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Hermite-Hadamard Inequalities for Generalized $\zeta-$Conformable Integrals Generated by Co-Ordinated Functions

Sümeyye Ermeydan Çiriş* and Hüseyin Yıldırım

Abstract
In this article, we introduce generalized $\zeta-$ conformable fractional integrals on co-ordinated functions and for the functions of two variables. Additionally, we derive a new Hermite-Hadamard inequality by utilizing the generalized Riemann-Liouville integrals, utilizing the generalized $\zeta-$conformable integral definition. Furthermore, we demonstrate some implications of the Hermite-Hadamard inequality and definitions introduced in this study. Consequently, we state and prove several related inequalities.

Keywords: *Co-ordinated functions, Generalized conformable integrals, Hermite-Hadamard inequality*

AMS Subject Classification (2020): *26A33; 41A55; 26D15*

****Corresponding author***

1. Introduction

Convex functions are a subject of extensive scientific research. One of the most outstanding inequality for convex functions was discovered by Hadamard in 1893. Additionally, many studies have focused on convex functions and the Hermite-Hadamard-type integral inequalities related to convex functions. Sarıkaya et al. define the general convex functions as the following inequality for $f:[\rho,\lambda]\subset\mathbb{R}\to\mathbb{R}$ on $[\rho,\lambda]$ in [1].

$$f\left(\theta\varphi\left(\tau\right)+\left(1-\theta\right)\varphi\left(\phi\right)\right)\leq\theta f\left(\varphi\left(\tau\right)\right)+\left(1-\theta\right)f\left(\varphi\left(\phi\right)\right). \tag{1.1}$$

Moreover, Cristescu defined and proved the Hermite-Hadamard-type integral inequality for general $\varphi-$convex functions in [2]. Then,

$$f\left(\frac{\varphi\left(\rho\right)+\varphi\left(\lambda\right)}{2}\right)\leq\frac{1}{\varphi\left(\rho\right)-\varphi\left(\lambda\right)}\int_{\varphi(\rho)}^{\varphi(\lambda)}f\left(z\right)dz\leq\frac{f\left(\varphi\left(\rho\right)\right)+f\left(\varphi\left(s\right)\right)}{2}. \tag{1.2}$$

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If f is a concave function, the inequality is reversed. This inequality is significant in fractional integrals and derivatives. There are many studies on the Hermite-Hadamard inequality in the literature (see, e.g., [3–5]).

Set et al. introduce $\varphi-$convex functions on co-ordinates, and they demonstrate their properties. Moreover, they obtain Hadamard type inequalities via $\varphi-$convex function on co-ordinates in [6]. We should give the following basic definition and basic theorem to use later.

Definition 1.1. [6] Let $\Delta := [\tau, \phi] \times [\theta, \mu] \subseteq [0, \infty) \times [0, \infty)$, $\tau < \phi$ and $\theta < \mu$. If $f : \Delta \to \mathbb{R}$ is said to be $\varphi-$convex on Δ for every two points $(\lambda, u), (\lambda, v), (y, u), (y, v) \in \Delta$ and $\rho, s \in [0, 1]$. Then, we get

$$\begin{aligned} &f(\rho\varphi_1(\lambda) + (1-\rho)\varphi_1(y), s\varphi_2(u) + (1-s)\varphi_2(v)) \\ &\leq \rho s f(\varphi_1(\lambda), \varphi_2(u)) + \rho(1-s) f(\varphi_1(\lambda), \varphi_2(v)) \\ &+ (1-\rho) s f(\varphi_1(y), \varphi_2(u)) + (1-\rho)(1-s) f(\varphi_1(y), \varphi_2(v)), \end{aligned} \tag{1.3}$$

for $\varphi_i : [\tau, \phi] \to [\theta, \mu]$, $i = 1, 2$ be a continuous function. A function $f : \Delta \to \mathbb{R}$ is $\varphi-$convex function on Δ is called co-ordinated $\varphi-$convex on Δ if the partial mappings $f_{\varphi_2} : [\tau, \phi] \to \mathbb{R}$, $f_{\varphi_2}(u) = f(u, \varphi_2)$ and $f_{\varphi_1} : [\theta, \mu] \to \mathbb{R}$, $f_{\varphi_1}(v) = f(\varphi_1, v)$ are $\varphi-$convex for all $\tau \leq \varphi_2 \leq \phi$ and $\theta \leq \varphi_1 \leq \mu$.

Theorem 1.1. [6] *If $f : \Delta = [\tau, \phi] \times [\theta, \mu] \subset \mathbb{R}^2 \to \mathbb{R}$ is $\varphi-$convex on the co-ordinates on Δ with $f \in L[\Delta]$, Then, we obtain*

$$\begin{aligned} &f\left(\tfrac{\varphi(\tau)+\varphi(\phi)}{2}, \tfrac{\varphi(\theta)+\varphi(\mu)}{2}\right) \\ &\leq \tfrac{1}{(\varphi(\phi)-\varphi(\tau))(\varphi(\mu)-\varphi(\theta))} \int_{\varphi(\tau)}^{\varphi(\phi)} \int_{\varphi(\theta)}^{\varphi(\mu)} f(\rho, s)\, ds d\rho \\ &\leq \tfrac{f(\varphi(\tau),\varphi(\theta))+f(\varphi(\tau),\varphi(\mu))+f(\varphi(\phi),\varphi(\theta))+f(\varphi(\phi),\varphi(\mu))}{4}. \end{aligned} \tag{1.4}$$

Furthermore, we demonstrate that the generalized $\zeta-$conformable fractional integration operator ${}^{\alpha}_{\sigma}J^{\beta}_{\tau+}$ is well-defined on $X^p_{\varrho}(\tau, \phi)$ for $p > \varrho$. We can write the following definition and theorem.

Definition 1.2. [7, 8] Let $\zeta(\lambda)$ be an increasing and positive monotone function on $[0, \infty)$. Furthermore, if we consider $\zeta'(\lambda)$ is continuous on $[0, \infty)$ and $\zeta(0) = 0$, the space $X^p_{\zeta}(0, \infty)$ is the following form for $(1 \leq p < \infty)$,

$$\|f\|_{X^p_{\zeta}} = \left(\int_0^{\infty} |f(t)|^p \zeta'(\lambda)\, dt\right)^{\frac{1}{p}} < \infty \tag{1.5}$$

and if we choose $p = \infty$,

$$\|f\|_{X^{\infty}_{\zeta}} = ess \sup_{1 \leq t < \infty} \left[f(t)\zeta'(\lambda)\right]. \tag{1.6}$$

Additionally, If we take $\zeta(\lambda) = \lambda$ $(1 \leq p < \infty)$ the space $X^p_{\zeta}(0, \infty)$, we have the $L_p[0, \infty)-$space. Moreover, if we take $\zeta(\lambda) = \frac{\lambda^{\sigma+1}}{\sigma+1}$ $(1 \leq p < \infty,\ \sigma \geq 0)$ the space $X^p_{\zeta}(0, \infty)$, we have the $L_{p,\sigma}[0, \infty)-$space.

Definition 1.3. Let $f \in X_{\zeta}(0, \infty)$, ζ be an increasing and positive monotone function on $[0, \infty)$ and also derivative ζ' be continuous on $[0, \infty)$ and $\zeta(0) = 0$. The left and right generalized conformable fractional integrals of order $\beta \in \mathbb{C}$, $\mathbb{R}(\beta) \geq 0$ and $\alpha > 0$,

$${}^{\alpha}_{\zeta}J^{\beta}_{\tau+}f(\lambda) = \tfrac{1}{\Gamma(\beta)} \int_{\tau}^{\lambda} \left[\tfrac{(\zeta(\lambda)-\zeta(\tau))^{\alpha}-(\zeta(\rho)-\zeta(\tau))^{\alpha}}{\alpha}\right]^{\beta-1} \tfrac{\zeta'(\rho)f(\rho)d\rho}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha}} \tag{1.7}$$

and

$${}^{\alpha}_{\zeta}J^{\beta}_{\phi-}f(\lambda) = \tfrac{1}{\Gamma(\beta)} \int_{\lambda}^{\phi} \left[\tfrac{(\zeta(\phi)-\zeta(\lambda))^{\alpha}-(\zeta(\phi)-\zeta(\rho))^{\alpha}}{\alpha}\right]^{\beta-1} \tfrac{\zeta'(\rho)f(\rho)d\rho}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha}}, \tag{1.8}$$

respectively.

Bozkurt et al. showed conformable derivatives and conformable integrals for the functions of two variables in [9]. Based on this article, we define the following the definition.

Definition 1.4. Let $f \in X_\zeta([\tau,\phi]\times[\theta,\mu])$ and $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta,\gamma \in \mathbb{C}$, $Re(\beta)>0$, $Re(\gamma)>0$. Meanwhile, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. Then, we have the generalized $\zeta-$conformable integrals of order β,γ of $f(\rho,s)$,

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\tau^+,\theta^+}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\\&\times\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.9}$$

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\phi^-,\theta^+}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y\left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\\&\times\frac{\zeta'(\rho)}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.10}$$

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\tau^+,\mu^-}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu\left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\\&\times\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.11}$$

and

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\phi^-,\mu^-}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu\left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\\&\times\frac{\zeta'(\rho)}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho.\end{aligned}\tag{1.12}$$

Remark 1.1. In here, when we get $\zeta(\lambda)=\frac{\lambda^{\sigma+1}}{(\sigma+1)^{\frac{1}{\alpha}}}$ in Definition 4, then we can write equations as follows,

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\tau^+,\theta^+}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\\&\times\frac{\rho^\sigma}{(\rho^{\sigma+1}-\tau^{\sigma+1})^{1-\alpha_1}}\cdot\frac{s^\sigma}{(s^{\sigma+1}-\theta^{\sigma+1})^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.13}$$

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\phi^-,\theta^+}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha_1}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\\&\times\frac{\rho^\sigma}{(\phi^{\sigma+1}-\rho^{\sigma+1})^{1-\alpha_1}}\cdot\frac{s^\sigma}{(s^{\sigma+1}-\theta^{\sigma+1})^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.14}$$

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\tau^+,\mu^-}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu\left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\\&\times\frac{\rho^\sigma}{(\rho^{\sigma+1}-\tau^{\sigma+1})^{1-\alpha_1}}\cdot\frac{s^\sigma}{(\mu^{\sigma+1}-s^{\sigma+1})^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.15}$$

and

$$\begin{aligned}&{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\phi^-,\mu^-}\\&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha_1}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\\&\times\frac{\rho^\sigma}{(\phi^{\sigma+1}-\rho^{\sigma+1})^{1-\alpha_1}}\cdot\frac{s^\sigma}{(\mu^{\sigma+1}-s^{\sigma+1})^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho.\end{aligned}\tag{1.16}$$

Remark 1.2. If we take $\zeta(\lambda)=\lambda$ in Definition 3, then we have the following equations in [9]

$$\begin{aligned}{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+}&=\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\\&\times\frac{1}{(\rho-\tau)^{1-\alpha_1}}\cdot\frac{1}{(s-\theta)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho,\end{aligned}\tag{1.17}$$

$$
\begin{aligned}
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+} &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y \left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\
&\times\frac{1}{(\phi-\rho)^{1-\alpha_1}}\cdot\frac{1}{(s-\theta)^{1-\alpha_2}} f(\rho,s)\,dsd\rho,
\end{aligned} \tag{1.18}
$$

$$
\begin{aligned}
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-} &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu \left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\
&\times\frac{1}{(\rho-\tau)^{1-\alpha_1}}\cdot\frac{1}{(\mu-s)^{1-\alpha_2}} f(\rho,s)\,dsd\rho,
\end{aligned} \tag{1.19}
$$

and

$$
\begin{aligned}
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-} &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu \left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\
&\times\frac{1}{(\phi-\rho)^{1-\alpha_1}}\cdot\frac{1}{(\mu-s)^{1-\alpha_2}} f(\rho,s)\,dsd\rho.
\end{aligned} \tag{1.20}
$$

Remark 1.3. [10] If we take $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Definition 3, then we get

$$ {}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y (\lambda-\rho)^{\beta-1}(y-s)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.21} $$

$$ {}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y (\rho-\lambda)^{\beta-1}(y-s)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.22} $$

$$ {}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu (\lambda-\rho)^{\beta-1}(s-y)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.23} $$

and

$$ {}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu (\rho-\lambda)^{\beta-1}(s-y)^{\gamma-1} f(\rho,s)\,dsd\rho. \tag{1.24} $$

Kiriş et al. studied Hermite-Hadamard inequalities for co-ordinated convex function via generalized conformable fractional integrals in [11]. Moreover, Çiriş and et al. defined generalized $\sigma-$conformable integrals by co-ordinated functions [12]. In addition, considering Definition 4, we can obtain Definition 5.

Definition 1.5. Let $f\in X_\zeta([\tau,\phi]\times[\theta,\mu])$ and $\alpha_1\neq 0$, $\alpha_2\neq 0$, $\beta,\gamma\in\mathbb{C}$, $Re(\beta)>0$, $Re(\beta)>0$. Now, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. In here,

$$
\begin{aligned}
&\left({}^{\alpha_1}_{\zeta}J^{\beta}_{\tau^+}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) \\
&= \frac{1}{\Gamma(\beta)}\int_\tau^\lambda \left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}},\ \lambda>\tau,
\end{aligned} \tag{1.25}
$$

$$
\begin{aligned}
&\left({}^{\alpha_1}_{\zeta}J^{\beta}_{\phi^-}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) \\
&= \frac{1}{\Gamma(\beta)}\int_\lambda^\phi \left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}},\ \lambda<\phi,
\end{aligned} \tag{1.26}
$$

$$
\begin{aligned}
&\left({}^{\alpha_2}_{\zeta}J^{\gamma}_{\theta^+}f\right)\left(\frac{\tau+\phi}{2},y\right) \\
&= \frac{1}{\Gamma(\gamma)}\int_\theta^y \left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)\zeta'(s)ds}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}},\ y>\theta,
\end{aligned} \tag{1.27}
$$

and

$$
\begin{aligned}
&\left({}^{\alpha_2}_{\zeta}J^{\gamma}_{\mu^-}f\right)\left(\frac{\tau+\phi}{2},y\right) \\
&= \frac{1}{\Gamma(\gamma)}\int_y^\mu \left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)\zeta'(s)ds}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}},\ y<\mu,
\end{aligned} \tag{1.28}
$$

we have equations.

Remark 1.4. If we take $\zeta(\lambda)=\frac{\lambda^{\sigma+1}}{(\sigma+1)^{\frac{1}{\alpha}}}$ in Definition 5, then we can write as the following,

$$
\begin{aligned}
&\left({}^{\alpha_1}_{\sigma}J^{\beta}_{\tau^+}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) \\
&= \frac{1}{\Gamma(\beta)}\int_\tau^\lambda \left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\frac{\rho^\sigma f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{1-\alpha_1}},\ \lambda>\tau,
\end{aligned} \tag{1.29}
$$

$$\begin{aligned}
&\left({}_{\sigma}^{\alpha_1}J_{\phi-}^{\beta}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)\\
&=\frac{1}{\Gamma(\beta)}\int_{\lambda}^{\phi}\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\frac{\rho^{\sigma}f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{1-\alpha_1}},\ \lambda<\phi,
\end{aligned}\tag{1.30}$$

$$\begin{aligned}
&\left({}_{\sigma}^{\alpha_2}J_{\theta+}^{\gamma}f\right)\left(\frac{\tau+\phi}{2},y\right)\\
&=\frac{1}{\Gamma(\gamma)}\int_{\theta}^{y}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\frac{s^{\sigma}f\left(\frac{\tau+\phi}{2},s\right)ds}{\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{1-\alpha_2}},\ y>\theta,
\end{aligned}\tag{1.31}$$

and

$$\begin{aligned}
&\left({}_{\sigma}^{\alpha_2}J_{\mu-}^{\gamma}f\right)\left(\frac{\tau+\phi}{2},y\right)\\
&=\frac{1}{\Gamma(\gamma)}\int_{y}^{\mu}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\frac{s^{\sigma}f\left(\frac{\tau+\phi}{2},s\right)ds}{\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{1-\alpha_2}},\ y<\mu.
\end{aligned}\tag{1.32}$$

Remark 1.5. [11] If we get $\zeta(\lambda)=\lambda$ in Definition 4, then we obtain

$$\begin{aligned}
\left({}^{\alpha_1}J_{\tau+}^{\beta}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)&=\frac{1}{\Gamma(\beta)}\int_{\tau}^{\lambda}\left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{(\rho-\tau)^{1-\alpha_1}},\ \lambda>\tau,\\
\left({}^{\alpha_1}J_{\phi-}^{\beta}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)&=\frac{1}{\Gamma(\beta)}\int_{\lambda}^{\phi}\left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{(\phi-\rho)^{1-\alpha_1}},\ \lambda<\phi,\\
\left({}^{\alpha_2}J_{\theta+}^{\gamma}f\right)\left(\frac{\tau+\phi}{2},y\right)&=\frac{1}{\Gamma(\gamma)}\int_{\theta}^{y}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)ds}{(s-\theta)^{1-\sigma}},\ y>\theta,\\
\left({}^{\alpha_2}J_{\mu-}^{\gamma}f\right)\left(\frac{\tau+\phi}{2},y\right)&=\frac{1}{\Gamma(\gamma)}\int_{y}^{\mu}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)ds}{(\mu-s)^{1-\sigma}},\ y<\mu.
\end{aligned}\tag{1.33}$$

In this study, we will examine Hermite-Hadamard inequalities for co-ordinated convex mappings by means of the generalized $\zeta-$conformable fractional integral operator. In addition, we are going to prove several important Theorems utilizing the Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals and by means of definitions which we define.

2. Hermite-Hadamard inequality

In this section, we will derive Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals.

Theorem 2.1. *Let $f\in X_{\zeta}([\tau,\phi])$ and f is $\varphi-$convex function. ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. We obtain the inequalities as follows utilizing the generalized $\zeta-$conformable fractional integrals for $\mathbb{R}(\beta)>0$ and $\alpha_1\in(0,1]$,*

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\\
&\le\frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_2})^{\beta}\right]}\\
&\times\left[{}_{\zeta}^{\alpha_1}J_{(w_1)+}^{\beta}f(w_2)+{}_{\zeta}^{\alpha_1}J_{(w_4)-}^{\beta}f(w_3)\right]\\
&\le\frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}.
\end{aligned}\tag{2.1}$$

In here, we have

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}=w_1,$$

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}=w_2,$$

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}=w_3$$

and

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}=w_4.$$

Proof. By definition of $\varphi-$convex function, we get

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\\
&=f\left[\frac{1}{2}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)+\frac{1}{2}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right]\\
&\leq\frac{1}{2}\left[f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)+f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right]\\
&\leq\frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}.
\end{aligned}$$

Here, we can write

$$\begin{aligned}
f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)&\leq\frac{1}{2}\left[f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\right.\\
&\quad\left.+f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right]\\
&\leq\frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}.
\end{aligned}\tag{2.2}$$

Moreover, if we multiply $\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}$ both of inequalities in (2.2) and we integrate from 0 to 1, then we acquire

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\\
&\leq\frac{1}{2}\left[\int_0^1 f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\right.\\
&\quad\left.+\int_0^1 f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\right]\\
&\leq\frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}.
\end{aligned}\tag{2.3}$$

Furthermore, we get I_1 as the following,

$$I_1=\int_0^1 f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}.$$

By changing the variable with,

$$\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)=\zeta(u),\tag{2.4}$$

we have

$$\begin{aligned}
I_1&=\int_{w_1}^{w_2}\left(\frac{1-\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(u)-\zeta(\tau))^{1-\alpha_1}}\\
&=\frac{2^{\alpha_1\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(u)-\zeta(\tau))^{1-\alpha_1}}\\
&=\frac{2^{\alpha_1\beta}\Gamma(\beta)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\cdot{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f(w_2).
\end{aligned}$$

At the same way, if we take I_2 as the following,

$$I_2=\int_0^1 f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}},$$

By changing the variable with,

$$\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)=\zeta(u),\tag{2.5}$$

we can write

$$\begin{aligned}
I_2&=\int_{w_3}^{w_4}\left(\frac{1-\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}(\zeta(\phi)-\zeta(u))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(\phi)-\zeta(u))^{1-\alpha_1}}\\
&=\frac{2^{\alpha_1\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(u))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(\phi)-\zeta(u))^{1-\alpha_1}}\\
&=\frac{2^{\alpha_1\beta}\Gamma(\beta)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\cdot{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f(w_3).
\end{aligned}$$

Additionally, if we get I_3 as the following, then we obtain

$$\begin{aligned}
I_3&=\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\\
&=\frac{1}{\beta\alpha_1^{\beta}}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right].
\end{aligned}$$

If we use I_1, I_2 and I_3 in (2.3) then, we have

$$\begin{aligned} & f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right) \\ & \leq \frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\ & \times\left[{}_{\zeta}^{\alpha_1}J_{(w_1)^+}^{\beta}f(w_2)+{}_{\zeta}^{\alpha_1}J_{(w_4)^-}^{\beta}f(w_3)\right] \\ & \leq \frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}. \end{aligned}$$

The proof is completed. □

If we take $\zeta(\lambda)=\lambda$ in Theorem 2, then we obtain

$$f\left(\frac{\tau+\phi}{2}\right)\leq\frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2}\right)+{}^{\alpha_1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2}\right)\right]\leq\frac{f(\tau)+f(\phi)}{2},$$

which is proved in [13].

Theorem 2.2. *Let $f\in X_{\zeta}([\tau,\phi]\times[\theta,\mu])$ and f is a $\zeta-$conformable co-ordinated $\varphi-$convex function. Moreover, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. Additionally, we have for $\alpha_1\neq 0$, $\alpha_2\neq 0$, $\beta,\gamma\in\mathbb{C}$, $Re(\beta)>0$, $Re(\gamma)>0$,*

$$\begin{aligned} & f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\ & \times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right) \\ & \leq\frac{1}{4}\left[\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma}f\right)(w_2,q_2)\right. \\ & \quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_4^-}^{\beta,\gamma}f\right)(w_2,q_3) \\ & \quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_1^+}^{\beta,\gamma}f\right)(w_3,q_2) \\ & \quad\left.+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_4^-}^{\beta,\gamma}f\right)(w_3,q_3)\right] \\ & \leq\frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4} \\ & \times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right). \end{aligned} \tag{2.6}$$

In here, we write

$$\begin{aligned} w_1 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_1 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}-\frac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2} \\ w_2 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_2 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}-\frac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\ w_3 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_3 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}+\frac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\ w_4 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_4 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}+\frac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2}. \end{aligned}$$

Proof. We can write the equality

$$\begin{aligned} & f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\ & =f\left[\frac{1}{4}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right)\right. \\ & \quad+\frac{1}{4}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right) \\ & \quad+\frac{1}{4}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right) \\ & \quad\left.+\frac{1}{4}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\right]. \end{aligned}$$

By Definition 1, we have

$$\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\leq \tfrac{1}{4}\Big[f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\Big]\\
&\leq \tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}.
\end{aligned}\tag{2.7}$$

If we multiply by $\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)}{(1-\zeta(s))^{1-\alpha_2}}$ both of the inequalities in (2.7) and integrating $[0,1]\times[0,1]$ with respect to s and ρ, then we obtain

$$\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\quad\times\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\leq\tfrac{1}{4}\Big[\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\quad\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\Big]\\
&\leq\tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}\\
&\quad\times\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right).
\end{aligned}\tag{2.8}$$

By changing variables,

$$\begin{aligned}
&\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi)=\zeta(u),\\
&\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)=\zeta(v)
\end{aligned}\tag{2.9}$$

we have

$$\begin{aligned}
&\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&=\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\tfrac{1-\left(\frac{2(\zeta(u)-\zeta(\tau))}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\tfrac{1-\left(\frac{2(\zeta(v)-\zeta(\theta))}{\zeta(\mu)-\zeta(\theta)}\right)^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\\
&\times\left(\tfrac{2^{\alpha_1}(\zeta(u)-\zeta(\tau))^{\alpha_1-1}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1}}\right)\left(\tfrac{2^{\alpha_2}(\zeta(v)-\zeta(\theta))^{\alpha_2}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2}}\right)\zeta'(u)\zeta'(v)f(\zeta(u),\zeta(v))\,dudv\\
&=\left(\tfrac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1\beta}\left(\tfrac{2}{\zeta(\mu)-\zeta(\theta)}\right)^{\alpha_2\gamma}\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\tfrac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\\
&\times\left(\tfrac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(v)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{f(\zeta(u),\zeta(v))\zeta'(u)\zeta'(v)dudv}{(\zeta(\phi)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\theta))^{1-\alpha_2}}\\
&=\tfrac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_1^+}f\right)(w_2,q_2).
\end{aligned}\tag{2.10}$$

In the same way, we have

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\&\times f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\\&=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_4^-}f\right)(w_2,q_3),\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\&\times f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right)\\&=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_4^-,q_1^+}f\right)(w_3,q_2)\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\&\times f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\\&=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_4^-,q_4^-}f\right)(w_3,q_3).\end{aligned}\tag{2.13}$$

By simple calculations, we have

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\&=\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right).\end{aligned}\tag{2.14}$$

By using (2.10)-(2.14) in (2.8), we obtain

$$\begin{aligned}&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\&\times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right)\\&\leq\frac{1}{4}\left[\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_1^+}f\right)(w_2,q_2)\right.\\&\quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_4^-}f\right)(w_2,q_3)\\&\quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_4^-,q_1^+}f\right)(w_3,q_2)\\&\quad\left.+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_4^-,q_4^-}f\right)(w_3,q_3)\right.\\&\leq\frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}\\&\times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right).\end{aligned}$$

The proof is completed. □

Remark 2.1. [11] If $\zeta(\lambda)=\lambda$ in Theorem 3, we obtain

$$\begin{aligned}&f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\left(\frac{1}{\alpha_1^{\beta}\alpha_2^{\gamma}\beta\gamma}\right)\leq(\frac{2^{\alpha_1\beta-1}2^{\alpha_2\gamma-1}\Gamma(\beta)\Gamma(\gamma)}{(\phi-\tau)^{\alpha_1\beta}(\mu-\theta)^{\alpha_2\gamma}}\\&\times\left[\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right.\\&\left.+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right]\\&\leq\frac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}\left(\frac{1}{\alpha_1^{\beta}\alpha_2^{\gamma}\beta\gamma}\right).\end{aligned}$$

Remark 2.2. By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 3, we have

$$\begin{aligned}
f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) &\leq \tfrac{2^{\beta-1}2^{\gamma-1}\Gamma(\beta+1)\Gamma(\gamma+1)}{(\phi-\tau)^{\beta}(\mu-\theta)^{\gamma}} \\
&\times\Big[\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\tau^+,\theta^+}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\tau^+,\mu^-}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) \\
&+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\phi^-,\theta^+}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\phi^-,\mu^-}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\Big] \\
&\leq \tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}$$

Remark 2.3. By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$, $\alpha_2=1$, $\beta=1$ and $\gamma=1$ in Theorem 3, we have

$$\begin{aligned}
&f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) \\
&\leq \tfrac{1}{(\phi-\tau)(\mu-\theta)}\int_{\tau}^{\theta}\int_{\theta}^{\mu}f(t,s)\,dsdt \\
&\leq \tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}$$

Theorem 2.3. *Let $f:\Delta=[\tau,\phi]\times[\theta,\mu]\subset\mathbb{R}^2\to\mathbb{R}$ for $0\leq\tau<\phi$ and $0\leq\theta<\mu$. Furthermore, f is $\zeta-$conformable co-ordinated $\varphi-$convex function and $f\in X_{\zeta}(\Delta)$. ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. We can obtain as the following inequality for $\alpha_1\neq0$, $\alpha_2\neq0$, $\beta,\gamma\in\mathbb{C}$, $Re(\beta)>0$, $Re(\gamma)>0$,*

$$\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
&\leq\tfrac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times\left[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&+\tfrac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}\right]} \\
&\times\left[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&\leq\tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times\left[{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&+\tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times\left[{}^{\alpha_1,\alpha_2}_{\zeta}J^{\gamma,\beta}_{w_4^-,q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1,\alpha_2}_{\zeta}J^{\gamma,\beta}_{w_4-,q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&\leq\tfrac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\Big[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right) \\
&\qquad+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\Big] \\
&+\tfrac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\Big[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(w_1,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(w_4,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
&\qquad+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(w_1,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(w_4,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\Big] \\
&\leq\tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}.
\end{aligned}$$

Here, we have

$$\begin{aligned}
w_1 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}-\tfrac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_1 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}-\tfrac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_2 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}-\tfrac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_2 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}-\tfrac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_3 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}+\tfrac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_3 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}+\tfrac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_4 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}+\tfrac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_4 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}+\tfrac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2}.
\end{aligned}$$

Proof. If $f:\Delta\to\mathbb{R}$ is co-ordinated $\varphi-$convex function and also $g_{\zeta(\lambda)}:[q_1,q_4]\to\mathbb{R}$, $g_{\zeta(\lambda)}(\zeta(\rho))=f(\zeta(\lambda),\zeta(\rho))$ is

$\varphi-$convex on $[q_1, q_4]$ for all $w_1 \leq \zeta(\lambda) \leq w_4$, then, we obtain by utilizing Theorem 2,

$$\begin{aligned} & g_{\zeta(\lambda)}\left(\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\ & \leq \frac{2^{\alpha_2\gamma-1}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\ & \times\left[{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}g_{\zeta(\lambda)}\left(q_2\right)+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}g_{\zeta(\lambda)}\left(q_3\right)\right] \\ & \leq \frac{g_{\zeta(\lambda)}\zeta(\theta)+g_{\zeta(\lambda)}\zeta(\mu)}{2}. \end{aligned}$$

Here, we can write

$$\begin{aligned} & f\left(\zeta\left(\lambda\right), \frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\ & \leq \frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\ & \quad\times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right. \\ & \quad\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right] \\ & \leq \frac{f(\zeta(\lambda),\zeta(\theta))+f(\zeta(\lambda),\zeta(\mu))}{2}. \end{aligned} \tag{2.15}$$

If we multiply both sides of (2.15) by

$$\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\lambda)}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}},$$

and if we integrate with respect to λ on $[w_1, w_2]$, then we get

$$\begin{aligned} & \frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}} \\ & \leq\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\ & \times\left[\left[\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right.\right. \\ & \left.\times\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right] \\ & +\left[\int_{w_1}^{w_2}\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right. \\ & \left.\left.\times\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\right] \\ & \leq\frac{2^{\alpha_1\beta-2}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right. \\ & \qquad\left.+\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right]. \end{aligned} \tag{2.16}$$

Similarly, if we multiply both sides of (2.15) by

$$\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\lambda)}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}},$$

and if we integrate with respect to λ on $[w_3, w_4]$, then we can have

$$
\begin{aligned}
&\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\\
&\leq\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[\left[\int_{w_3}^{w_4}\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right.\right.\\
&\times\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right]\\
&+\int_{w_3}^{w_4}\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\Bigg]\\
&\times\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\Bigg]\\
&\leq\frac{2^{\alpha_1\beta-2}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right.\\
&\qquad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right].
\end{aligned}
\tag{2.17}
$$

If f is co-ordinated $\varphi-$convex function, then $g_{\zeta(\rho)}:[w_1,w_4]\to\mathbb{R}$, $g_{\zeta(\rho)}(\zeta(\lambda))=f(\zeta(\lambda),\zeta(\rho))$ is $\varphi-$convex function, then, by Theorem 2, we get

$$
\begin{aligned}
&g_{\zeta(\rho)}\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\\
&\leq\frac{2^{\alpha_1\beta-1}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}g_{\zeta(\rho)}(w_2)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}g_{\zeta(\rho)}(w_3)\right]\\
&\leq\frac{g_{\zeta(\rho)}\zeta(\tau)+g_{\zeta(\rho)}\zeta(\phi)}{2}.
\end{aligned}
$$

Here, we can write,

$$
\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\\
&\leq\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\quad\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\quad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\\
&\leq\frac{f(\zeta(\tau),\zeta(\rho))+f(\zeta(\phi),\zeta(\rho))}{2}.
\end{aligned}
\tag{2.18}
$$

Moreover, by multiplying both side of (2.18) by

$$
\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}},
$$

and by integrating with respect to ρ on $[q_1, q_2]$, then we obtain

$$\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\\
\le&\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
\times&\left[\left[\int_{q_1}^{q_2}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\right.\\
\times&\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right]\\
+&\left[\int_{q_1}^{q_2}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\\
\times&\left.\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\right]\\
\le&\frac{2^{\alpha_2\gamma-2}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right.\\
&\qquad\left.+\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right].
\end{aligned}\tag{2.19}$$

Furthermore, if we multiply both sides of (2.18) by

$$\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(\rho)}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}},$$

and if we integrate with respect to ρ on $[q_3, q_4]$, then we get

$$\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\\
\le&\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
\times&\left[\left[\int_{q_3}^{q_4}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\right.\\
\times&\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\\
+&\left[\int_{q_3}^{q_4}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\\
\times&\left.\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\right]\\
\le&\frac{2^{\alpha_2\gamma-2}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_1}}\right.\\
&\qquad\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right].
\end{aligned}\tag{2.20}$$

In here, if we add the inequalities (2.16)-(2.20) and divide by 2, then we have

$$
\begin{aligned}
&\frac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&+\frac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&\leq\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\times\left[{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_4^-}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&+\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4-,q_4^-}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&\leq\frac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\right.\\
&\qquad\left.+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\right]\\
&+\frac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(w_1,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(w_4,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right.\\
&\qquad\left.+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(w_1,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(w_4,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right].
\end{aligned}\tag{2.21}
$$

We give some results for special circumstances, if we get $\zeta(\lambda)=\frac{\zeta(\tau)+\zeta(\phi)}{2}$ on the left side of the (2.15) inequality, then we obtain,

$$
\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\leq\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\quad\times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right.\\
&\quad\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right].
\end{aligned}\tag{2.22}
$$

Similarly, if we take $\zeta(\rho)=\frac{\zeta(\theta)+\zeta(\mu)}{2}$ on the left side of the (2.18) inequality, then we get,

$$
\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\leq\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\quad\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\quad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right].
\end{aligned}\tag{2.23}
$$

If we do the necessary calculations for (2.22) and (2.23), we can obtain

$$
\begin{aligned}
& f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}, \frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
& \leq \frac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
& \times\left[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+} f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}, \frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-} f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}, \frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
& +\frac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}\right]} \\
& \times\left[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+} f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}, \frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-} f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}, \frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right].
\end{aligned} \tag{2.24}
$$

The inequality in (2.24)is the first inequality of Theorem 4.

Finally, if we get $\zeta(\rho)=\zeta(\theta)$ on the right-hand side of the (2.18) which we get by using the second inequality in (2.1), then we obtain

$$
\begin{aligned}
& \frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
& \times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right. \\
& \left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right] \\
& \leq \frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\phi),\zeta(\theta))}{2}.
\end{aligned} \tag{2.25}
$$

In same way, if we take $\zeta(\rho)=\zeta(\mu)$ in (2.18), then we can write

$$
\begin{aligned}
& \frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
& \times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right. \\
& \left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right] \\
& \leq \frac{f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\mu))}{2}.
\end{aligned} \tag{2.26}
$$

In a similar way, if we take $\zeta(\lambda)=\zeta(\tau)$ on the right-hand side of the (2.15) inequality, then we have

$$
\begin{aligned}
& \frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}\right]} \\
& \times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right. \\
& \left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right] \\
& \leq \frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))}{2}.
\end{aligned} \tag{2.27}
$$

Moreover, if we take $\zeta(\lambda)=\zeta(\phi)$ in (2.15), then we get

$$
\begin{aligned}
& \frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}\right]} \\
& \times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right. \\
& \left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right] \\
& \leq \frac{f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{2}.
\end{aligned} \tag{2.28}
$$

When we make the necessary calculations for (2.25), (2.26), (2.27) and (2.28), then we obtain the $4th$ inequality of Theorem 4. □

Corollary 2.1. *[11] If $\zeta(\lambda)=\lambda$ in Theorem 4, we acquire,*

$$\begin{aligned} & f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right) \\ & \leq \frac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{\alpha_1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & +\frac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\mu-\theta)^{\alpha_2\gamma}}\left[{}^{\alpha_2}J_{\theta^+}^{\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\mu^-}^{\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & \leq \frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}.(\mu-\theta)^{\alpha_2\gamma}} \\ & \times\left[{}^{\alpha_1,\alpha_2}J_{\tau^+,\theta^+}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{\alpha_1,\alpha_2}J_{\tau^+,\mu^-}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \end{aligned} \tag{2.29}$$

$$\begin{aligned} & +\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}.(\mu-\theta)^{\alpha_2\gamma}} \\ & \times\left[{}^{\alpha_1,\alpha_2}J_{\phi^-,\theta^+}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{\alpha_1,\alpha_2}J_{\phi^-,\mu^-}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & \leq \frac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{\alpha_1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\mu\right)\right. \\ & \qquad\left.+{}^{\alpha_1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{\alpha_1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\mu\right)\right] \\ & +\frac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\mu-\theta)^{\alpha_2\gamma}}\left[{}^{\alpha_2}J_{\theta^+}^{\gamma}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\theta^+}^{\gamma}f\left(\phi,\frac{\theta+\mu}{2}\right)\right. \\ & \qquad\left.+{}^{\alpha_2}J_{\mu^-}^{\gamma}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\mu^-}^{\gamma}f\left(\phi,\frac{\theta+\mu}{2}\right)\right] \\ & \leq \frac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}. \end{aligned} \tag{2.30}$$

Corollary 2.2. *By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 4, we write as the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} & f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\leq\frac{2^{\beta-2}\Gamma(\beta+1)}{(\phi-\tau)^{\beta}}\left[{}^{1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & \qquad+\frac{2^{\gamma-2}\Gamma(\gamma+1)}{(\mu-\theta)^{\gamma}}\left[{}^{1}J_{\theta^+}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{1}J_{\mu^-}^{\beta}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & \leq\frac{2^{\gamma-1}.2^{\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)}{(\phi-\tau)^{\beta}.(\mu-\theta)^{\gamma}}\left[{}^{1,1}J_{\tau^+,\theta^+}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{1,1}J_{\tau^+,\mu^-}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & +\frac{2^{\gamma-1}.2^{\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)}{(\phi-\tau)^{\beta}.(\mu-\theta)^{\gamma}}\left[{}^{1,1}J_{\phi^-,\theta^+}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+{}^{1,1}J_{\phi^-,\mu^-}^{\beta,\gamma}f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right] \\ & \leq\frac{2^{\beta-3}\Gamma(\beta+1)}{(\phi-\tau)^{\beta}}\left[{}^{1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\tau^+}^{\beta}f\left(\frac{\tau+\phi}{2},\mu\right)\right. \\ & \qquad\left.+{}^{1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\phi^-}^{\beta}f\left(\frac{\tau+\phi}{2},\mu\right)\right] \\ & \qquad+\frac{2^{\gamma-3}\Gamma(\gamma+1)}{(\mu-\theta)^{\gamma}}\left[{}^{1}J_{\theta^+}^{\gamma}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{1}J_{\theta^+}^{\gamma}f\left(\phi,\frac{\theta+\mu}{2}\right)\right. \\ & \qquad\left.+{}^{1}J_{\mu^-}^{\gamma}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{1}J_{\mu^-}^{\gamma}f\left(\phi,\frac{\theta+\mu}{2}\right)\right] \\ & \leq\frac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}. \end{aligned} \tag{2.31}$$

Corollary 2.3. *By choosing $\zeta(\lambda)=\lambda$, $\beta=1$, $\gamma=1$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 4, we write as the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} & f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right) \\ & \leq\frac{1}{2(\phi-\tau)}\left[\int_{\tau}^{\phi}f\left(t,\frac{\theta+\mu}{2}\right)dt\right]+\frac{1}{2(\mu-\theta)}\left[\int_{\phi}^{\mu}f\left(\frac{\tau+\phi}{2},s\right)ds\right] \\ & \leq\frac{1}{(\phi-\tau).(\mu-\theta)}\left[\int_{\tau}^{\theta}\int_{\theta}^{\mu}f(t,s)\,dsdt\right] \\ & \leq\frac{1}{4(\phi-\tau)}\left[{}^{1}J_{\tau^+}^{1}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\tau^+}^{1}f\left(\frac{\tau+\phi}{2},\mu\right)\right. \\ & \qquad\left.+{}^{1}J_{\phi^-}^{1}f\left(\frac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\phi^-}^{1}f\left(\frac{\tau+\phi}{2},\mu\right)\right] \\ & +\frac{1}{4(\mu-\theta)}\left[{}^{1}J_{\theta^+}^{1}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{1}J_{\theta^+}^{1}f\left(\phi,\frac{\theta+\mu}{2}\right)\right. \\ & \qquad\left.+{}^{1}J_{\mu^-}^{1}f\left(\tau,\frac{\theta+\mu}{2}\right)+{}^{1}J_{\mu^-}^{1}f\left(\phi,\frac{\theta+\mu}{2}\right)\right] \\ & \leq\frac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}. \end{aligned} \tag{2.32}$$

3. Conclusion

There are many studies on Hermite-Hadamard inequalities and fractional integrals [14–20]. In this study, we derive the Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals. Moreover, we derive two distinct definitions for these integrals: one for functions with two variables and another for co-ordinated functions. Expanding on these definitions, we highlight several significant findings and illustrate their implications and applications. Furthermore, we discuss important consequences within the broader mathematical context.

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References

[1] Sarikaya, M. Z., Büyükeken, M., Kiriş, M. E.: *On some generalized integral inequalities for $\varphi-$convex functions*. Studia Universitatis Babeş-Bolyai Mathematica. **60**(3), 367-377 (2015).

[2] Cristescu, G.: *Hadamard type inequalities for ϕ-convex functions*. Annals of the University of Oradea, Fascicle of Management and Technological Engineering, C-Rom Edition, III (XIII), (2004).

[3] Akkurt, A., Sarikaya, M. Z., Budak, Yildirim, H.: *On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals.* Journal of King Saud University-Science. **29**(2017), 380-387 (2017).

[4] Sarikaya, M. Z., Set, E., Ozdemir, M. E., Dragomir, S. S.: *New some Hadamard's type inequalities for co-ordinated convex functions*. Tamsui Oxford Journal of Mathematical Sciences. **28**(2), 137-152 (2012).

[5] Bakula, M. K.: *An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates*. Australian Journal of Mathematical Analysis and Applications. **11**(1), 1-7 (2014).

[6] Set, E., Sarıkaya, M.Z., Akdemir, A. O.: *Hadamard type inequalities for $\varphi-$convex functions on the co-ordinates*. Tbilisi Mathematical Journal. **7**(2), 51-60 (2014).

[7] Yıldırım, H., Kırtay, Z.: *Ostrowski inequality for generalized fractional integral and related inequalities*. Malaya Journal of Matematik. **2**(3), 322-329 (2014).

[8] Kaçar, E., Kaçar, Z., Yildirim, H.: *Integral inequalities for Riemann-Liouville fractional integral of a function with respect to another function*. Iranian Journal of Mathematical Sciences and Informatics. **13**, 1-13 (2018).

[9] Bozkurt, M., Akkurt, A., Yildirim, H.: *Conformable derivatives and integrals for the functions of two variables*. Konuralp Journal of Mathematics. **9**(1), 49-59 (2021).

[10] Sarikaya, M. Z.: *On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integral*. Integral Transforms and Special Functions. **25**(2), 134-147 (2014).

[11] Kiriş, M.E., Bayrak, G.: *New version of Hermite-Hadamard inequality for co-ordinated convex function via generalized conformable integrals*. Filomat. **38**(16), 5575-5589 (2024).

[12] Çiriş, S. E., Yildirim, H.: *Hermite-Hadamard inequalities for generalized $\sigma-$conformable integrals generated by co-ordinated functions*. Chaos, Solitons & Fractals. **181**, 114628 (2024).

[13] Hyder, A. A., Almoneef, A. A., Budak, H., Barakat, M. A.: *On new fractional version of generalized Hermite-Hadamard inequalities*. Mathematics. **10**(18), 33-37 (2022).

[14] Celik, B., Set, E., Akdemir, A. O., Ozdemir, M. E.: *Novel Generalizations for Gruss type inequalities pertaining to the constant proportional fractional integrals*. Applied and Computational Mathematics. **22**(2), 275-291 (2023).

[15] Yıldız, C., Bakan, E., Donmez, H.: *New general inequalities for exponential type convex function*. Turkish Journal of Science. **8**(1), 11-18 (2023).

[16] Andric, M.: *Fejer type inequalities for (h, g; m)-convex functions*. *TWMS Journal of Pure and Applied Mathematics.* **14**(2), 185-194 (2023).

[17] Desta, H. D., Budak, H., Kara, H.: *New perspectives on fractional Milne-type inequalities: :nsights from twice-differentiable functions*. *Universal Journal of Mathematics and Applications.* **7**(1), 30-37 (2024).

[18] Dragomir, S.: *Refinements and reverses of tensorial and Hadamard product inequalities for Selfadjoint operators in Hilbert Spaces related to Young's result*. Communications in Advanced Mathematical Sciences. **7**(1), 56-70 (2024).

[19] Turker, R. Kavurmacı, Onalan, H.: *Generalized inequalities for Quasi-Convex functions via generalized Riemann-Liouville fractional integrals*. Turkish Journal of Science.**7**(3), 219-230 (2022).

[20] Kırmacı, U. S.: *On some Cauchy type mean-value theorems with applications*. *Communications in Advanced Mathematical Sciences.* **7**(3), 147-156 (2024).

Affiliations

SÜMEYYE ERMEYDAN ÇİRİŞ
ADDRESS: Kahramanmaraş Sütçü İmam University, Dept. of Mathematics, 46000, Kahramanmaraş-Türkiye
E-MAIL: sumeyye_ermeydan@hotmail.com
ORCID ID:0009-0000-2472-5311

HÜSEYIN YILDIRIM
ADDRESS: Kahramanmaraş Sütçü İmam University, Dept. of Mathematics, 46000, Kahramanmaraş-Türkiye
E-MAIL: hyildir@ksu.edu.tr
ORCID ID:0000-0001-8855-9260