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ANALYSIS OF THERMAL PERFORMANCE OF ANNULAR FINS WITH VARIABLE THERMAL CONDUCTIVITY BY HOMOTOPY ANALYSIS METHOD

Cihat ARSLANTÜRK
Department of Mechanical Engineering, Faculty of Engineering, Ataturk University, 25240 Erzurum, Turkey
e-mail : carslan@atauni.edu.tr

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Abstract: The homotopy analysis method (HAM) is used to analyze the thermal performance of annular fins with temperature-dependent thermal conductivity. Since the HAM algorithm contains a parameter that controls the convergence and accuracy of the solution, its results can be verified internally by calculating the residual error. The HAM solution appears in terms of algebraic expressions which are not only easy to compute but also give highly accurate results covering a wide range of values of the parameters rather than the small values dictated by the perturbation solution. In this work, the fin efficiency of nonlinear annular fins is obtained as a function of thermo-geometric fin parameter, thermal conductivity parameter and radii ratio. The data from the present solutions is correlated for suitable ranges of problem parameters. The resulting correlation equations can assist thermal design engineers for designing of annular fins with temperature-dependent thermal conductivity.
Keywords: Annular fin, Efficiency, Homotopy analysis method.

ISIL İLETKENLİĞİ SICAKLIKLA DEĞİŞEN DAİRESEL KANATLARIN PERFORMANSININ HOMOTOPİ ANALİZ YÖNTEMİ İLE İNCELENMESİ

Özet: Bu çalışmada, ısıl iletkenliği sıcaklıkla değişen dairesel kanatların ısıl performansını incelemek için homotopi analiz yöntemi kullanılmıştır. Yöntemin algoritması, çözümün doğruluğu ve yakınsaklığını kontrol eden bir parametre içerdiğinden, sonuçlar hatanın çözüm anında hesaplanmasıyla doğrulanabilir. HAM, pertürbasyon yöntemlerinin aksine çok geniş problem parametreleri aralıkları için yakınsak çözümler vermektedir. Bu çalışmada dairesel kanatların verimleri üç problem parametresi cinsinden elde edilmiştir. Çözümden elde edilen veri yardımıyla kanat verimi için korelasyon denklemleri üretilmiştir. Korelasyon denklemleri ısı iletim katsayıları sıcaklıkla değişen dairesel kanatların tasarımı için kullanılabilir.
Anahtar Kelimeler: Dairesel kanat, Homotopi analiz yöntemi, Verim.

NOMENCLATURE

A	cross-sectional area of the fin, (m^2)
Bi	Biot number, ($=hr_i/k_\infty$)
h	heat transfer coefficient, [$W/(m^2K)$]
$\hbar$	auxiliary parameter
H	auxiliary function
k	thermal conductivity, [$W/(mK)$]
b	fin length, (m)
L	auxiliary linear operator
N	nonlinear operator
p	embedding parameter
Q	heat transfer rate, (W)
r	radial coordinate, (m)
t	fin thickness, (m)
T	temperature, (K)
Greek symbols	
β	a constant describing the variation of thermal conductivity, [$=\kappa(T_b-T_\infty)$]
η	fin efficiency
κ	the slope of the thermal conductivity-temperature curve, (1/K)
ξ	dimensionless coordinate, [$=(r-r_i)/r_i$]
λ	radii ratio, ($=r_o/r_i$)
ψ	thermo-geometric fin parameter, ($=2hr_i^2/tk_\infty$)
u	a dependent variable
θ	dimensionless temperature, [$=(T-T_\infty)/(T_b-T_\infty)$]
θ_m	*m*th order approximation
τ	an independent variable
χ	two-valued function
Subscripts	
b	base
i	inner
o	outer
∞	ambient fluid

INTRODUCTION

Extended surfaces are ubiquitous in engineering applications where there is a need to enhance heat transfer between a hot surface and an adjoining coolant. Applications in which fins of longitudinal and radial configurations and spines are employed range from small electronic components to large power and process heat exchangers. A considerable amount of research has been conducted into the variable thermal parameters which are associated with fins in operating in practical situations. An extensive review on this topic is presented by Aziz (1992) and Kraus et al. (2001). Aziz and Hug (1975) used the regular perturbation method and a numerical solution method to compute a closed form solution for a straight convective fin with temperature dependent thermal conductivity. Razelos and Imre (1983) considered the variation of the convective heat transfer coefficient from the base of a convecting fin to its tip. Yu and Chen (1999) assumed that the linear variation of the thermal conductivity and then, solved the nonlinear conducting-convecting-radiating heat transfer equation by the differential transformation method. Convective straight fins with temperature dependent thermal conductivity have been studied by some researchers using different methods (Arslanturk, 2005; Coskun and Atay, 2008; Inc, 2008; Domairry and Fazeli, 2009; Joneidi et al., 2009). Arslanturk (2005) gave correlation equations for efficiency of straight fins with variable thermal conductivity by using Adomian decomposition method. In another study performed by Arslanturk (2009), correlation equations were given for optimal design of annular fins with temperature dependent thermal conductivity by using Adomian decomposition method. Coskun and Atay (2008) used variational iteration method and finite element method for analyzing the efficiency of straight fins. Homotopy analysis method is employed to evaluate the efficiency of convective straight fins (Inc, 2008; Domairry and Fazeli, 2009). The same problem has been solved by differential transformation method by Joneidi et al. (2009). Differential quadrature optimization of convective-radiative fins based on two-dimensional heat transfer analysis was presented by Malekzadeh et al. (2007).

The basic idea of homotopy in topology provided an idea to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) which is used for solving nonlinear fin problem in the present work, the method was proposed by Liao in 1992 (Liao, 1992; Liao, 2003; Liao, 2004). The method is now widely used to solve many types of nonlinear problems (Abbasbandy, 2008; Khani et al., 2009a; Khani et al., 2009b; Sajid et al., 2009). Since the HAM algorithm contains a parameter that controls the convergence and accuracy of the solution, its results can be verified internally by calculating the residual error.

In the present paper, the energy balance for a differential fin element is developed. The resulting nonlinear differential equation is solved by HAM to evaluate the temperature distribution within the fin. Using the temperature distribution, the efficiency of the fins is expressed through a term called thermo-geometric fin parameter, Ψ, thermal conductivity parameter, β, describing the variation of the thermal conductivity, and radii ratio λ. Since the resulting analytical expression for the fin efficiency is complicated, the data from the expression has been correlated for a wide range of problem parameters. The correlation equations of compact form are useful for designing of the annular fins with variable thermal conductivity.

PROBLEM DESCRIPTION

An annular fin with temperature-dependent thermal conductivity as shown in Fig. 1 is considered in this work. The fin of thickness t, base radius r_i, and tip radius r_o is exposed to a convective environment at the constant ambient temperature T_∞ and heat transfer coefficient h. The base temperature T_b of the fin is constant, and the fin tip is insulated. Since the fin is assumed to be thin, the temperature distribution within the fin does not depend on axial direction.

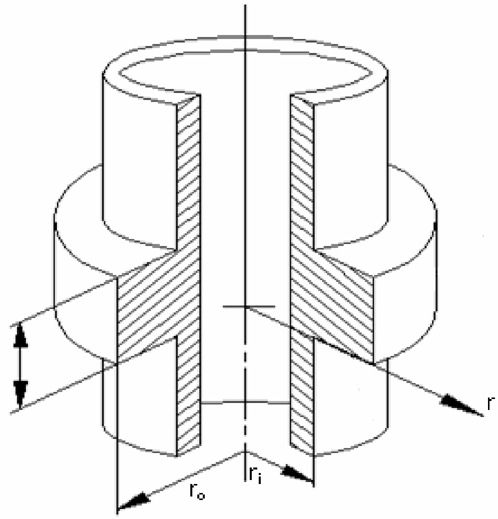

Figure 1. Schematic of an annular fin

The energy balance equation is given

$$t\frac{d}{dr}\left[k(T)r\frac{dT}{dr}\right] = 2hr(T - T_\infty) \qquad (1)$$

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to

$$k(T) = k_\infty[1 + \kappa(T - T_\infty)] \qquad (2)$$

where k_∞ is the thermal conductivity at the ambient fluid temperature of the fin, κ is the parameter describing the variation of thermal conductivity.

In order to simplify the parameter studies, the following non-dimensional variables are defined:

$$\theta = \frac{T - T_\infty}{T_b - T_\infty} \quad \psi^2 = \frac{2hr_i^2}{k_\infty t} \quad \beta = \kappa(T_b - T_\infty)$$
$$Bi = \frac{hr_i}{k_\infty} \quad \xi = \frac{r - r_i}{r_i} \quad \lambda = \frac{r_o}{r_i} \quad \delta = \frac{t}{r_i} \tag{3}$$

By using the aforementioned non-dimensional variables, the governing equation and its associated boundary conditions become as follows:

$$\frac{d^2\theta}{\partial\xi^2} + \beta\left(\frac{d\theta}{d\xi}\right)^2 + \beta\theta\frac{d^2\theta}{\partial\xi^2} + \frac{\beta}{(1+\xi)}\theta\frac{d\theta}{\partial\xi} + \frac{1}{(1+\xi)}\frac{d\theta}{d\xi} - \psi^2\theta = 0, \quad 0 < \xi < \lambda - 1 \tag{4a}$$

$$\theta = 1 \quad \text{at} \quad \xi = 0 \tag{4b}$$

$$\frac{d\theta}{d\xi} = 0 \quad \text{at} \quad \xi = \lambda - 1 \tag{4c}$$

THE HOMOTOPY ANALYSIS METHOD

In 1992, Liao (1992) employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method. A systematically description of the basic ideas of the homotopy analysis method, can be found in the literature (Liao, 2003).

Consider the following differential equation:

$$N[u(\tau)] = 0 \tag{5}$$

Where N is a nonlinear operator, τ denotes an independent variable, and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao (2003) constructed the so-called zero-order deformation equation as

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = p\hbar H(\tau)N[\phi(\tau;p)] \tag{6}$$

Where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ a nonzero auxiliary parameter, $H(\tau) \neq 0$ an auxiliary function, L an auxiliary linear operator, $u_0(\tau)$ an initial guess of $u(\tau)$ and $\phi(\tau;p)$ is an unknown function. It is important to have enough freedom of choose auxiliary unknowns in HAM. Obviously, when p=0 and p=1, it holds

$$\phi(\tau;0) = u_0(\tau) \text{ and } \phi(\tau;1) = u(\tau) \tag{7}$$

Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess, $u_0(\tau)$, to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m \tag{8}$$

where,

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\bigg|_{p=0} \tag{9}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are quite properly chosen, the series Eq. (8) converges at p=1. Then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) \tag{10}$$

According to Eq. (8), the governing equation can be deduced from the zero-order deformation Eq. (5). The vector is defined as

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\} \tag{11}$$

Differentiating Eq (1) m times with respect to the embedding parameter p, and then setting p=0 and finally dividing them by $m!$, we will have the so-called *m*th-order deformation equation as

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}) \tag{12}$$

where,

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}\bigg|_{p=0} \tag{13}$$

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \tag{14}$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear equation (12) with the linear boundary conditions coming from the original problem, which can be easily solved using a symbolic computation software such as Maple or Mathematica.

THE FIN TEMPERATURE DISTRIBUTION

Consider Eqs. (4a)-(4c) and let us solve them through HAM. Following the Homotopy analysis method, the linear operator is defined as

$$L = \frac{d^2\theta}{d\xi^2} \tag{15}$$

and nonlinear operator is defined as

$$N = \frac{d^2\theta}{\partial\xi^2} + \beta\left(\frac{d\theta}{d\xi}\right)^2 + \beta\theta\frac{d^2\theta}{\partial\xi^2} + \frac{\beta}{(1+\xi)}\theta\frac{d\theta}{\partial\xi} + \frac{1}{(1+\xi)}\frac{d\theta}{d\xi} - \psi^2\theta \tag{16}$$

According to the governing equation and the initial condition (4a)-(4c), the solution can be expressed by a set of base functions

$$\left\{ \xi^{2n} \mid n = 1, 2, 3, \ldots \right\} \tag{17}$$

In the form

$$\theta(\xi) = \sum_{n=1}^{\infty} d_{2n} \xi^{2n} \tag{18}$$

where d_{2n} is a coefficient to be determined. And

$$L[c_1 + c_2 \xi] = 0$$

where c_1 and c_2 are constants that are obtained from boundary conditions when integrated from Eq. (21) according to the linear operator.

To obey both the rule of solution expression and the rule of the coefficient ergodicity, the corresponding auxiliary function can be determined uniquely $H(\tau) = 1$.
Then

$$(1-p)L[\phi(\xi;p) - u_0(\xi)] = p\hbar N[\phi(\xi;p)]$$

According to Eqs. (4a)-(4c) and the rule of solution expression Eq. (15) , it is straightforward that the initial approximation should be in the form

$$\theta_0 = 1 \tag{19}$$

From Eqs. (13) and (16), we have

$$\begin{aligned} R_m(\vec{\theta}_{m-1}) &= \theta''_{m-1}(\xi) + \frac{1}{(1+\xi)}\theta'_{m-1}(\xi) - \psi^2 \theta_{m-1}(\xi) \\ &+ \beta \sum_{i=0}^{m-1} \left[\theta_i(\xi)\theta''_{m-1-i}(\xi) + \theta'_i(\xi)\theta'_{m-1-i}(\xi) + \frac{1}{(1+\xi)}\theta_i(\xi)\theta'_{m-1-i}(\xi) \right] \end{aligned} \tag{20}$$

Then

$$L[\theta_m(\xi) - \chi_m \theta_{m-1}(\xi)] = \hbar R_m(\vec{\theta}_{m-1})$$
$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \tag{21}$$

Following the homotopy analysis method

$$\theta_1(\xi) = \hbar\psi^2(\lambda - 1)\xi - \frac{1}{2}\hbar\psi^2\xi^2 \tag{22}$$

$$\begin{aligned} \theta_2(\xi) &= \hbar^2\psi^2\lambda(1+\beta)\ln(1+\xi) + \frac{1}{3}\hbar\psi^2 \\ &\left\{ 3\hbar\lambda\left[\beta + 1 + \beta(1 + \ln(1+\xi)) - \psi^2(\lambda - 1) - (1+\beta)\ln(\lambda)\right] \right. \\ &\left. + \hbar\psi^2\lambda^3 - \hbar\psi^2 - 6\beta\hbar - 6\hbar + 3\lambda - 3 \right\}\xi - \frac{1}{2}\hbar\psi^2 \\ &[2\hbar(1+\beta) + 1]\xi^2 - \frac{1}{6}\hbar^2\psi^4(\lambda - 1)\xi^3 + \frac{1}{24}\hbar^2\psi^4\xi^4 \\ &\vdots \end{aligned} \tag{23}$$

In this way, we derive $\theta_m(\xi)$ for m=1, 2, 3,.., successively. At the *J*th-order approximation, we have the analytic solution of Eq. (4a), namely

$$\theta(\xi) = \sum_{m=0}^{J} \theta_m(\xi) \tag{24}$$

RESULTS AND DISCUSSION

We note that the explicit, analytic expression in Eq. (24) is the series solution of the problem. One can find the convergence region and rate of approximation by choosing the proper values of the auxiliary parameter ħ. To see this, the ħ-curves are plotted for different radii ratios, and thermal conductivity parameter in Fig. 2.

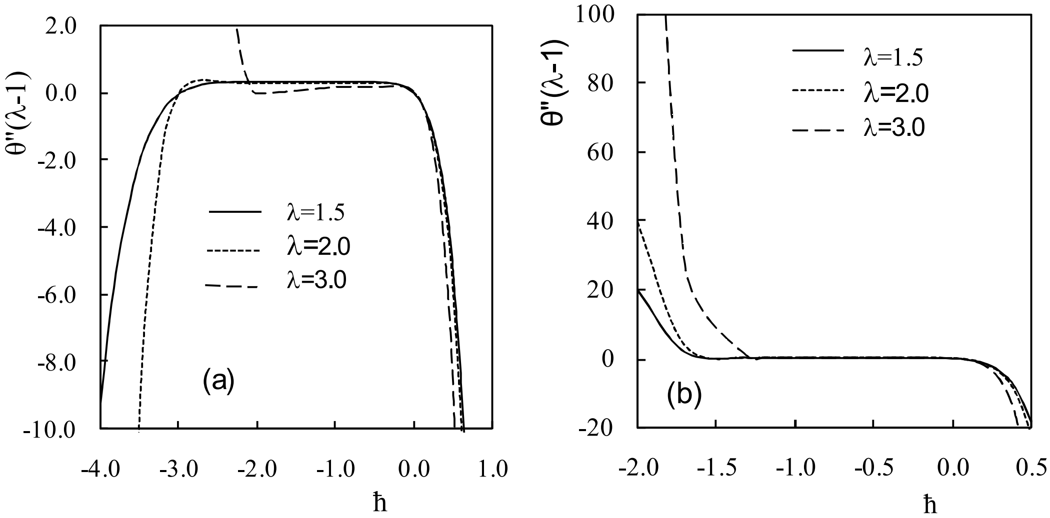

Figure 2. The ħ-curves for $\psi = 0.5$, for 10th-order approximation of $\theta''(\lambda - 1)$ (a) β = -0.3 (b) β = 0.3.

Fig. 3 shows the residual error for *J*th-order approximation as follows and clearly indicates that the HAM gives rapid convergence. From the Fig. 3, it can be seen that if an appropriate value for is taken, the residual error converge to zero.

$$\text{Residual Error} = (1+\beta\theta_J'')\theta_J'' + \beta(\theta_J')^2 + \frac{\beta}{(1+\xi)}\theta_J\theta_J'' + \frac{1}{(1+\xi)}\theta_J' - \psi^2\theta_J \tag{25}$$

In order to investigate the accuracy of the HAM solution with a finite number of terms, the problem is also solved numerically by using BVPFD subroutine in the IMSL library which solves a system of differential equations with boundary conditions at two points, using a variable order, variable step size finite difference method with deferred corrections and the corresponding results are compared with HAM solution. For the different values of the thermal conductivity parameters, the results of the present analysis are tabulated against the numerical solution in Table 1. A very good agreement between the results was observed, which confirms the validity of the HAM. The results of the comparison show that the difference is 0.05 % in the case of the strongest nonlinearity, i.e. β = - 0.3. Note we present HAM by 10th-order approximation of solution.

Fin efficiency is calculated from following expression:

$$\eta = \frac{Q}{Q_{ideal}} = \frac{-2\pi t\left[k(T) r \frac{dT}{dr}\right]_{r=r_i}}{2\pi h(r_o^2 - r_i^2)(T_b - T_\infty)} \tag{26}$$

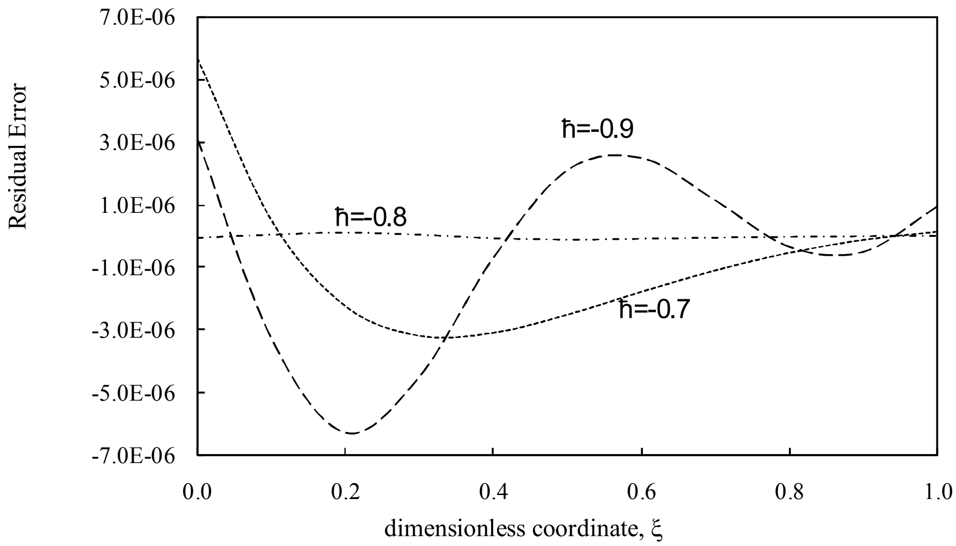

Figure 3. The residual error for Eq. (4a) when $\psi = 0.5$, β = 0.3 and λ= 2.0.

Table 1. Comparison of the FDM and HAM results (λ=2.0, ψ=1.0)

	β = -0.3		β = 0.0			β = 0.3	
ξ	FDM	HAM	FDM	HAM	Exact	FDM	HAM
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	0.88020	0.88020	0.90581	0.90582	0.90582	0.92201	0.92202
0.2	0.78675	0.78673	0.82851	0.82851	0.82851	0.85658	0.85658
0.3	0.71332	0.71336	0.76535	0.76536	0.76536	0.80211	0.80212
0.4	0.65569	0.65582	0.71427	0.71428	0.71428	0.75734	0.75735
0.5	0.61088	0.61108	0.67361	0.67362	0.67362	0.72124	0.72124
0.6	0.57677	0.57701	0.64210	0.64212	0.64212	0.69296	0.69296
0.7	0.55181	0.55205	0.61874	0.61876	0.61876	0.67181	0.67182
0.8	0.53487	0.53510	0.60274	0.60275	0.60275	0.65724	0.65725
0.9	0.52512	0.52533	0.59347	0.59348	0.59348	0.64876	0.64877
1.0	0.52196	0.52217	0.59046	0.59048	0.59047	0.64601	0.64602

After some treatments, fin efficiency is written in terms of dimensionless problem parameters as follows.

$$\eta = -\frac{2(1+\beta)}{\psi^2(\lambda^2-1)}\left[\frac{d\theta}{d\xi}\right]_{\xi=0} \tag{27}$$

Fig. 4 shows the fin efficiency as a function of thermo-geometric fin parameter for different thermal conductivity parameters and radii ratios. From the Fig.4, it has been observed that the thermal conductivity parameter has a strong influence over the fin performance. In order to use the present solutions by thermal design engineers, the fin efficiency expressed as a function of thermo-geometric fin parameter, ψ, and radii ratio, λ, for an attained thermal conductivity parameter, β. With this correlation equation for the fin efficiency, it is assumed that the equation has the following form for an attained thermal conductivity parameter. The ranges of problem parameters are taken as $0.1 < \psi < 2.5$ and $1.3 < \lambda < 3.5$, in the correlation equations. The coefficients in Eq. 28 are tabulated in Table 2 for the fin efficiency.

$$\eta = \frac{A + B\psi + C\ln(\lambda) + D\ln^2(\lambda) + E\ln^3(\lambda)}{1 + F\psi + G\psi^2 + H\psi^3 * I\ln(\lambda) + J\ln^2(\lambda) + K\ln^3(\lambda)}$$

The correlation coefficients for each of the correlations are higher than 0.9991. These correlations found as the result of multiple regression analysis represent the fin efficiency of annular fins and can be used the designing of annular fins with temperature dependent thermal conductivity.

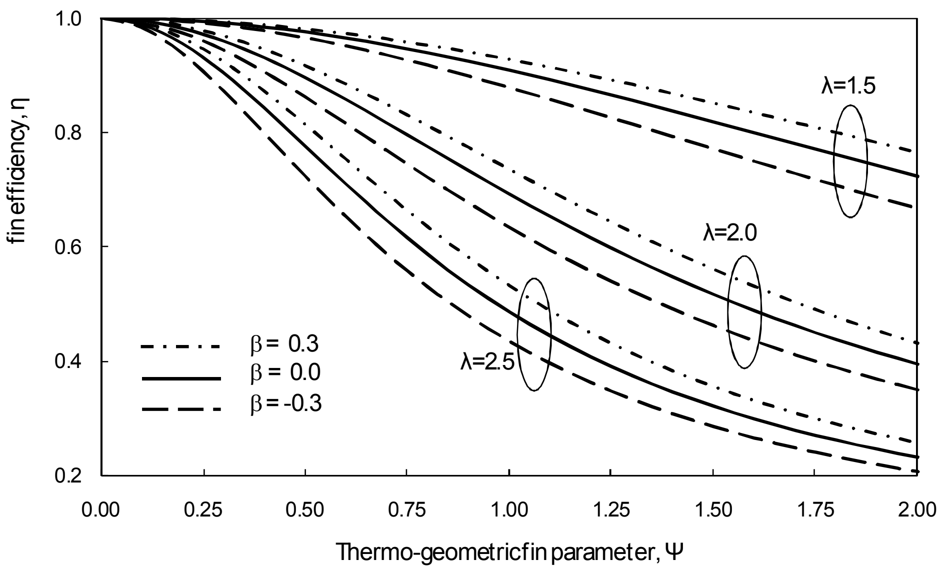

Figure 4. Fin efficiency as a function of thermo-geometric fin parameter for t different thermal conductivity parameter and radii ratio.

Table 2. Coefficients in Eq. 28 for the fin efficiency.

	β= -0.3	β= -0.2	β= -0.1	β=0.0	β=0.1	β=0.2	β=0.3
A	0.60792	0.60877	0.61213	0.61938	0.62828	0.63628	0.64567
B	0.03968	0.04249	0.04477	0.04681	0.04847	0.04953	0.05026
C	-1.29777	-1.34036	-1.38184	-1.42871	-1.47527	-1.51401	-1.55363
D	1.33981	1.39471	1.44450	1.49914	1.55168	1.59383	1.63827
E	-0.61488	-0.63777	-0.65723	-0.67859	-0.69847	-0.71369	-0.73079
F	-0.73564	-0.70262	-0.67107	-0.63696	-0.60366	-0.57578	-0.54824
G	0.37212	0.34334	0.31790	0.29283	0.26971	0.25101	0.23346
H	0.00947	0.01241	0.01486	0.01756	0.01997	0.02144	0.02279
I	-1.29531	-1.33785	-1.37920	-1.42646	-1.47351	-1.51219	-1.55193
J	1.31379	1.36772	1.41699	1.47059	1.52202	1.56420	1.60875
K	-0.58265	-0.60690	-0.62786	-0.64928	-0.66880	-0.68510	-0.70295

CONCLUSIONS

Convective annular fins with temperature-dependent thermal conductivity were analyzed using the homotopy analysis method. The HAM supplies reliable results in the form of analytical approximation converging very rapidly. The nonlinear differential equation which governs the fin temperature distribution was solved and then the fin efficiency was calculated. The results are expressed in terms of suitable dimensionless parameters and presented in terms of regression equations obtained by standard statistical techniques. These results can be used for designing straight fins with variable thermal conductivity.

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