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# THE NOTES ON SLANT HELICES ACCORDING TO EQUIFORM FRAME ON SYMPLECTIC SPACE

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**Abstract:** In this paper, first of all, we define basic definitions, some characterizations and theorems of symplectic space we calculated equiform frame in 4-dimensional symplectic space. Then, we obtain Frenet vectors and curvatures of a symplectic curve due to equiform frame. We have dealed with the properties of k-type slant helix according to equiform frame. It is seen that there exist k-type slant helices for all cases. In addition, we express some characterizations for k-type slant helix according to equiform frame geometry in symplectic regular curves. Finally, we give an example about symplectic space on graphics with symplectic frame on 4-dimensional symplectic space.

**Keywords:** Frenet vectors, Slant helix, Equiform frame

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## 1. Introduction

In recent years, by the coming theory of the curves, researches extend some special curves. Some of them are helices, slant helices, Bertrand curves, associated curves, adjoint curve etc. There are many studies on these special curves. Especially a general helices and slant helices are used for many applications. A general helices is defined that its tangent vector fields makes a constant angle with a fixed direction called the axis of general helices. The notion of slant helices defined by Izumiya and Takeuchi (2004) based on the property that the principal normal lines of the curve make a constant angle with a fixed direction. Moreover, Kula and Yaylı (2005) obtained slant helices and their spherical indicatries. Later, slant helices subject are also presented in 3-,4-, and n- dimensional space, respectively in Ali and Turgut, (2010), Ali et al. (2012), Çiçek Çetin and Bektaş (2020a and 2020b) and Yılmaz and Bektaş (2020) different dimensions. Afterwards, another property of helices are discussed in Kula and Yaylı (2005), Ali and Lopez, (2011) and Önder et al. (2008). Ferrandez et al. (2002) also researchers studied k-type slant helices in $E_1^4$. In particular, authors defined (k,m)-type slant helices in $E^4$, and discussed them for partially and pseudo null curves in $E_1^4$ in Yılmaz and Bektaş, (2018 and 2020). Çetin and Bektaş (2019) introduced some characteizations of symplectic space. Also, Bulut (2021a, 2021b and 2023), Bulut and Eker (2023), Bulut and Tartık (2021) and Bulut and Bektaş, (2020) authors deal with k -and (k,m)-type slant helices in different spacetime. Furthermore, there are many studies about symplectic curves in 4-dimensional symplectic space and equiform differential geometry (Struik, 1988; Abdel Aziz, 2015).

The word symplectic was first used by Weyl in the sense of complex to describe symplectic groups. Some scientist accepts symplectic geometry as the language of classical mechanics. In fact, the basis of Hamilton and Kahler manifolds, which play an important role in mathematics and theoretical physics, is based on symplectic geometry. The origin of Hamilton mechanics is symplectic geometry, and the base-spaces of classical systems play an important role in the structure of symplectic manifold. Symplectic geometry is at basis of optics. On the other hand, symplectic geometry also has important connections with dynamic systems, integrable systems, algebraic geometry and global analysis. Symplectic geometry is studied by geometers (Chern and Wang, 1947; Kamran and Olver, 2009). Symplectic spaces were first studied by Chern and Wang (1947) as local symplectic invariants of Euclidean subspaces. Symplectic space differs from Euclidean space in terms of metric and arc length. Kamran and Olver (2009) obtained the Frenet frame of curves using local symplectic invariants. Authors, introduced the concept of symplectic arc length for curves. And they constructed an adapted symplectic Frenet frame and expressed 2n - 1 local differential invariants that they called symplectic curvatures of the curve

According to our opinion, slant helices should be researched equiform frame in symplectic space. Because we know that symplectic curves play an important role in modern geometry. Also symlectic space are studied by some geometers. They proved that up to a rigid symplectic motion of $R^{2n}$, there exists a unique curve



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with symplectic curvatures.

In this paper, we focus on k-type and (k,m)- type slant helix according to equiform frame and we also consider some characterizations for (k,m)- type slant helix of symplectic regular curves.

## 2. Materials and Methods

Let us introduced some definition for symplectic space.

For, any vectors, $u=(x^1,x^2,\dots,x^n,y^1,\dots,y^n), v=(\xi^1,\dots,\xi^n,\eta^1,\dots,\eta^n)\in R^{2n}$

Symplectic inner product is given by

$$<u,v>=\varphi(u,v)=\sum_{i=1}^{n}(x_i\eta_i\wedge y_i\xi_i)$$

4-dimensional symplectic space $Sim=(R^4,\varphi)$ is the vector space $R^4$ equipped with the standard symplectic form, written as

$$\varphi=\sum dx_i\wedge dy_i$$

(Kamran and Olver, 2009).

Let $V$ be a vector space on the field of real numbers $R$. If, for each $u,v$

$$\varphi(u,v)=-\varphi(v,u)$$

It is called anti symmetric bilinear transformation.

Symplectic space with symplectic inner product can be written as

$$<u,v>=\varphi(u,v)=\textstyle\sum_{i=1}^{2}(x_i\eta_i-y_i\xi_i)$$
$$=x_1\eta_1+x_2\eta_2-y_1\xi_1-y_2\xi_2$$

here $u=\{x_1,x_2,y_1,y_2\}$ and $v=\{\xi_1,\xi_2,\eta_1,\eta_2\}$

Additionally, the tangent vectors $\{a_1,a_2,a_3,a_4\}$ satisfying the equations

$$\langle a_k,a_l\rangle=\langle a_{2+k},a_{2+l}\rangle=0 \ \ 1\le k,l\le 2,$$
$$\langle a_k,a_{2+l}\rangle=0 \ \ 1\le k\ne l\le 2,$$
$$\langle a_k,a_{2+l}\rangle=1 \ \ 1\le k\le 2$$

for symplectic frame, structure equations are defined by

$$1\le i,j\le n$$
$$da_i=\sum_{k=1}^{n}w_{ik}a_k+\sum_{k=1}^{n}\varphi_{ik}\,a_{k+n}$$
$$da_{i+n}=\sum_{k=1}^{n}\theta_{ik}a_k+\sum_{k=1}^{n}w_{ik}\,a_{k+n}$$

here, $\varphi_{ij}=\varphi_{ji},\theta_{ij}=\theta_{ji}$ (Kamran and Olver, 2009).

Let $z(t):R\to R^4$ be a symplectic regular curve parametrized by symplectic arc length with symplectic frame. Throughout this paper, we make some notations and calculations for later use. We show that $z$ to indicate differentiation with respect to the parameter $t$:

$$\dot{z}=\frac{dz}{dt}$$

**Definition 2.1.** Let $z(t)$ be a symplectic regular curve in $Sim=(R^4,\varphi)$. Then the following non-degeneracy condition is satisfied

$$\langle\dot{z},\ddot{z}\rangle\ne 0$$

for all $t\in R$.

**Definiton 2.2.** Let $z(t)$ be a symplectic regular curve, with symplectic arc length s. Then z(s) can be written as

$$s(t)=\int_{t_0}^{t}\langle\dot{z},\ddot{z}\rangle^{1/3}\,dt$$

for all $t\ge t_0$.

If

$$\int_{t_1}^{t_2}\langle\dot{z},\ddot{z}\rangle^{1/3}\,dt=t_2-t_1 \ \ (t_1,t_2\in I)\ t_1\le t_2$$

symplectic regular curve is said to parametrized by symplectic arc length. The symplectic arc length parameter corresponds to the equaffine arc length for plane curves.

Taking the extrerior differential of above equation, the symplectic arc length element is obtained by

$$ds=\langle\dot{z},\ddot{z}\rangle^{1/3}dt$$

and the arc length derivative operator is

$$D=\frac{d}{ds}=\langle\dot{z},\ddot{z}\rangle^{-1/3}\frac{d}{dt}$$

In our notion, the symplectic arc length derivative operator is defined by

$$z'=\frac{dz}{ds}$$

**Definiton 2.3.** A symplectic regular curve is parametrized by symplectic arc length if it satisfies

$$\langle\dot{z},\ddot{z}\rangle=1$$

for all $t\in R$.

Let $z(s)$ be a symplectic regular curve parametrized by arc length with $\{a_1,a_2,a_3,a_4\}$ a symplectic frame. Then, Frenet equations can be written (equation 1)

$$\begin{aligned}{a_1}'(s)&=a_3(s)\\ a_2'(s)&=H_2(s)a_4(s)\\ a_3'(s)&=k_1(s)a_1(s)\\ a_4'(s)&=a_1(s)+k_2(s)a_2(s)\end{aligned}\qquad(1)$$

where $H_2(s)=cost(\ne 0)$ (Valiquette, 2012).

## 3. k−Type Slant Helices According to Symplectic Equiform Frame

**Definiton 3.1.** Let $z:I\to R^4$ be a symplectic regular curve according to equiform frame $\{V_1,V_2,V_3,V_4\}$. We say that if there exists a (non-zero) constant vector field $U\in R^4$ such that

$$\langle V_{k+1},U\rangle=c=\ const,\ 0\le k\le 3$$

z is a $k$- type slant helix.

**Theorem 3.2.** Let $z:I\to R^4$ be a symplectic regular curve according to symplectic equiform frame $\{V_1,V_2,V_3,V_4\}$. Then z is 0 −type slant helix iff (equation 2)

$$\langle V_2, U\rangle = \frac{cK_1{}^2 - c\rho^2}{\rho^2} \tag{2}$$

**Proof .** Suppose that z is a 0 −type slant helix and let U be the constant vector field, we get (equation 3)

$$\langle V_1, U\rangle = c. \tag{3}$$

where $c$ is constant.
differentiating the relation (2) and using Frenet equation (1), we find,

$$K_1\langle V_1, U\rangle + \rho\langle V_3, U\rangle = 0$$

Or

$$K_1 c + \rho\langle V_3, U\rangle = 0$$

and

$$\langle V_3, U\rangle = -\frac{cK_1}{\rho}$$

Differentiating above equation and using Frenet equation (2), we hence find,

$$K_1\langle V_3, U\rangle + \langle V_1, U\rangle + \rho\langle V_2, U\rangle = \frac{cK_1'\rho + cK_1\rho'}{\rho^2}$$

So, we obtain,

$$\langle V_2, U\rangle = \frac{cK_1{}^2 - c\rho^2}{\rho^2}$$

Hence the theorem is proven.
**Theorem 3.3.** Let $z\colon I \to R^4$ be a symplectic regular curve according to equiform frame $\{V_1, V_2, V_3, V_4\}$. Then z is 1−type slant helix iff

$$\langle V_1, U\rangle = \frac{-cK_1'K_2 + cK_1K_2' + cK_1 - cK_3K_2{}^2}{K_2{}^2}$$

**Proof.** Suppose that z is a 1 −type slant helix and let U be the constant vector field, we have equation 4:

$$\langle V_2, U\rangle = c \tag{4}$$

By taking derivative on both sides of (4), and using Frenet equation (1), we find,

$$K_1\langle V_2, U\rangle + K_2\langle V_4, U\rangle = 0$$

and .

$$\langle V_4, U\rangle = -\frac{K_1\, c}{K_2}$$

Similarly, differentiating above and using Frenet equations (1), we find,

$$\rho\langle V_1, U\rangle + K_3\langle V_2, U\rangle + K_1\langle V_4, U\rangle = \frac{-cK_1{}'K_2 + cK_1K_2{}'}{K_2{}^2}$$

or

$$\langle V_1, U\rangle = \frac{-cK_1'\,K_2 + cK_1K_2' + cK_1 - cK_3K_2{}^2}{K_2{}^2}$$

The theorem is proven.
**Theorem 3.4.** Let $z\colon I \to R^4$ be a symplectic regular curve respect to symplectic equiform frame $\{V_1, V_2, V_3, V_4\}$. Then z(s) is 2 −type slant helix iff

$$\langle V_4, U\rangle = -\frac{cK_1{}^2 - c\rho}{\rho K_2}$$

**Proof.** Suppose that z(s) is a 2 −type slant helix and let U be the constant vector field, we have equation 5:

$$\langle V_3, U\rangle = c \tag{5}$$

differentiating equation (5) and using Frenet equation (1), we obtain that,

$$K_1\langle V_3, U\rangle + \langle V_1, U\rangle + \rho\langle V_2, U\rangle = 0$$

or

$$\langle V_1, U\rangle + \rho\langle V_2, U\rangle = -cK_1$$
$$\langle V_3, U\rangle = -\frac{K_1\, c_1}{\rho}$$

Similarly, differentiating above equation and using Frenet equation (1), we obtain,

$$K_1\langle V_1, U\rangle + \rho\langle V_3, U\rangle + \rho K_1\langle V_2, U\rangle + \rho K_2\langle V_4, U\rangle = 0$$

or,

$$K_1(\langle V_1, U\rangle + \rho\langle V_2, U\rangle) = -cK_1{}^2$$

Thus, we can easily seen that,

$$\langle V_4, U\rangle = -\frac{cK_1{}^2 - c\rho}{\rho K_2}$$

So, the theorem is proven.
**Theorem 3.1.5.** Let $z\colon I \to R^4$ be a symplectic regular curve according to symplectic equiform frame $\{V_1, V_2, V_3, V_4\}$. Then z is 3 −type slant helix iff

$$\langle V_2, U\rangle = \frac{-cK_1' - cK_3K_2 + cK_1{}^2 - \rho^2\langle V_3, U\rangle}{-cK_1{}^2 - K_3K_1 + \rho K_3{}'}$$
$$\langle V_1, U\rangle = -\frac{K_3c_1 + K_1c_4}{\rho}$$

**Proof.** Assume that z is a 3 −type slant helix and let U be the constant vector field, we have equation 6:

$$\langle V_4, U\rangle = c \tag{6}$$

Differentiating equation (6) and using Frenet equations (1), we find,

$$\rho\langle V_1, U\rangle + K_3\langle V_2, U\rangle + K_1\langle V_4, U\rangle = 0$$

Similarly, differentiating above equation and using Frenet equations (1), we get,

$$\rho\langle V_1, U\rangle + K_3\langle V_2, U\rangle = -cK_1$$

Differentiating again, we obtain

$$\rho'\,\langle V_1, U\rangle + \rho^2\langle V_3, U\rangle + K_3'\,\langle V_2, U\rangle = -cK_1' - cK_3K_2 + cK_1{}^2$$

Thus, we get,

$$\langle V_2, U\rangle = \frac{-cK_1' - cK_3K_2 + cK_1{}^2 - \rho^2\langle V_3, U\rangle}{-cK_1{}^2 - K_3K_1 + \rho K_3{}'}$$

Thus, the theorem is proven.
**Example 3.8**. Let’s take the following curve

$$z(s) = \frac{1}{\sqrt{5}}\left(shs, \frac{1}{2}s^2 + 2s, chs, \frac{1}{2}s^2 - 2s\right).$$

Figure 1. Then,

$$a_1(s) = z'(s) = \frac{1}{\sqrt{5}}(chs, s+2, shs, s-2)$$

and

$$a_3(s) = z''(s) = \frac{1}{\sqrt{5}}(shs, 1, chs, 1).$$

Thus $z(s)$ is a symplectic regular curve. In addition

$$a_2(s) = \frac{1}{\sqrt{5}}(\frac{4}{5}chs, -\frac{1}{5}(s+2), \frac{4}{5}shs, -\frac{1}{5}(s-2),$$

$$a_4(s) = \frac{5}{4\sqrt{5}}(4shs, -1, 4chs, -1).$$

Since $\rho = 1$, we obtain Frenet vectors respect to equiform frame ,

$$a_1(s) = V_1(s) = \frac{1}{\sqrt{5}}(chs, s+2, shs, s-2)$$

$$a_2(s) = V_2(s) = \frac{1}{\sqrt{5}}(\frac{4}{5}chs, -\frac{1}{5}(s+2), \frac{4}{5}shs, -\frac{1}{5}(s-2)$$

$$a_3(s) = V_3(s) = \frac{1}{\sqrt{5}}(shs, 1, chs, 1)$$

$$a_4(s) = V_3(s) = \frac{5}{4\sqrt{5}}(4shs, -1, 4chs, -1).$$

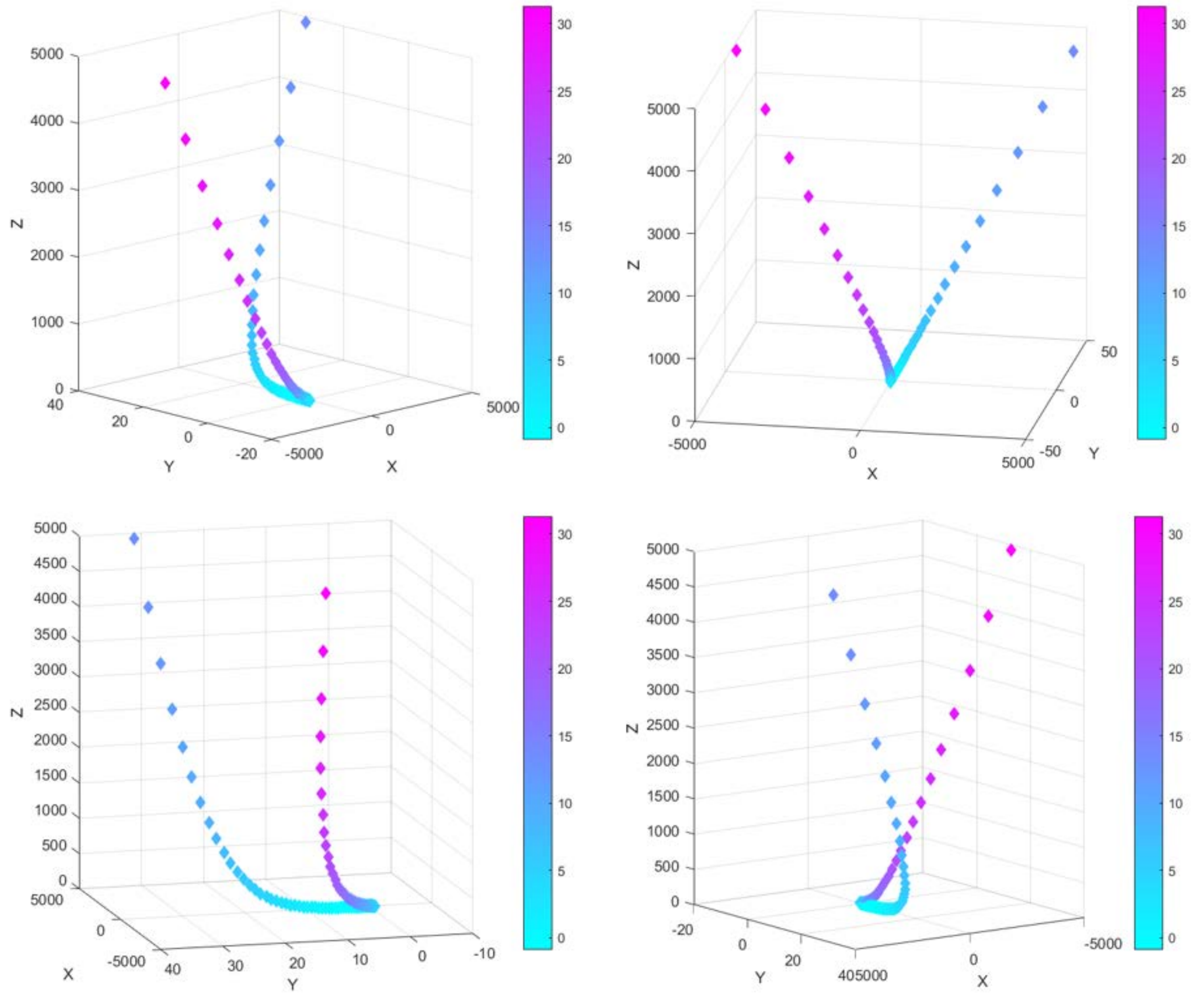


**Figure 1.** Symplectic regular curve $z(s)$.

This curve in $R^4$ is plotted with a code that represents the fourth dimension with a color scale.

## 4. Conclusion

In this study, we especially examined slant helices obtained by symplectic equiform frame. Moreover, we defined basic definition and theorem with symplectic equiform frame. It is seen that there exist k-type slant helices for all cases. Since helices are natural twist structures, examining the symplectic spiral can help to research on connection of contact and symplectic geometries.

**Author Contributions**

The percentage of the author contributions is presented below. The author reviewed and approved the final version of the manuscript.

| | E.Ç.Ç. |
|---|---|
| C | 100 |
| D | 100 |
| S | 100 |
| DCP | 100 |
| DAI | 100 |
| L | 100 |
| W | 100 |
| CR | 100 |
| SR | 100 |
| PM | 100 |
| FA | 100 |

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

**Conflict of Interest**

The author declared that there is no conflict of interest.

**Ethical Consideration**

Ethics committee approval was not required for this study because of there was no study on animals or humans.

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