kaybeden kanatların sabit kanat hacmi için optimum boyutlarını bulmaktır. Doğrusal olmayan kanat denklemi Adomian yöntemi ile çözülerek kanat içindeki sıcaklık dağılımı elde edilmiştir. Bulunan sıcaklık profili yardımıyla sabit hacme sahip bir kanattan çevreye olan ısı geçişini maksimum yapan kanat geometrisi belirlenmiştir. Belirlenen optimizasyon koşulu kanat parametresi ile ısı iletim katsayısının sıcaklıkla değişimini tanımlayan ısı iletim katsayısı parametresi arasında kurulan bir korelasyon denklemidir. Bu denklem ışınımla ısı yayan dikdörtgen kesitli düz kanatların optimum tasarımı için son derece kullanışlı bir araçtır.
Anahtar Kelimeler: Değişken ısı iletim katsayısı, Kanat, Optimizasyon.

NOMENCLATURE

A	cross-sectional area of the fin, (m^2)
b	fin length, (m)
C	integral constant representing dimensionless temperature at the fin tip
k	thermal conductivity, [W/(mK)]
k_o	thermal conductivity at the outer space temperature, [W/(mK)]
L	the highest order derivative
L^{-1}	inverse operator of L
N	nonlinear operator
Q	heat transfer rate, (W)
T	temperature, (K)
w	semi-thickness of the fin, m
x	axial distance measured from fin tip, m
β	a constant describing the variation of thermal conductivity
ε	fin emissivity
η	fin efficiency
λ	the slope of the thermal conductivity temperature curve, (1/K)
ψ	fin parameter
σ	the Stefan-Boltzmann constant, [W/(m^2K^4)]
θ	dimensionless temperature
ξ	dimensionless axial distance measured from the fin tip

Subscripts and superscripts

f	fin
*	optimum

INTRODUCTION

Extended surfaces are extensively used in various industrial applications. An extensive review on this topic is presented by Kraus et al (2001). Fins are employed to enhance the heat transfer between the primary surface and its convective, radiating or convective-radiating environment. Physical situations that involve only conduction and radiation are fairly common. A review on combined heat conduction and radiation can be seen in the literature (Özışık, 1973). Some examples include heat losses through the walls of a vacuum, heat transfer through super insulation made up of separated layers of highly reflective material and heat losses in satellite and spacecraft structures. The basic mechanism of heat transfer in a space radiator and a fin array is conduction combined with radiation in a nonparticipating medium, and the heat transfer characteristics of simple, one-dimensional, radiating fins have been studied extensively. Bartas and Sellers (1960) studied a heat rejecting system consisting of parallel tubes joined by web plates that served as extended surfaces. Wilkins Jr. (1960) gave expressions for the optimum proportion of triangular fins radiating to space at absolute zero. Chung and Zhang (1991) determined the optimum shape and minimum mass of a thin fin with diffuse reflecting surfaces using a variational calculus approach. Karlekar and Chao (1963) presented an optimization procedure for achieving maximum dissipation from a longitudinal fin system of trapezoidal profile with mutual irradiation. Schnurr (1976) used a nonlinear optimization approach for determining the minimum weight design for radiating fin arrays used in space applications. Hrymark et al. (1985) presented an efficient numerical method to discover the optimal shape for a fin subject to both convective and radiative heat loss. Krishnaprakas (1996) presented the optimum design of a diffusely reflecting rectangular plate fin array extending from a plane wall employing a nonlinear optimization method. The optimum dimensions of trapezoidal profile radiating and convective-radiating circular fins and rectangular radiating fins with radiant interaction between the fin and its base were determined by Razelos and Krikkis (2001). However, all the papers dealing with fins or fin systems assumed a constant thermal conductivity. Fin or fin systems which are employed in spacecraft applications have high temperature differences between fin bases and their tips. Therefore, the variation of the thermal conductivity of fin material with fin temperature should be taken into consideration.

In this work, Adomian decomposition method has been used to evaluate the temperature distribution within radiating rectangular fins with temperature-dependent thermal conductivity. The data obtained the decomposition solution has been correlated for a wide range of fin parameter and the thermal conductivity parameter describing the variation of the thermal conductivity. The result has been presented a simple correlation equation between two affecting parameters which are fin parameter and thermal conductivity parameter. The correlation equation can readily be used for designing of the radiating rectangular fins with variable thermal conductivity.

PROBLEM DESCRIPTION

A rectangular fin of thickness 2w and length b is shown in Figure 1. The dimension in the direction normal to the cross-section is large. Both surfaces of the fin are radiating to the vacuum of outer space at a very low temperature, which is assumed equal to zero absolute. The fin is diffuse-gray with emissivity ε, and has temperature-dependent thermal conductivity k, which depends on temperature linearly. The base temperature T_b of the fin is constant, and the fin tip is insulated. The radiative interaction between the fin and its base is neglected. Since the fin is assumed to be thin, the temperature distribution within the fin does not depend on y-direction.

The energy balance equation for the differential element shown in Figure 1 is given

$$2w \frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] - 2\varepsilon\sigma T^4 = 0 \qquad (1)$$

where k(T) and σ are the thermal conductivity and the Stefan-Boltzmann constant, respectively.

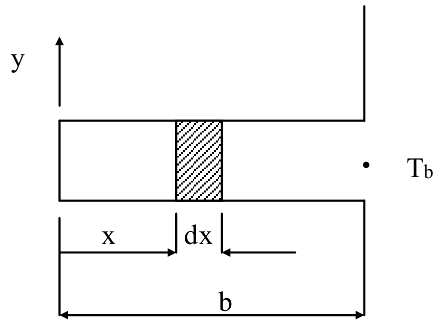

Figure 1. Schematic of a radiating rectangular fin.

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to

$$k(T)=k_o(1+\lambda T) \qquad (2)$$

where k_o is the thermal conductivity at the outer space temperature of the fin, λ the slope of the thermal conductivity-temperature curve.

Employing the following dimensionless parameters

$$\theta = \frac{T}{T_b} \qquad \psi = \frac{\varepsilon\sigma T_b^3 b^2}{k_o w} \qquad \xi = \frac{x}{b} \qquad \beta = \lambda T_b \qquad (3)$$

the formulation of the problem reduces to

$$\frac{d^2\theta}{\partial\xi^2} + \beta\left(\frac{d\theta}{d\xi}\right)^2 + \beta\theta\frac{d^2\theta}{\partial\xi^2} - \Psi\theta^4 = 0 \qquad (4a)$$

with boundary conditions

$$\frac{d\theta}{d\xi}=0 \quad \text{at} \quad \xi=0 \tag{4b}$$

$$\theta=1 \quad \text{at} \quad \xi=1 \tag{4c}$$

THE ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method, proposed by Adomian initially with the aims to solve frontier physical problem, has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions etc. For nonlinear models, the method has shown reliable results in supplying analytical approximation that converges very rapidly (Adomian, 1988; Adomian, 1994).

We consider a general nonlinear equation

$$Lu+Ru+Nu=g, \tag{5}$$

where L is the highest order derivative which is assumed to be easily invertible, R the linear differential operator of less order than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to the both sides of Eq. (5), and using the given conditions we obtain

$$u=f(x)-L^{-1}(Ru)-L^{-1}(Nu), \tag{6}$$

where the function f(x) represents the terms arising from integrating the source term g(x), and from the using given conditions, all of which are assumed to be prescribed.

For nonlinear differential equations, the nonlinear operator Nu=F(u) is represented by an infinite series of the so-called Adomian polynomials

$$F(u)=\sum_{m=0}^{\infty}A_m \tag{7}$$

The polynomials A_m are generated for all kind of nonlinearity so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The standard Adomian method defines the solution u(x) by the series

$$u=\sum_{m=0}^{\infty}u_m \tag{8}$$

F(u) is expanded Taylor series about u_0

$$F(u)=F(u_0)+F'(u_0)(u-u_0)+ \\ F''(u_0)\frac{(u-u_0)^2}{2!}+F'''(u_0)\frac{(u-u_0)^3}{3!}+... \tag{9}$$

and is rewritten Eq. (8) as $u-u_0= u_1+ u_2+ u_3+$……, substituting it into Eq. (9) and then equating two expressions for F(u) found in Eq. (9) and Eq. (7) defines formulas for the Adomian polynomials

$$F(u)=A_1+A_2+...=F(u_0)+F'(u_0)(u_1+u_2+...) \\ +F''(u_0)\frac{(u_1+u_2+...)^2}{2!}+... \tag{10}$$

By equating terms in Eq. (10) the first few Adomian's polynomials $A_0+A_1+A_2\ldots$ are given

$$A_0=F(u_0), \tag{11a}$$

$$A_1=u_1F'(u_0), \tag{11b}$$

$$A_2=u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0), \tag{11c}$$

$$A_3=u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0), \tag{11d}$$

$$\vdots$$

Now that the A_k are known, Eq. (7) can be substituted in Eq. (6) to specify the terms in the expansion for the solution of Eq. (8).

FIN TEMPERATURE DISTRIBUTION

Following Adomian decomposition analysis, the linear operator is defined as:

$$L=\frac{d^2}{d\xi^2}. \tag{12}$$

Consequently, Eq. (4a) can be written as follows:

$$L\theta=-\beta\theta\frac{d^2\theta}{\partial\xi^2}-\beta\left(\frac{d\theta}{d\xi}\right)^2+\Psi\theta^4=-\beta NA-\beta NB+\Psi NC \tag{13}$$

where

$$\theta=\sum_{m=0}^{\infty}\theta_m \tag{14a}$$

$$NA=\theta\frac{d^2\theta}{d\xi^2}=\sum_{m=0}^{\infty}A_m \tag{14b}$$

$$NB=\left(\frac{d\theta}{d\xi}\right)^2=\sum_{m=0}^{\infty}B_m \tag{14c}$$

$$NC=\theta^4=\sum_{m=0}^{\infty}C_m \tag{14d}$$

are nonlinear terms. Hence, using Eq. (11) gives

$$A_0=\theta_0\frac{d^2\theta_0}{d\xi^2} \tag{15a}$$

$$A_1=\theta_1\frac{d^2\theta_0}{d\xi^2}+\theta_0\frac{d^2\theta_1}{d\xi^2} \tag{15b}$$

$$A_2=\theta_2\frac{d^2\theta_0}{d\xi^2}+\theta_1\frac{d^2\theta_1}{d\xi^2}+\theta_0\frac{d^2\theta_2}{d\xi^2} \tag{15c}$$

$$A_3 = \theta_3 \frac{d^2\theta_0}{d\xi^2} + \theta_2 \frac{d^2\theta_1}{d\xi^2} + \theta_1 \frac{d^2\theta_2}{d\xi^2} + \theta_0 \frac{d^2\theta_3}{d\xi^2} \quad (15d)$$

$\vdots$

$$B_0 = \left(\frac{d\theta_0}{d\xi}\right)^2 \quad (16a)$$

$$B_1 = 2\frac{d\theta_0}{d\xi}\frac{d\theta_1}{d\xi} \quad (16b)$$

$$B_2 = \left(\frac{d\theta_1}{d\xi}\right)^2 + 2\frac{d\theta_0}{d\xi}\frac{d\theta_2}{d\xi} \quad (16c)$$

$$B_3 = 2\frac{d\theta_1}{d\xi}\frac{d\theta_2}{d\xi} + 2\frac{d\theta_0}{d\xi}\frac{d\theta_3}{d\xi} \quad (16d)$$

$\vdots$

and

$$C_0 = \theta_0^4 \quad (17a)$$

$$C_1 = 4\theta_0^3\theta_1 \quad (17b)$$

$$C_2 = 6\theta_0^2\theta_1^2 + 4\theta_0^3\theta_2 \quad (17c)$$

$$C_3 = 4\theta_0\theta_1^3 + 12\theta_0^2\theta_1\theta_2 + 4\theta_0^3\theta_3 \quad (17d)$$

$\vdots$

With the boundary condition given in Eq. (4b), $\theta(0)$ is any arbitrary constant, C.

Applying the inverse operator L^{-1} to both sides of Eq. (13) we obtain

$$L^{-1}L\theta = -\beta L^{-1}NA - \beta L^{-1}NB + \Psi L^{-1}NC, \quad (18)$$

and

$$\theta = \theta_0 - \beta L^{-1}NA - \beta L^{-1}NB + \Psi L^{-1}NC. \quad (19)$$

If L is a second-order operator, L^{-1} is a twofold indefinite integral, i.e.

$$L^{-1}L\theta = \theta - \theta(0) - \xi\frac{d\theta(0)}{d\xi}. \quad (20)$$

thus,

$$\theta_0 = \theta(0) + \xi\frac{d\theta(0)}{d\xi}. \quad (21)$$

The next iterates are determined recursively by

$$\theta_{m+1} = -\beta L^{-1}A_m - \beta L^{-1}B_m + \Psi L^{-1}C_m \quad (22)$$

Therefore, the first five iterates are expressed as:

$$\theta_0 = C \quad (23a)$$

$$\theta_1 = \frac{1}{2}C^4\psi\xi^2 \quad (23b)$$

$$\theta_2 = -\frac{1}{2}\beta C^5\Psi\xi^2 + \frac{1}{6}C^7\Psi^2\xi^4 \quad (23c)$$

$$\theta_3 = \frac{1}{2}\beta^2C^6\Psi\xi^2 - \frac{11}{24}\beta C^8\Psi^2\xi^4 + \frac{13}{180}C^{10}\Psi^3\xi^6 \quad (23d)$$

$$\theta_4 = -\frac{1}{2}\beta^3C^7\Psi\xi^2 + \frac{25}{24}\beta^2C^9\Psi^2\xi^4 - \frac{17}{45}\beta C^{11}\Psi^3\xi^6 + \frac{71}{2520}C^{13}\Psi^4\xi^8 \quad (23e)$$

$\vdots$

Summing those iterates the dimensionless temperature distribution is calculated from the Eq. (5). The coefficient C representing the temperature at the fin tip, that must lie in the interval (0, 1), can be evaluated from the boundary condition given in Eq. (4c) using the Newton-Raphson method. Taking the fourteen terms in the series and applying the boundary condition given in Eq. (4c), the values of coefficient C relative to the thermal conductivity parameters, β, and the fin parameter, ψ, are calculated.

The problem is also solved numerically by using MAPLE which uses a finite difference technique (FDT) with Richardson extrapolation (Ascher and Petzold, 1998), and the corresponding results are compared with the Adomian solution. The results of the comparison show that the difference is 2.7 % in a case of the strongest nonlinearity, i.e. β=1.0 and ψ=100.

FIN EFFICIENCY AND OPTIMIZATION

The heat transfer rate from the fin surfaces is found by applying the Stefan-Boltzmann law.

$$Q = \int_0^b 2\varepsilon\sigma T^4 dx \quad (24)$$

Fin efficiency is defined as the energy radiated away by the fin divided by the energy that would be radiated if the entire fin were at the base temperature (Siegel and Howell, 1972).

$$Q_{ideal} = 2b\varepsilon\sigma T_b^4 \quad (25)$$

Employing the dimensionless parameters in Eq. (3), fin efficiency is expressed as

$$\eta = \frac{Q}{Q_{ideal}} = \int_0^1 \theta^4 d\xi \quad (26)$$

Because the resulting complicated fin efficiency expression, i.e. Eq. (26), the fin efficiency was expressed as a function of fin parameter for an attained thermal conductivity parameter as following correlation equation.

$$\eta = \frac{a + c\ln(\Psi) + e\ln(\Psi)^2}{1 + b\ln(\Psi) + d\ln(\Psi)^2} \quad (27)$$

The coefficients in Eq. (27) are given in Arslanturk (2006). Knowing the fin efficiency, we can determine the heat transfer rate from the fin surfaces from the Eq. (25) and (26).

$$Q = 2b\varepsilon\sigma T_b^4 \eta(\psi) \tag{28}$$

It is important to determine the conditions that will yield maximum heat dissipation for given value of thermal conductivity parameter and fixed fin profile, that is, A=2bw=constant.

When the value of thermal conductivity parameter is fixed, the fin efficiency is a function of thermo-geometric fin parameter ψ only from Eq. (27). Then Eq. (28) can be written as

$$Q = 2\varepsilon\sigma T_b^4 \frac{A}{2w} \eta(\psi) \tag{29}$$

since A=2bw, and ψ can be related to w by

$$\psi = \frac{\varepsilon\sigma T_b^3 A^2}{4k_o w^3} \tag{30}$$

In Eq. (30) the fin thickness w is the only variable. To maximize Q we differentiate Eq. (29) with respect to w and equate the resulting expression to zero.

$$\frac{\partial Q}{\partial w} = \varepsilon\sigma T_b^4 A \frac{\partial}{\partial w}\left[\frac{\eta(\psi)}{w}\right] = 0 \tag{31a}$$

or

$$-\frac{1}{w^2}\eta(\psi) + \frac{1}{w}\frac{\partial\eta(\psi)}{\partial\psi}\frac{\partial\psi}{\partial w} = 0 \tag{31b}$$

Differentiation of Eq. (30) with respect to w yields

$$\frac{\partial\psi}{\partial w} = -\frac{3\varepsilon\sigma T_b^3 A^2}{4k_o w^4} = -\frac{3\psi}{w} \tag{32}$$

Substituting $\partial\psi/\partial w$ into Eq. (31b), we obtain

$$\psi_{opt} = -\frac{1}{3}\frac{\eta(\psi)}{(d\eta/d\psi)} \tag{33}$$

where ψ_{opt} is the value of the fin parameter that will give maximum heat transfer rate Q for given value of thermal conductivity parameter β. Substituting the efficiency equation given in Eq. (27) into Eq. (33) it is obtained an algebraic equation for optimum thermo-geometric fin parameter. Solving the algebraic equation by Newton-Raphson method, it is found the optimum value of the fin parameter as a function of thermal conductivity parameter. The results have been reported in terms of a simple correlation equation of compact form using standard statistical techniques.

$$\psi^* = -0.2105\beta^3 + 0.1928\beta^2 + 1.1064\beta + 1.1311 \tag{34}$$

The range of thermal conductivity parameter is taken as $-0.8<\beta<0.8$, in the correlation equation. The correlation coefficient for the regression equation is $R^2=0.9998$.

CONCLUSION

Radiating rectangular fins with temperature-dependent thermal conductivity were optimized in the present work using the Adomian decomposition method. The optimization results were expressed a simple correlation equation obtained by standard statistical techniques. This equation yields the relation between the optimum fin parameter and thermal conductivity parameter describing the variation of the thermal conductivity. The present equation can be employed for all practical engineering purposes.

REFERENCES

Adomian, G., *Solving frontier problems of physics: The decomposition method*, Kluwer, Boston, 1988.

Adomian, G., *Nonlinear stochastic system theory and applications to physics*, Kluwer, Boston, 1994.

Arslanturk, C., Optimum design of space radiators with temperature-dependent thermal conductivity, *Applied Thermal Engineering*, 26, 1149-1157, 2006.

Ascher, U., and Petzold, L., Computer methods for ordinary differential equations and differential-algebraic equations, *SIAM*, Philadelphia, 1998.

Bartas, J.G., and Sellers W.H., Radiation fin effectiveness, *Journal of Heat Transfer* 82, 73-75, 1960.

Cherruault, Y., Convergence of Adomian's method, *Kybernets*,. 18, 31-38,1989.

Cherruault, Y., Saccomandi, G., and Some, B., New results for convergence of Adomian's method applied to integral equations, *Mathematical and Computer Modelling,* 16, 85-93, 1992.

Chung B.T.F., and Zhang B.X., Minimum mass longitudinal fins with radiation interaction at the base, *J. Franklin Institute,* 328, 143-161, 1991.

Hrymak, A., McRae, G., and Westerberg, A., Combined analysis and optimization of extended heat transfer surfaces, *Journal of Heat Transfer* 107, 527-532, 1985.

Karlekar, B.V., and Chao B.T., Mass minimization of radiating trapezoidal fins with negligible base cylinder interaction, *Int. J. Heat and Mass Transfer,* 6, 33-48, 1963.

Kraus, D.A., Aziz, A., and Welty, J., *Extended surface heat transfer*, Wiley and Sons., 2001.

Krishnaprakas, C.K., Optimum design of radiating rectangular plate fin array extending from a plane wall, *Journal of Heat Transfer,* 118, 490–493, 1996.

Özışık, M.N., *Radiative transfer and interactions with conduction and convection,* John Wiley and Sons., New York, 1973.

Razelos, P., Krikkis, R.N., Optimum design of longitudinal rectangular fins with base-to-fin radiant interaction, *Heat Transfer Engineering,* 22, 3-17, 2001.

Répaci, A., Nonlinear dynamical systems: on the accuracy of Adomian's decomposition method, *Applied Math. Letters,* 3, 35-39, 1993.

Schnurr, N.M., Shapiro, A.B., and Townsend, M.A., Optimization of radiating fin arrays with respect to weight, *Journal of Heat Transfer,* 98, 643–648, 1976.

Siegel, R., Howell, J.R., *Thermal radiation heat transfer*, McGraw-Hill, New York, 1972.

Wilkins, J.E.Jr., Minimizing the mass of thin radiating fins, *J. Aero. Space Science,* 27, 145-146, 1960.