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Efficient Computational Techniques for Fractional Order Delay-Integro Differential Equations

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Kesirli Mertebeden Gecikmeli-İntegro Diferansiyel Denklemler için Etkili Yöntemler

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Abstract
In this paper, we present Legendre - collocation method, together with the Gauss–Legendre quadrature integration for solving fractional order delay-integro differential equations (FDIDE) with Caputo fractional derivative. The properties of shifted Legendre polynomials are used to solve the FDIDE to system of equations. The equation system obtained is solved by using Newton iteration method based on our present method with numerical examples is shown both applicability and efficiency of method. The results obtained by the collocation method are compared with exact solution and is shown to be compatible. The Maple and MATLAB programs are used for the calculations required in the study.

Keywords *Fractional Delay Integro Differential Equation; Shifted Legendre Polynomials; Caputo Fractional Derivative; Gauss–Legendre Quadrature; Collocation Method.*

Öz
Bu çalışmada, kesirli mertebeden gecikmeli integro diferansiyel denklemlerin nümerik çözümleri için Gauss-Legendre quadrature integrasyonu ile birlikte Caputo kesirli türevi ve Legendre kolokasyon yöntemi uygulanmıştır. Ötelenmiş Legendre polinomları yardımıyla denklem sistemi elde edilmiş ve kesirli mertebeden gecikmeli-integro diferansiyel denklemleri nümerik olarak çözülmüştür. Elde edilen denklem sistemi Newton iterasyon yöntemi kullanılarak çözülmüştür. Yöntemin uygulanabilirliği ve etkinliği sayısal örneklerle gösterilmiştir. Elde edilen sonuçlar tam çözümler ile karşılaştırılmış ve uyumlu olduğu gösterilmiştir. Tüm hesaplamalar için Maple ve MATLAB programları kullanılmıştır.

Anahtar Kelimeler *Kesirli Gecikmeli İntegro Diferansiyel Denklem; Ötelenmiş Legendre Polinomları; Caputo Kesirli Türevi; Gauss-Legendre Quadrature; Kolokasyon Metodu.*

1. Introduction

The fractional calculus has a quite old history but has been extensively studied over the last decade. The fractional calculus is generalization of normal derivative and integration of fractional order. Fractional differential equations can be used for many physical models. These physical models have been applied in such areas as damping, electromagnetism, viscoelasticity, optimal control problem, diffusion, robotics, heat conduction, acoustics, signal processing, etc. (Balatif et al. 2015, Baleanu et al. 2012, Engheta 1997, Podlubny 1999, Soczkiewicz 2002).

Delay (difference) differential equations are a private kind of differential equations. A delay differential equation is equations that has delayed argument which derivative at any time depend upon the solution at previous times. An integro differential equation is equations which contain both derivatives and integrals of an unknown function. The applications of integro differential equations play an important role in applied mathematics. The exact solutions of some integro differential equations cannot be found, so some numerical methods can be required. Integro differential equations use commonly in many applied branches that physics, biology, mechanics, engineering. There are many different methods for solving integro differential equations. Spectral method (Yousefi et al. 2019), wavelet method (Rajagopal et al. 2020). Delay-integro differential equations are equations involving both derivative and integral operations on an unknown function at the same time having delay argument. Delay and integro differential equations use in different fields such as immunology, physiology, ecology, neural networks, epidemiology, electrodynamics, etc. (Baker et al. 1999, Driver 1997, Marchuk 1997, Hethcote et al. 1989, Waltman 1974). Fractional order differential equations can be used for different phenomena models in fluid-dynamics, models of earthquakes, chemistry, acoustics, diffusion processes and psychology (Baillie 1996, Cushing 1977, Mainardi 1997). FDIDE are quite complex equations. Last decades,

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many numerical methods have been presented to approximate the solution of FDIDE. Taylor method (Bellour and Bousselsal 2014), collocation method (Fazeli and Hojjati 2015, Gu 2020), perturbation method (Panda et al. 2021), Split-step theta method (Liu et al. 2019), spline method (Qin et al. 2018) have been used to solve FDIDE.

In this paper, Legendre collocation method is applied to solving fractional linear and non-linear FDIDE.

$$D^{\alpha} y(x) = Q\big(x, y(x), y(g(x))\big) + \psi\Big(x, y(x), \int_a^x \mathrm{K}\big(t, y(g(t))\big)dt \tag{1}$$

with the initial conditions

$$y(x) = y_0 \tag{2}$$

where $g(x) = ax + b$ in form $a, b \in R$, $0 \le x \le 1$, $0 < \alpha \le 1$. D^{α} represents fractional derivative in the Caputo sense. Q and ψ are represent nonlinear or linear, continuous functions.

This paper is structured as follows: In section 2, we introduce some preliminaries of Legendre polynomials and definitions regarding to Caputo's fractional derivatives and we give the procedure for solving FDIDE and collocate formulated equation at some suitable points together with initial conditions. In section 3, an illustrative example is given. Lastly, a brief conclusion is presented in Section 4.

2. Materials and Methods

Essential definitions preliminaries and definition on fractional calculus are given, in this section.

Definition: (Alipour and Baleanu 2013) The fractional order derivative of $f(x)$ by means of Caputo is given as follows,

$$D^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} y(t)dt \tag{3}$$

for $n-1 < \alpha \le n$, $n \in N$, $t > 0$, $y \in C_{-1}^{n}$

Similar to integer order derivative, Operator of fractional derivative in the Caputo sense is a linear operation (Bhrawy et al. 2015),

$$D^{\alpha}(\sigma f(x) + \psi g(x)) = \sigma \mathrm{D}^{\alpha} f(x) + \psi \mathrm{D}^{\alpha} g(x) \tag{4}$$

$$\mathrm{D}^{\alpha} x^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, n-1 < \alpha < n, \\ \qquad \beta > n-1, \quad \beta \in R. \\ 0 \qquad , n-1 < \alpha < n, \\ \qquad \beta \le n-1, \quad \beta \in \mathrm{N}. \end{cases} \tag{5}$$

The Caputo derivative provides the following

$$D^{\alpha} C = 0, \quad C \text{ is a constant} \tag{6}$$

2.1 The Shifted Legendre Polynomials

The Legendre polynomials are orthogonal polynomials on the interval $[-1,1]$ and written by following recurrence relation (Sokhanvar and Askarı-Hemmat 2015),

$$L_{i+1}(z) = \frac{(2i+1)}{(i+1)} z L_i(z) - \frac{i}{(i+1)} L_{i-1}(z) \quad , i = 1, 2, ...$$

where $L_0(z) = 1$ and $L_1(z) = z$. Change of variable $z = 2x - 1$ is performed and interval $[-1,1]$ is converted to interval $[0,1]$. Shifted Legendre polynomials are denoted by $P_i(x)$. The shifted Legendre polynomials are defined as follows:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)} P_i(z) - \frac{i}{(i+1)} P_{i-1}(x)$$

$P_0(x) = 1$ and $P_1(x) = 2x - 1$. $P_i(x)$ is given in analytical form as following

$$P_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} x^k \tag{7}$$

Note that $P_i(0) = (-1)^i$ and $P_i(1) = 1$. From orthogonality condition

$$\int_0^1 P_i(x) P_j(x) dx = \begin{cases} \dfrac{1}{2i+1}, & \textit{for} \quad i = j \\ 0 \quad , & \qquad\quad i \ne j \end{cases}$$

The function $y(x)$, integrable in interval $[0,1]$. $y(x)$ may be expressed as follows

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x)$$

where coefficients c_j can be given as follows:

$$c_j = (2j+1) \int_0^1 y(x) P_j(x) dx, \qquad j = 1, 2, ...$$

We consider the first $m+1$ terms of the shifted Legendre polynomials (Sokhanvar and Askarı-Hemmat 2015). We can write as follows

$$y_m(x)=\sum_{j=0}^{m}c_jP_j(x)$$

Theorem 1: (Khader and Hendy 2012) Suppose that $y(x)$ is approximated the shifted Legendre polynomials as follows

$y_m(x)=\sum_{j=0}^{m}c_jP_j(x)$, for $0<\alpha\le 1$, then

$$D^{\alpha}(y_m(x))=\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}{}^{(\alpha)}x^{k-\alpha}$$

$\Omega_{i,k}{}^{(\alpha)}$ can be written as follows

$$\Omega_{i,k}{}^{(\alpha)}=\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)}$$

Proof: Fractional derivative in Caputo sense is a linear operation we get

$$D^{\alpha}(y_m(x))=\sum_{i=0}^{m}c_iD^{(\alpha)}(P_i(x)) \tag{8}$$

Substituting equations (4), (5) and (6) in equation (7); we obtain

$$D^{\alpha}(P_i(x))=0,\quad i=0,1,\dots,\lceil\alpha\rceil-1,\quad 0<\alpha\le 1 \tag{9}$$

$$\begin{aligned}D^{\alpha}P_i(x)&=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2}D^{\alpha}x^k\\&=\sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)}x^{k-\alpha}\end{aligned} \tag{10}$$

So, Theorem 1 is proved.

2.2. Legendre-Collocation method

Let we handle the FDIDE given in equation (1). In this section, we will use the Legendre collocation method to solve the FDIDE (1) together with the initial conditions. In order can implement proposed method, approximate solution $y(x)$ can be written as follows:

$$y_m(x)=\sum_{j=0}^{m}c_jP_j(x) \tag{11}$$

Using equations (1), (11) and Theorem 1, we can be written as follows

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}{}^{(\alpha)}x_p{}^{k-\alpha}=Q\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\sum_{j=0}^{m}c_jP_j(g(x_p))\right)+\psi\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\int_a^x\mathrm{K}\left(t,\sum_{j=0}^{m}c_jP_j(g(t))\right)dt\right). \tag{12}$$

where $g(x_p)=ax_p+b$ in form $a,b\in R$. We can collocate equation (12) at $(m+1-\lceil\alpha\rceil)$ points x_p, for $p=0,1,\dots,\ m-\lceil\alpha\rceil$ and rewrite equation (12) as follows:

$$\begin{aligned}&\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}{}^{(\alpha)}x_p{}^{k-\alpha}=\\&Q\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\sum_{j=0}^{m}c_jP_j(g(x_p))\right)+\\&\psi\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\int_a^x\mathrm{K}\left(t,\sum_{j=0}^{m}c_jP_j(g(t))\right)dt\right).\end{aligned} \tag{13}$$

We will use zeros of the shifted Legendre polynomial $P_{m+1-\lceil\alpha\rceil}(x)$ for proper collocation point. We can use the Gauss-Legendre quadrature for integral in equation (13), thus we transform $\lfloor 0,x_p\rfloor$ into $[-1,1]$, use the change of variable

$$\mu=\frac{2}{x_p}t-1$$

Equation (13), for $p=0,1,\dots,\ m-\lceil\alpha\rceil$, if rewritten for μ ,

$$\begin{aligned}&\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}{}^{(\alpha)}x_p{}^{k-\alpha}=\\&Q\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\sum_{j=0}^{m}c_jP_j(g(x_p))\right)+\\&\psi\left(\begin{aligned}&x_p,\sum_{j=0}^{m}c_jP_j(x_p),\\&\frac{x_p}{2}\int_{-1}^{1}\mathrm{K}\left(\frac{x_p}{2}(\mu+1),\sum_{j=0}^{m}c_jP_j\left(g\left(\frac{x_p}{2}(\mu+1)\right)\right)\right)d\tau\end{aligned}\right).\end{aligned} \tag{14}$$

Using the Gauss-Legendre quadrature formula for $p=0,1,\dots,\ m-\lceil\alpha\rceil$, we can rewrite equation (14),

$$\begin{aligned}&\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}{}^{(\alpha)}x_p{}^{k-\alpha}\approx\\&Q\left(x_p,\sum_{j=0}^{m}c_jP_j(x_p),\sum_{j=0}^{m}c_jP_j(g(x_p))\right)+\\&\psi\left(\begin{aligned}&x_p,\sum_{j=0}^{m}c_jP_j(x_p),\\&\frac{x_p}{2}\sum_{j=0}^{m}w_q\mathrm{K}\left(\frac{x_p}{2}(\mu_q+1),\sum_{j=0}^{m}c_jP_j\left(g\left(\frac{x_p}{2}(\mu_q+1)\right)\right)\right)\end{aligned}\right).\end{aligned} \tag{15}$$

where w_q are the correspondent weights of roots of the Legendre polynomial. μ_q are $(m+1)$ roots of the Legendre polynomial $L_{m+1}(t)$ (Saadatmandi and Dehghan 2011). Also, if we substitute equation (11) in the initial condition (2), we can write $\lceil\alpha\rceil$ equations by

$$\sum_{i=0}^{m}(-1)^i c_i = y_0 \tag{16}$$

Equation (15) and together with $\lceil\alpha\rceil$ equation of initial condition, obtain $(m+1)$ linear or non-linear equations. A system of equations is obtained. The equation system obtained is solved by using Newton iteration method for the unknown c_i, $i=0,1,\ldots,m$. As a result, $y(x)$ given in equation (1) may be found.

3. Results and Discussions

The Legendre collocation method was implemented for some fractional delay integro differential equation, in this section. In our implementation, the method was calculated using the Maple and MATLAB. In the examples, we show the applicability of the proposed method.

Example 1: Consider FDIDE,

$$D^{\alpha}y(x)=1-\frac{1}{2}xy(x)+2y(x)+2\int_0^x\left(y\left(\frac{t}{2}\right)\right)^2 dt \tag{17}$$

with the initial conditions

$$y(0)=0 \tag{18}$$

The exact solution, when $\alpha=1$, is $y(x)=xe^x$.

If the present method for $m=3$ is applied and numerical solution as follows,

$$y_3(x)=\sum_{j=0}^{3}c_j P_j(x) \tag{19}$$

Using equation (15) we get

$$\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Omega_{i,k}^{(1.0)}x_p^{\,k-1.0}\approx Q\left(x_p,\sum_{j=0}^{3}c_jP_j(x_p),\sum_{j=0}^{3}c_jP_j(g(x_p))\right)+\psi\left(\begin{array}{l}x_p,\sum_{j=0}^{3}c_jP_j(x_p),\\ \frac{x_p}{2}\sum_{j=0}^{3}w_q\mathrm{K}\left(\frac{x_p}{2}(\mu_q+1),\sum_{j=0}^{3}c_jP_j\left(g\left(\frac{x_p}{2}(\mu_q+1)\right)\right)\right)\end{array}\right).$$

$$p=0,1,2. \tag{20}$$

x_p are zeros of shifted Legendre polynomials $P_3(x)$ and values of roots as follows

$x_0=0.5000000$
$x_1=0.1127017$
$x_2=0.8872983.$

μ_q are roots of the Legendre polynomial $L_4(t)$ and values of roots as follows

$\mu_0=0.861136$, $\mu_1=0.339981$, $\mu_2=-0.339981$, $\mu_3=-0.861136$

w_q are the corresponding weights and their values are:

$w_0=0.3478548$, $w_1=0.6521451$, $w_2=0.6521451$, $w_3=0.3478548$

By using equations (18) and (19) we get:

$$c_0-c_1+c_2-c_3=0 \tag{21}$$

By using equations (20) and (21), we obtain $(m+1)$ equations. Solving together with the equation (20) and equation (21) we find the approximate c_i values for $m=3$.

$c_0=1.000652$, $c_1=1.310492$, $c_2=0.3652749$, $c_3=0.05543499$

From equation (20) and (21), we can find approximate solution $y(x)$

$$y(x)=-0.9\times10^{-7}+1.094555x+0.528599x^2+1.108700x^3$$

This problem is solved for different values of α. We compared exact and approximate solution in the case $\alpha=1,\ 0.90$ when $m=3$. The computational results of $y(x)$ for different α and $m=3$ together with the exact solution at $\alpha=1$ are given in Figure 1 and in Table 1.

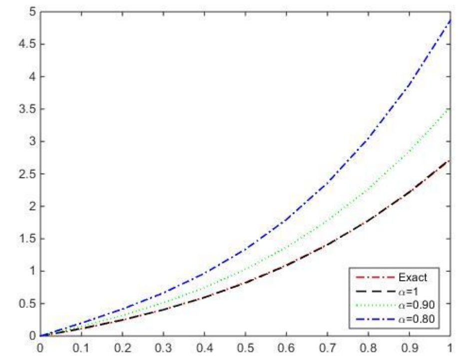

Figure 1. Approximate solutions of $y(x)$ for different values of α and exact solution in Example 1.

Table 1. The comparison numerical and exact solutions when $\alpha = 1, 0.90$, $m = 3$ in Example 1.

		Legendre Collocation Method	
x	Exact Solution	$\alpha = 1$	Absolute Error
0.0	0.00000	0.000000	0.000000
0.1	0.11051	0.115850	0.005340
0.2	0.24428	0.248924	0.004644
0.3	0.40495	0.405875	0.000925
0.4	0.59673	0.593354	0.003376
0.5	0.82436	0.818014	0.006346
0.6	1.09327	1.086508	0.006762
0.7	1.40963	1.405486	0.004144
0.8	1.78043	1.781601	0.001171
0.9	2.21364	2.221507	0.007867
1.0	2.71828	2.731854	0.013524

Example 2: Consider FDIDE,

$$D^{0.5}y(x) - 2y(x) + y(x-1) + \int_0^x y(t)dt = f(x)$$

with the initial conditions

$$y(0) = 0$$

where the source term is defined as $f(x) = \frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} - 2x^2 + (x-1)^2 + \frac{1}{3}x^2$

Exact solution of this example at $\alpha = 0.5$ is $y(x) = x^2$ We found approximate solution of this problem with Legendre collocation method. The results are displayed in Figure 2 and Figure 3. In these figures show comparison of exact solution with approximate solution. In Figure 2, we compare the exact solution and approximate solutions found when $m = 3$. Approximate solutions when $m = 3$ and variable values of α is shown in Figure 3 While it can be seen, approximate solution approaches exact solution when α is close to 0.5.

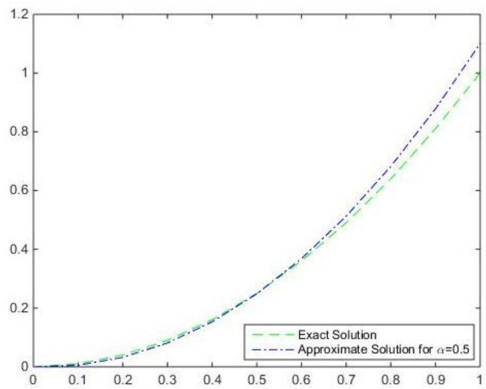

Figure 2. Comparison of exact and approximate solution in Example 2.

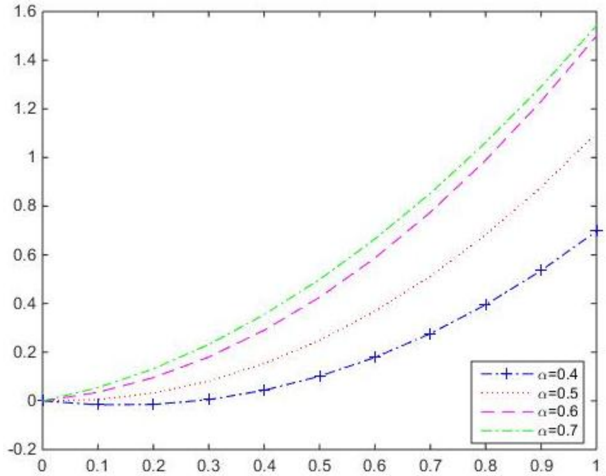

Figure 3. The comparison of $y(x)$ by the proposed method for $m = 3$ and different values of α in Example 2.

Example 3: Consider the FDIDE (AlHabees et al. 2016),

$$D^{0.9}y(x) = \frac{\Gamma(4)}{\Gamma(3.1)}x^{2.1} - \frac{x^2}{5}e^x y(x) - \frac{(0.5)^2}{2}y\left(\frac{x}{2}\right) + \int_0^x te^x y(t)dt + \int_0^{\frac{x}{2}} ty(t)dt$$

with the initial condition

$$y(0) = 0$$

Exact solution is $y(x) = x^3$.

In Table 2, we give the absolute errors for $\alpha = 1$ and $m = 3$. Exact and approximate solutions for variable values of α and $m = 3$ is shown in Figure 4.

Table 2. Exact and approximate solutions when $\alpha = 1, 0.90$., $m = 3$ in Example 3.

		Legendre Collocation Method	
x	Exact Solution	$\alpha = 1$	Absolute Error
0.0	0.00000	0.0000000	0.000000
0.1	0.00100	0.0020914	0.001091
0.2	0.00800	0.0069041	0.001106
0.3	0.02700	0.0207973	0.006202
0.4	0.06400	0.0501302	0.013870
0.5	0.12500	0.1012620	0.023738
0.6	0.21600	0.1805520	0.035451
0.7	0.34300	0.2943593	0.048647
0.8	0.51200	0.4490433	0.062957
0.9	0.72900	0.6509631	0.078760
1.0	1.00000	0.9064780	0.094747

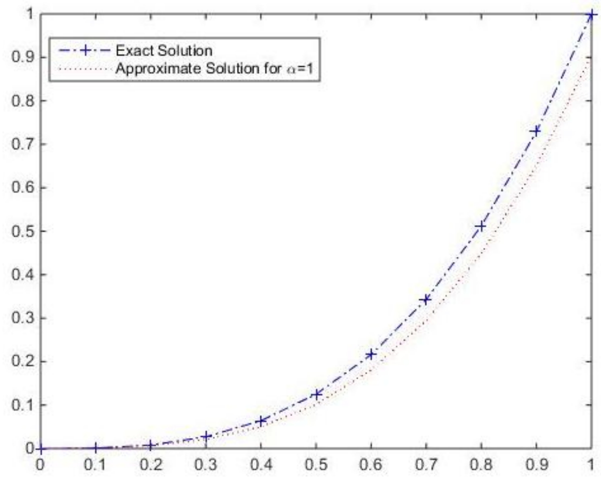

Figure 4. Comparison of exact and approximate solution at $m=3$ in Example 3.

Example 4: Consider the FDIDE (Nemati et al. 2020),

$$D^{\alpha} y(x) = y(x-1) + \int_{x-1}^{x} y(t)dt$$

with the initial conditions

$y(0)=1$

The exact solution when $\alpha=1$, is $y(x)=e^{x}$.

The exact solution of example 4 are compared with approximate solutions in the case $m=3, 4, 5$ when $\alpha=1$, in Figure 5. Help of Figure 5, it may be seen that approximate solution approaches to exact solution while *m* rises. In Figure 6, the exact and approximate solutions found in case $m=5$ for $\alpha=1, 0.90, 0.80$ is displayed.

As can be seen Figure 6 approximate solution approaches to the exact solution the case $\alpha=1$ while α is close to 1.

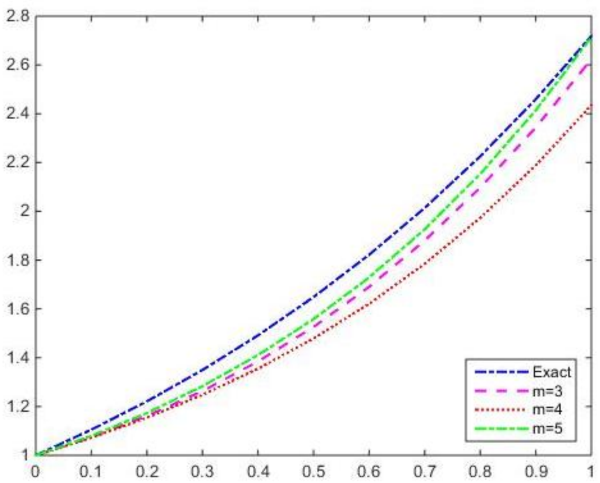

Figure 5. Exact solution and approximate solutions for $\alpha=1$ and $m=3, 4, 5$ in Example 4.

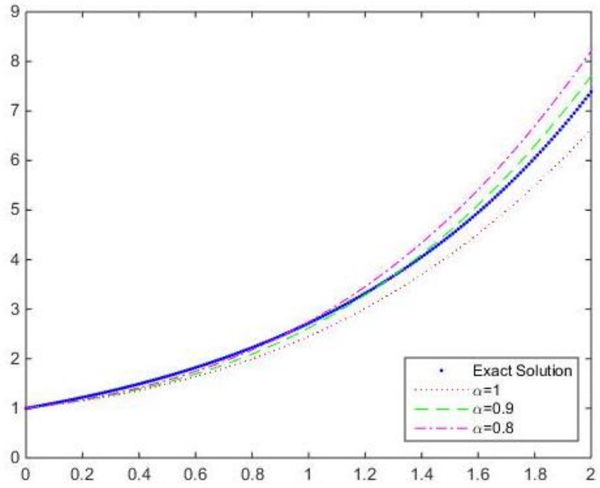

Figure 6. Exact solutions and approximate solutions for $\alpha=1, 0.90, 0.80$ in Example 4.

4. Conclusions

In this study, we introduced Legendre collocation method for solving fractional delay integro differential equation. We transformed these equations into a system of algebraic equations using the properties of shifted Legendre polynomials and the Gauss-Legendre quadrature rule. We computed the unknown coefficients for solving obtained system. There are many methods used in the literature to solve fractional delay integro differential equation. Legendre collocation method may be used for linear and nonlinear fractional delay integro differential equations. We showed the efficiency and accuracy of the method with four numerical examples. Numerical results are compared with exact solution and approximate solution. The numerical examples demonstrate advantage of using the Legendre collocation method for solving the fractional delay integro differential equation. From the table and figure, it has seen that the present method gives good results. By increasing the m values, we can increase the accuracy rate for the fractional delay integro differential equation. From results the obtained in examples, we may conclude that the present method gives a result close to the exact solution. The numerical results were computed using the Maple and MATLAB software's.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement
Author 1: Research, Analysis, Writing, Figures, Data, Review and Editing.
Author 2: Research, Analysis, Writing, Supervision.

Declaration of Competing Interest
The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement
All data generated or analyzed during this study are included in this published article.

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