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Inference for two Weibull populations under joint generalized progressive type-I hybrid censoring with a simulation study and applications

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Abstract

The generalized progressive censoring scheme has been considered one of the most general cases of censoring schemes. In this study, we consider two Weibull populations under a jointly generalized progressive hybrid censoring scheme as a more flexible extension of the exponential distribution. The methods presented in this paper let experimenters evaluate life testing studies in the case of the most generalized censoring scheme based on a flexible distribution that has increasing, constant, and decreasing failure rates. The maximum likelihood method is used to obtain point estimates of the unknown parameters and the corresponding approximate confidence intervals by using asymptotic theory and bootstrap sampling. The Bayesian inferences are handled under informative and non-informative priors. The highest posterior density credible intervals are also obtained for the Bayesian estimations. We further obtained results with a challenging task an optimal censoring scheme using the A-optimality, D-optimality, and F-optimality criterion to let researchers determine the optimal censoring plan before conducting experiments or collecting data. Following the numerical results within this paper, A-optimality and D-optimality proposed the same scheme, while F-optimality proposed a scheme similar to them. In the last part of the study, we provide simulation studies under different censoring plans and use a numerical example to exemplify the theoretical outcomes. It is observed that the best estimation performances are obtained by informative Bayesian methods.

Mathematics Subject Classification (2020). 60E05, 62F15, 62N01, 62N02, 62F10

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1. Introduction

Due to advanced technological improvements and customer expectations on product quality, it is very rare to get a significant number of failures in a short period of time. This results in a challenging task to efficiently collect sufficient failure time data, within a limited time, from a life testing experiment. Therefore, censoring is widely used in reliability engineering. Different censoring schemes are available in the literature for various life testing experiments. Time censoring (Type-I) and failure censoring (Type-II) are the most basic censoring schemes. Whereas, in time censoring, we pre-fixed the experimental time and in the failure censoring scheme, we pre-determined the number of failures, say, m, before starting the experiment.

In particular, if the experimenters want to remove the experimental units during the test, then a progressive censoring scheme is a good choice for such cases. The progressive Type-II censoring scheme can be briefly described as follows. Suppose, n number of items are put in a life-testing experiment. Also, before the experiment, we prefixed an integer $m < n$. Further, the progressive censoring plans $R = (R_1, R_2, \cdots, R_m)$ with $R_i \geq 0$ are pre-specified in such a way that it satisfies the linear equation $n = m + \sum_{i=1}^{m} R_i$. Now, R_1 surviving units are removed from the experiment at the time of the first failure. Again, at the time of the second failure, R_2 number of surviving units is removed from the remaining $(n - R_1 - 1)$ units and the experiment continues till the m^{th} failure occurs. Finally, the test terminates at the time of the m^{th} failure, when all remaining surviving units are excluded from the test. For more details on the progressive censoring scheme, one can see [7] and the citations therein. The major drawback of this censoring scheme is to get m failure at a reasonable time, for highly reliable products.

Kundu and Joarder [26] introduced progressive hybrid censoring schemes which is beneficial over the progressive Type-II censoring scheme. Although, there may be some cases where very few failures may occurred before the pre-fixed time point T. To overcome such a scenario, Cho et al. [12] proposed a generalized progressive hybrid censoring scheme (GPHCS) which always ensures a fixed number of failures at the end of the experiment as in progressive Type-II censoring. We now describe this scheme as follows.

Suppose n identical units are put in a lifetime experiment in which the associated lifetimes are described by independent and identically distributed (i.i.d.) random variables $(X_1, X_2, \cdots, X_n)$. Further, suppose pre-fixed a time point T and pre-determined integers k, r such that $1 \leq k < r \leq n$. Also, a predetermined censoring plan, $(R_1, R_2, \cdots, R_r)$, as described in the above for progressive Type-II censoring scheme. Now, at the time of the first failure, say $X_{1:r:n}$, remove R_1 surviving units randomly from the experiment. Similarly, at second failure, $X_{2:r:n}$, remove R_2 units from remaining $(n - R_1 - 2)$ surviving units and continues until the termination time

$$T^* = \max\{X_{k:r:n}, \min\{T, X_{r:r:n}\}\},$$

occurred and removed all the remaining units from the experiment (see Figure 1). Therefore, the possible types of failure times are observed in the following pattern

$$\begin{aligned} &X_{1:r:n}, X_{2:r:n}, \cdots, X_{k:r:n}, \quad \text{if} \quad T < X_{k:r:n} < X_{r:r:n} && \text{(Case-I)} \\ &X_{1:r:n}, X_{2:r:n}, \cdots, X_{D:r:n}, \quad \text{if} \quad X_{k:r:n} < T < X_{r:r:n} && \text{(Case-II)} \\ &X_{1:r:n}, \cdots, X_{k:r:n}, \cdots, X_{r:r:n}, \quad \text{if} \quad X_{k:r:n} < X_{r:r:n} < T, && \text{(Case-III)} \end{aligned}$$

where D denotes the number of observed failures until pre-determined time point T such that $X_{D:r:n} < T < X_{D+1:r:n}$. Notice that in this censoring scheme, we can always get k failures at termination time T^*, which was the main drawback of the progressive hybrid censoring scheme. Some recent contributions on GPHCS can be found in the papers by [23], [17], [29], [5], [21] and [42].

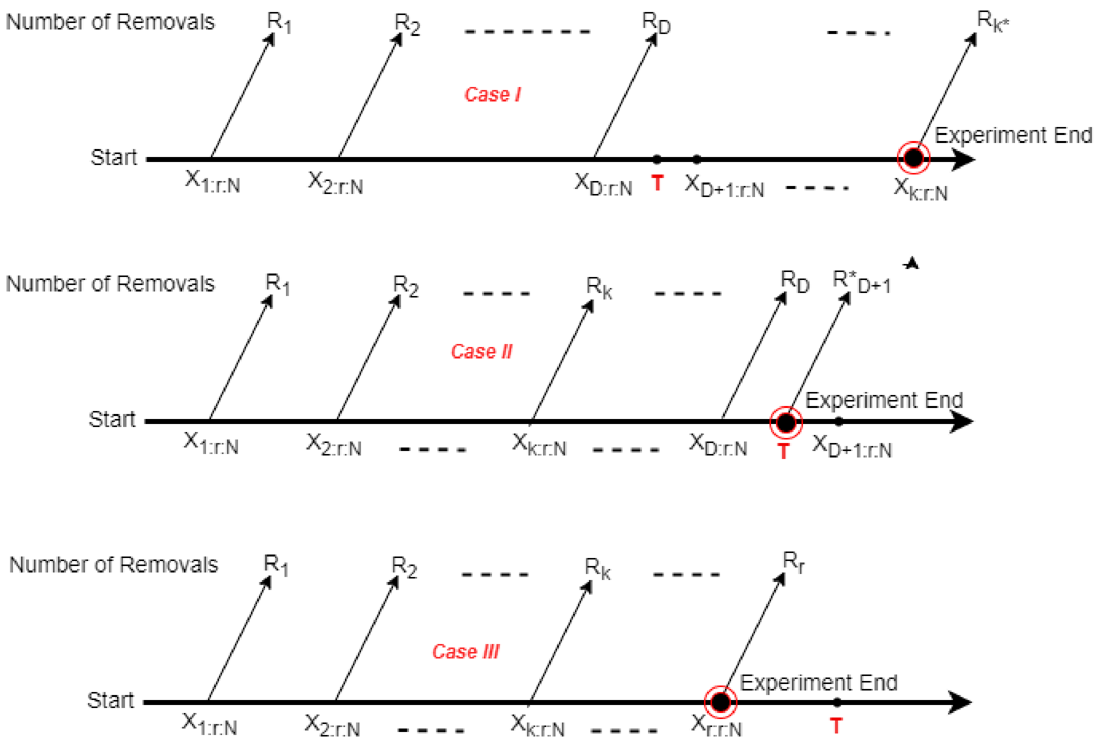

Figure 1. Schematic representation of generalized progressive hybrid censoring scheme.

Instead of a single sample progressive censoring scheme, Balakrishnan and Rasouli [8] proposed a joint Type-II censoring scheme for two exponential populations, they have shown that the usefulness of the joint censoring schemes compared to single sample censoring schemes. Recently, in the literature, many authors considered different kinds of joint censoring schemes like [9] proposed joint Type-II censoring for k-exponential populations. Ashour and Abo-Kasem [6] handled joint progressive Type-I censored scheme for two exponential populations. Doostparast et al. [16] studied Bayesian estimations for joint progressive Type-II censored data. Krishna and Goel [25] studied jointly Type-II censored Lindley distributions. Rasouli et al. [34] proposed the joint progressive Type-II censoring scheme, Mondal and Kundu [30] studied the Weibull distribution based on the balanced joint progressive censoring scheme. The balanced adaptive progressive censoring scheme proposed by [39], has the advantage of having lower experimental time over the balanced joint progressive censoring scheme proposed by [31]. Abo-Kasem et al. [1] studied a new two sample generalized Type-II hybrid censoring scheme and Elshahhat et al. [18] handled a new jointly hybrid censored Rayleigh populations. Almuhayfith [4] studied joint adaptive progressive type-II censored under comparative generalized inverted exponential distributions, Shi and Gui [37] handled two Gompertz populations under a balanced joint progressive Type-II censoring scheme, Rahman et al. [33] and Alam et al. [3] studied multiple censoring approaches under partially accelerated life test plans. Recently, Çetinkaya et al. [11] proposed a jointly generalized progressive hybrid censoring scheme (J-GPHCS) under two exponential distributions and made some inferential results based on maximum likelihood and the Bayesian inference methods. Following this paper, joint Type-II generalized progressive hybrid censoring scheme is studied by [35].

It is known that exponential distribution is important for the analysis of lifetime which has constant hazard. However, decreasing and increasing failure rates have a great deal in reliability analysis. The flexibility of the Weibull distribution let the experimenters apply life tests under increasing and decreasing hazards with also constant hazard. On the other hand, the Weibull distribution is an expansion of the exponential distribution and it can be reduced the exponential model when its shape parameter equal to 1. Therefore, the idea of J-GPHCS for two exponential distributions can be extended for Weibull distributions to put forward to provide more flexible censoring plan.

In this paper, we consider two independent Weibull populations with different shape and scale parameters when the data is coming from joint generalized progressive Type-I hybrid censoring scheme (J-GPHCS). In Section 2, we have described the model and obtained the maximum likelihood estimators (MLEs) of the unknown model parameters in Section 3. In Section 4, we also obtain the asymptotic confidence intervals as well as bootstrap confidence intervals (ACI) for the unknown model parameters. We further consider the Bayes estimators, using the squared error loss function (SELF), of the unknown parameters under the assumptions of independent gamma priors in Section 5. Furthermore, in section 6, the optimal censoring plan has been calculated by using different optimality criteria. To assess the efficiency of the estimates, simulation studies are performed in Section 7. Also, we illustrate the proposed methods through one real-life data analysis in the same section, while Section 8 ends with some concluding remarks.

2. Model Description

In the framework of the proposed study, we assume that $X = (X_1, X_2, \cdots, X_m)$ are independently and identically distributed (i.i.d.) lifetimes of the test units of Sample-I of size m, similarly, $Y = (Y_1, Y_2, \cdots, Y_n)$, test units from Sample-II of size n, are also i.i.d. and X_i's and Y_j's are independently distributed. We assume that X and Y have Weibull distributions such as $WE(\alpha, \beta_1)$ and $WE(\lambda, \beta_2)$. Then, the probability density function (pdf) and cumulative distribution function (cdf) based on Sample-I and Sample-II are given by

$$f(x; \alpha, \beta_1) = \alpha\beta_1 x^{\beta_1 - 1} e^{-\alpha x^{\beta_1}}, \quad F(x; \alpha, \beta_1) = 1 - e^{-\alpha x^{\beta_1}}$$
$$g(y; \lambda, \beta_2) = \lambda\beta_2 y^{\beta_2 - 1} e^{-\lambda y^{\beta_2}}, \quad G(y; \lambda, \beta_2) = 1 - e^{-\lambda y^{\beta_2}}$$

where $x > 0, y > 0$ and $\alpha, \lambda, \beta_1, \beta_2 > 0$.

Then, it is assumed that $W_{(1:\eta:N)}, W_{(2:\eta:N)}, \cdots, W_{(\eta:\eta:N)}$ denotes the combined ordered samples form of test samples (X_i, Y_j) where $N = m + n$ and η is the number of observed failures. Thus, J-GPHCS can be described as follows. We assume T to be the prefixed time point and also to have fixed two integers r, k such that $1 \le k < r \le N$ in advance. Firstly, at the time of the first failure (which may be from sample I or sample II), the R_1 surviving units are randomly withdrawn from the remaining $(N-1)$ surviving units. Similarly, at the time of the second failure (either from Sample-I or Sample-II), R_2 units are randomly removed from the remaining $(N - R_1 - 2)$ surviving units, and this scheme continues until the termination point $T^* = \max\Big\{W_{k:r:N}, \min\{T, W_{r:r:N}\}\Big\}$ has arrived. Under the J-GPHCS the possible values of T^* would be

$$T^* = \begin{cases} W_{k:r:N} & ,\text{if} \quad T < W_{k:r:N} < W_{r:r:N}, \\ T & ,\text{if} \quad W_{k:r:N} < T < W_{r:r:N}, \\ W_{r:r:N} & ,\text{if} \quad W_{k:r:N} < W_{r:r:N} < T, \end{cases}$$

and the corresponding number of observed failures are $\eta = k$ for Case-I, $\eta = D$ for Case-II and $\eta = r$ for Case-III where D denote the number of observed failures until pre-determined time point T for Case-II such that $W_{D:r:N} < T < W_{D+1:r:N}$. The schematic representation of the J-GPHCS model can be seen in Figure 2.

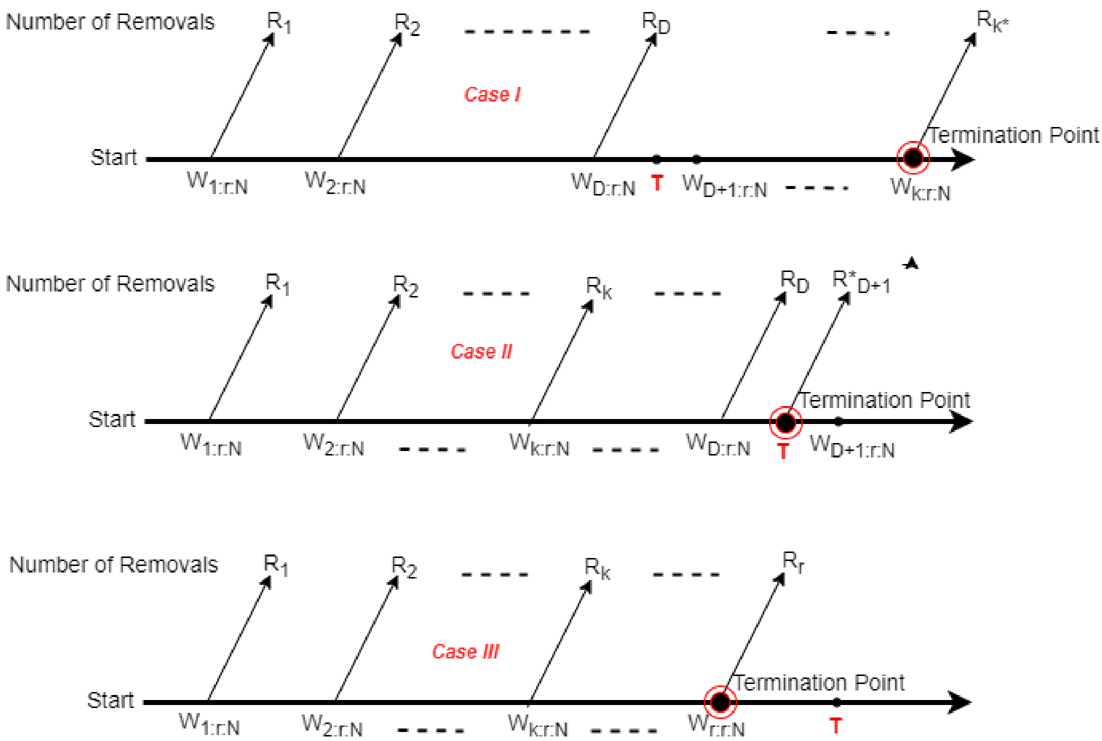

Figure 2. Schematic representation of the J-GPHCS.

In this jointly censoring scheme, the total number of failures η for different cases and the progressive censoring scheme $R_1, R_2, \cdots, R_\eta$ are pre-specified. Furthermore, suppose $R_i = s_i + q_i$ $i = 1, 2, \cdots, \eta$, where s_i and q_i denote the number of units withdrawn at the time of $i-$th failure that belongs to Sample-I and Sample-II, respectively. Note that s_i and q_i are random variables. Therefore, $R = (R_1, R_2, \cdots, R_\eta)$ can be decomposed as $R = S + Q$, where $S = (s_1, s_2, \cdots, s_\eta)$ and $Q = (q_1, q_2, \cdots, q_\eta)$. In particular, if $\eta = r$ then the J-GPHCS reduces to a joint Type II progressive censoring scheme established by [34]. Let $Z = (Z_1, Z_2, \cdots, Z_\eta)$ for which Z_i, $(\forall i = 1, 2, \cdots, \eta)$ takes two values either 1 or 0 depending on W_i from Sample-I or Sample-II, respectively. Then, under J-GPHCS the likelihood function of (W, Z, R) based on the parameters $\theta = (\alpha, \lambda, \beta_1, \beta_2)$ where can be written as

$$L_W(\theta) = C \begin{cases} \prod_{i=1}^{k} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} & \text{, Case-I} \\ \prod_{i=1}^{D} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} \left[\bar{F}(T)\right]^{R_S^*} \left[\bar{G}(T)\right]^{R_Q^*} & \text{, Case-II} \\ \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} & \text{, Case-III,} \end{cases}$$

where $\bar{F} = 1 - F$, $\bar{G} = 1 - G$, $D_1 = \sum_{i=1}^{D} z_i$, $D_2 = \sum_{i=1}^{D}(1 - z_i)$, $\sum_{i=1}^{\eta} s_i + \sum_{i=1}^{\eta} q_i = \sum_{i=1}^{\eta} R_i$, $R_S^* = m - D_1 - \sum_{i=1}^{D} s_i$, $R_Q^* = n - D_2 - \sum_{i=1}^{D}(R_i - s_i)$ and constant

$$C = \begin{cases} \prod_{j=1}^{k} \left[\sum_{i=j}^{r}(s_i + z_i)\right]\left[\sum_{i=j}^{r}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-I} \\ \prod_{j=1}^{D} \left[\sum_{i=j}^{r}(s_i + z_i)\right]\left[\sum_{i=j}^{r}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-II} \\ \prod_{j=1}^{r} \left[\sum_{i=j}^{r}(s_i + z_i)\right]\left[\sum_{i=j}^{r}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-III.} \end{cases}$$

On the other hand, if we take $s_i = 0$, $q_i = 0$ for $i = 1, 2, \cdots, m-1$ and $s_\eta = m - \sum_{i=1}^{\eta} z_i$, $q_\eta = n - \sum_{i=1}^{\eta}(1 - z_i)$ i.e., if we will not remove any surviving items in between the test, only the removals can occur at the stopping time of the test, then we obtain the model proposed by [38].

3. Maximum Likelihood Estimation

This section deals with the classical maximum likelihood estimations of unknown model parameters of the jointly censored Weibull distributions under J-GPHCS. From the expression of $L_W(\theta)$ the likelihood function of $\theta = (\alpha, \lambda, \beta_1, \beta_2)$ is obtained as

$$L(\theta|\mathbf{w},\,\mathbf{z}) \propto \alpha^{D_1}\lambda^{D_2}\beta_1^{D_1}\beta_2^{D_2}e^{\beta_1\sum_{i=1}^{\eta} z_i\ln(w_i)}e^{\beta_2\sum_{i=1}^{\eta}(1-z_i)\ln(w_i)}e^{-\alpha\xi(\beta_1)-\lambda\xi(\beta_2)} \tag{3.1}$$

where $D_1 = \sum_{i=1}^{\eta} z_i$, $D_2 = \eta - \sum_{i=1}^{\eta} z_i$,

$$\xi(\beta_1) = \sum_{i=1}^{\eta}(z_i + s_i)w_i^{\beta_1} + \delta T^{\beta_1}R_S^* \quad \text{and} \quad \xi(\beta_2) = \sum_{i=1}^{\eta}(1 - z_i + q_i)w_i^{\beta_2} + \delta T^{\beta_2}R_Q^*$$

where $\delta = 1$ for Case-II or $\delta = 0$, otherwise. Thus, the log-likelihood function may then be written as

$$\begin{aligned}\ell(\theta|\mathbf{w},\,\mathbf{z}) \propto\ & D_1\big(\log(\alpha) + \log(\beta_1)\big) + D_2\big(\log(\lambda) + \log(\beta_2)\big) + \beta_1\sum_{i=1}^{\eta} z_i\ln(w_i)\\ &+ \beta_2\sum_{i=1}^{\eta}(1 - z_i)\ln(w_i) - \alpha\xi(\beta_1) - \lambda\xi(\beta_2).\end{aligned} \tag{3.2}$$

We take derivatives of $\ell(\theta|\mathbf{w},\,\mathbf{z})$ with respect to θ_i for $i = 1, 2, 3, 4$ and we obtain

$$\begin{aligned}\frac{\partial\ell(\theta|\mathbf{w},\,\mathbf{z})}{\partial\alpha} &= \frac{D_1}{\alpha} - \xi(\beta_1),\\ \frac{\partial\ell(\theta|\mathbf{w},\,\mathbf{z})}{\partial\beta_1} &= \frac{D_1}{\beta_1} + \sum_{i=1}^{\eta} z_i\ln(w_i) - \alpha\xi'(\beta_1),\\ \frac{\partial\ell(\theta|\mathbf{w},\,\mathbf{z})}{\partial\lambda} &= \frac{D_2}{\lambda} - \xi(\beta_2) = 0,\\ \frac{\partial\ell(\theta|\mathbf{w},\,\mathbf{z})}{\partial\beta_2} &= \frac{D_2}{\beta_2} + \sum_{i=1}^{\eta}(1 - z_i)\ln(w_i) - \alpha\xi'(\beta_2),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\xi'(\beta_1) &= \sum_{i=1}^{\eta}(s_i + z_i)w_i^{\beta_1}\ln(w_i) + \delta T^{\beta_1}\ln(T)R_S^*,\\ \xi'(\beta_2) &= \sum_{i=1}^{\eta}(q_i + 1 - z_i)w_i^{\beta_2}\ln(w_i) + \delta T^{\beta_2}\ln(T)R_Q^*.\end{aligned}$$

Remark 3.1. If it is assumed that $X = (X_1, X_2, \cdots, X_m)$ are i.i.d. by the Weibull(α, β_1) and $Y = (Y_1, Y_2, \cdots, Y_n)$ are i.i.d. by Weibull(λ, β_2), then under the J-GPHCS, the MLEs of α and λ for fixed β_1 and β_2, denoted by $\hat{\alpha}$ and $\hat{\lambda}$, are exist and given by

$$\hat{\alpha} = \frac{D_1}{\xi(\beta_1)} \quad \text{and} \quad \hat{\lambda} = \frac{D_2}{\xi(\beta_2)}.$$

Remark 3.2. If it is assumed that $X = (X_1, X_2, \cdots, X_m)$ are i.i.d. by the Weibull(α, β_1) and $Y = (Y_1, Y_2, \cdots, Y_n)$ are i.i.d. by the Weibull(λ, β_2), then under the J-GPHCS, the MLEs of β_1 and β_1, denoted by $\hat{\beta_1}$ and $\hat{\beta_2}$ are exists and can be obtained with the solutions of the two following non-linear equations separately

$$\begin{aligned}H(\beta_1) &= \frac{D_1}{\beta_1} + \sum_{i=1}^{\eta} z_i\ln(w_i) - D_1\frac{\xi'(\beta_1)}{\xi(\beta_1)} = 0,\\ H(\beta_2) &= \frac{D_2}{\beta_2} + \sum_{i=1}^{\eta}(1 - z_i)\ln(w_i) - D_2\frac{\xi'(\beta_2)}{\xi(\beta_2)} = 0.\end{aligned} \tag{3.4}$$

These non-linear equations $H(\beta_1)$ and $H(\beta_2)$ cannot be solved analytically and an iterative method is needed to solve such equations, such as the Newton-Raphson method.

The proofs of the existence and uniqueness of θ_i for $i = 1, 2, 3, 4$ are presented in the Appendix.

4. Approximate Confidence Intervals for MLEs

In this section, we discuss the asymptotic confidence interval and bootstrap confidence intervals for the unknown model parameter θ.

4.1. Asymptotic Confidence Interval

In this section, based on the asymptotic normality results of the MLEs, we provide asymptotic confidence intervals for the MLEs of the unknown parameters by using the observed Fisher information matrix. We assume the parameter vector $\theta = (\alpha, \lambda, \beta_1, \beta_2)$. Then, the observed Fisher information matrix is given by

$$F = (f_{ij}) = -\left(\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right) = -\begin{pmatrix} \frac{\partial\ell^2}{\partial\alpha^2} & \frac{\partial\ell^2}{\partial\alpha\partial\lambda} & \frac{\partial\ell^2}{\partial\alpha\partial\beta_1} & \frac{\partial\ell^2}{\partial\alpha\partial\beta_2} \\ & \frac{\partial\ell^2}{\partial\lambda^2} & \frac{\partial\ell^2}{\partial\lambda\partial\beta_1} & \frac{\partial\ell^2}{\partial\lambda\partial\beta_2} \\ & & \frac{\partial\ell^2}{\partial\beta_1^2} & \frac{\partial\ell^2}{\partial\beta_1\beta_2} \\ & & & \frac{\partial\ell^2}{\partial\beta_2^2} \end{pmatrix}_{(\hat{\alpha},\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2)} \tag{4.1}$$

$$= \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ & f_{22} & f_{23} & f_{24} \\ & & I_{33} & f_{34} \\ & & & f_{44} \end{pmatrix}$$

The elements of the Fisher information matrix are obtained as in the following

$$f_{11} = \frac{D_1}{\alpha^2}, \quad f_{22} = \frac{D_2}{\lambda^2}, \quad f_{13} = f_{31} = \xi^{'}(\beta_1), \quad f_{24} = f_{42} = \xi^{'}(\beta_2)$$

$$f_{33} = \frac{D_1}{\beta_1^2} - \alpha\xi^{''}(\beta_1), \quad f_{44} = \frac{D_2}{\beta_1^2} - \lambda\xi^{''}(\beta_2)$$

where

$$\xi^{''}(\beta_1) = \sum_{i=1}^{\eta}(s_i + z_i)w_i^{\beta_1}\ln^2(w_i) + \delta T^{\beta_1}\ln^2(T)R_S^*$$

$$\xi^{''}(\beta_2) = \sum_{i=1}^{\eta}(q_i + 1 - z_i)w_i^{\beta_2}\ln^2(w_i) + \delta T^{\beta_2}\ln^2(T)R_Q^*$$

or $f_{ij} = 0$, otherwise. Then, the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2)^T$ is given by $\hat{\theta} - \theta \sim N_4(0, F^{-1})$. Therefore, the $100(1-\gamma)\%$ ACI of θ_i is given by

$$\theta_i \pm Z_{1-\frac{\gamma}{2}}\sqrt{v_{ii}},$$

where v_{ii} is the $(i,i)^{th}$ element of the inverse of the observed Fisher information matrix and z_γ denotes the 100γth percentile of the standard normal distribution $N(0,1)$.

4.2. Bootstrap Confidence Intervals

This section deals with bootstrap confidence intervals as alternatives to ACIs. For this purpose, we obtain the parametric percentile bootstrap method (boot-p) and the studentized bootstrap method (boot-t) to obtain approximate confidence intervals for the MLEs of the unknown parameters. The corresponding algorithms defined by [40] can be described as follows. For boot-p confidence intervals

Step 1: Generate a sample $\mathbf{X}$ of size m from Weibull(α, β_1) and sample $\mathbf{Y}$ of size n from Weibull(λ, β_2) respectively.

Step 2: Evaluate the censored sample $\{w_1, w_2, \cdots, w_\eta; z_1, z_2, \cdots, z_\eta\}$ where η is the number of failures for different cases of J-GPHCS. Then, calculate the MLEs of $\hat{\theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2)$ using the remarks 3.1-3.2.

Step 3: Generate bootstrap samples X_i^* and Y_j^* for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ based on Weibull$(\hat{\alpha}, \hat{\beta}_1)$ and Weibull$(\hat{\lambda}, \hat{\beta}_2)$, respectively. Then compute the bootstrap estimates of θ which is denoted by $\hat{\theta}^*$.

Step 4: Repeat steps 2-3 for B times and obtain bootstrap estimates $\hat{\theta}_j^*$ for $j = 1, 2, \cdots, B$. For boot-p intervals;

Step 5: Let $G_1(\Delta) = P(\hat{\theta}_i^* \leq \Delta)$, for $i = 1, 2$, be the cumulative distribution function (CDF) of $\hat{\theta}_i^*$. Define $\hat{\theta}_{i_{Boot-p}} = G_1^{-1}(\Delta)$ for a given Δ. Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2$, is given by

$$\left(\hat{\theta}^{*(\gamma/2)}_{i_{Boot-p}}, \hat{\theta}^{*(1-\gamma/2)}_{i_{Boot-p}}\right).$$

where $\hat{\theta}^{*(\gamma)}_{i_{Boot-p}}$ is the γ percentile of $\hat{\theta}_i^*$, $i = 1, 2, \cdots, B$. For boot-t intervals;

Step 5: Compute $Var(\hat{\theta}_i^*)$ from the inverse of the observed Fisher information matrix.

Step 6: Calculate $\Psi^* = \frac{\hat{\theta}_i^* - \hat{\theta}_i}{\sqrt{Var(\hat{\theta}_i^*)}}$, $i = 1, 2, 3, 4$.

Step 7: Let $G_2(\Delta) = P(\Psi^* \leq \Delta)$ be the CDF of Ψ^*. From the values obtained Ψ^*, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2, 3, 4$ can be obtained as follows. For a given Δ, define

$$\hat{\theta}_{iBoot-t} = \hat{\theta}_i + G_2^{-1}(\Delta)\sqrt{Var(\hat{\theta}_i)}.$$

Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2$, is given by

$$\left(\hat{\theta}^{*(\gamma/2)}_{i_{Boot-t}}, \hat{\theta}^{*(1-\gamma/2)}_{i_{Boot-t}}\right).$$

where $\hat{\theta}^{*(\gamma)}_{i_{Boot-t}}$ is the γ percentile of $\hat{\theta}^*_{i_{Boot-t}}$, $i = 1, 2, \cdots, B$.

5. Bayesian Inference

In this section, the Bayesian inference of the unknown parameters is given as an alternative method to the likelihood inference. In this purpose, we assume that the parameters $\theta = (\alpha, \lambda, \beta_1, \beta_2)$ are random variables and we can assume vague priors such as $\pi(\theta_i) \propto 1/\theta_i$, $i = 1, \cdots, 4$ as non-informative prior distributions. On the other hand, the gamma distribution is versatile for adjusting different shapes of the density function. It has a log-concave density function in the interval $(0, \infty)$. Due to the importance of the gamma distributions, we have chosen gamma priors to obtain Bayes estimates in the informative case. Further, Jeffery's prior can be obtained as a special case of the gamma prior.

5.1. Non-Informative Prior

In this case, we prefer to use the following vague priors for the unknown parameters

$$\pi(\alpha, \lambda, \beta_1, \beta_2) \propto \frac{1}{\alpha\lambda\beta_1\beta_2}$$

which are appropriate [22]. Then, the joint posterior distribution for α, λ,, β_1 and β_2 is given by

$$L(\theta, \mathbf{w}, \mathbf{z}) = \mathbf{L}(\mathbf{w}, \mathbf{z}|\alpha, \lambda, \beta_{\mathbf{1}}, \beta_{\mathbf{2}})\pi(\alpha)\pi(\lambda)\pi(\beta_{\mathbf{1}})\pi(\beta_{\mathbf{2}}),$$

and the joint posterior density of α, λ, β_1 and β_2 given data is obtained by

$$L(\theta|\mathbf{w},\mathbf{z}) = \frac{L(\mathbf{w},\mathbf{z}|\alpha,\lambda,\beta_1,\beta_2)\pi(\alpha)\pi(\lambda)\pi(\beta_1)\pi(\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{w},\mathbf{z}|\alpha,\lambda,\beta_1,\beta_2)\pi(\alpha)\pi(\lambda)\pi(\beta_1)\pi(\beta_2)d\alpha\, d\lambda\, d\beta_1\, d\beta_2} \tag{5.1}$$

Then, the Bayes estimate of θ in the case of the vague prior, denoted by θ^{B_0}, under the squared error loss function, can be obtained as the mean of the posterior function in equation (5.1) as given in the following

$$E(\theta|\mathbf{w},\mathbf{z}) = \int_0^\infty \cdots \int_0^\infty \theta_i\; L(\theta_i|\mathbf{w},\mathbf{z})d\theta_i \tag{5.2}$$

Thus, the joint posterior density function of α, λ,, β_1 and β_2 is obtained as

$$\begin{aligned} L(\theta|\mathbf{w},\,\mathbf{z}) \propto\; & \alpha^{D_1-1}\lambda^{D_2-1}\beta_1^{D_1-1}\beta_2^{D_2-1}e^{-\alpha\xi(\beta_1)-\lambda\xi(\beta_2)} \\ & \times e^{\beta_1\sum_{i=1}^{\eta} z_i\ln(w_i)}e^{\beta_2\sum_{i=1}^{\eta}(1-z_i)\ln(w_i)}. \end{aligned} \tag{5.3}$$

Then, the conditional posterior density functions of the parameters can be obtained easily as in the following

$$\begin{aligned} \pi(\alpha|\beta_1) &\propto GA\Big(D_1,\xi(\beta_1)\Big), \\ \pi(\lambda|\beta_2) &\propto GA\Big(D_2,\xi(\beta_2)\Big), \\ \pi(\beta_1|\alpha) &\propto \beta_1^{D_1-1}\exp\Big\{\beta_1\sum_{i=1}^{\eta} z_i\ln(w_i) - \alpha\xi(\beta_1)\Big\}, \\ \pi(\beta_2|\lambda) &\propto \beta_2^{D_2-1}\exp\Big\{\beta_2\sum_{i=1}^{\eta}(1-z_i)\ln(w_i) - \lambda\xi(\beta_2)\Big\}, \end{aligned} \tag{5.4}$$

where GA denoted the gamma distribution with rate parameter. It is seen from (5.4) that the samples for α and λ can be easily obtained by using gamma densities. However, the densities from $\pi(\beta_1|\alpha)$ and $\pi(\beta_2|\lambda)$ can not be reduced analytically to well-known distributions and therefore it is not possible to generate samples directly by standard methods. We observed that the density plots of the conditional posterior densities of β_1 and β_2 are like to the Gaussian distribution as seen in Figure 3.

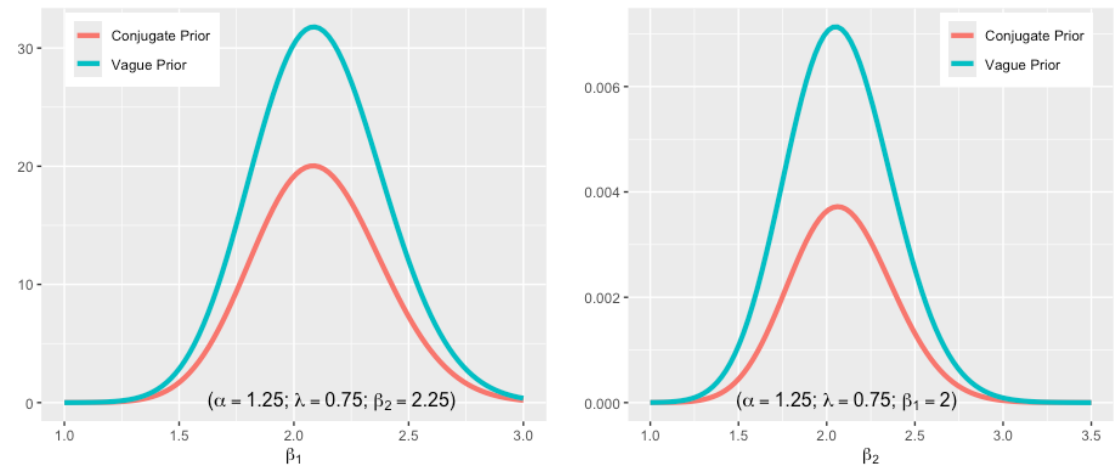

Figure 3. Plots for the conditional posterior probability density function of β_1 and β_2 in the case of $m = n = 35$ and CS-II.

Hence, the Metropolis-Hasting (M-H) algorithm with normal proposal distribution as suggested by [19] can be used for the Bayesian estimation of θ. The algorithm for Gibbs sampling with the M-H method can be described as follows:

Step 1: Start by using the initial values of $(\alpha^{(0)}, \lambda^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$
Step 2: Set $t = 1$
Step 3: Generate $\alpha^{(t)}$ from $GA\Big(D_1, \xi(\beta_1^{(t)})\Big)$.
Step 4: Generate $\lambda^{(t)}$ from $GA\Big(D_2, \xi(\beta_2^{(t)})\Big)$.
Step 5: Generate $\beta_1^{(t)}$ from $\pi(\beta_1|\alpha)$ by using the M-H algorithm with normal proposal as

- Let $v = \beta_1^{(t-1)}$ and generate w from the proposal as $w = N(\beta_1^{(t-1)}, \sigma_{\hat{\beta}_1})$.
- Let $p(v,w) = \min\left\{1, \frac{\pi(w|\alpha^{(t)})N(v,\sigma_{\hat{\beta}_1})}{\pi(v|\alpha^{(t)})N(w,\sigma_{\hat{\beta}_1})}\right\}$
- Generate U from $U(0,1)$, then accept proposal if $U \leq p(v,w)$ and set $\beta_1^{(t)} = w$ or otherwise $\beta_1^{(t)} = v$

Step 6: Calculate $\beta_2^{(t)}$ by using the same method given in Step 5.
Step 7: Set $t = t + 1$.
Step 8: Repeat Step 3 to Step 7, for M times.
Step 9: To compute the HPD credible intervals of θ, order the MCMC sample of $\theta^{(t)}$, $t = 1, 2, \ldots, M$ as $\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(M)}$. Then, following the algorithm proposed by [28], the $100(1-\gamma)\%$ HPD credible interval can be constructed by

$$\Big(\theta_{(j^*)}, \theta_{(j^*+(1-\gamma)M)}\Big) \tag{5.5}$$

where j^* is chosen such that

$$\theta_{(j^*+[(1-\gamma)M])} - \theta_{(j^*)} = \min_{1\leqslant j\leqslant\gamma}\Big(\theta_{(j+[(1-\gamma)M])} - \theta_{(j)}\Big), j^* = 1, 2, \ldots, M \tag{5.6}$$

In this equation, $[\gamma]$ denotes the largest integer less than or equal to γ. Then the HPD credible interval of θ is the interval which has the shortest length.

Then, the approximate posterior mean of θ under the squared error ($\hat{\theta}^{B_0}$) can be derived as

$$\hat{\theta}^{B_0} = \frac{1}{M-B}\sum_{t=B+1}^{M}\theta^{(t)} \tag{5.7}$$

where B is the burn-in period.

5.2. Informative Prior

In this section, we assume that the unknown parameters α, λ, β_1 and β_2 follow independent gamma priors such that $\theta_i \sim GA(a_i, b_i)$ for $i = 1, 2, 3, 4$ with density functions are given as in the following

$$\pi(\psi_i) \propto \psi_i^{a_i-1}e^{-b_i\psi_i}, \quad a_i,\ b_i > 0, \quad \text{for} \quad i = 1, 2, 3, 4.$$

where a_i and b_i are called the hyper-parameters. In the case of non-informative priors, very small and non-negative values of the hyper-parameters can be used, i.e. $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0.0001$, as suggested by [13] which are almost like Jeffrey's priors, but they are proper. This value of the hyper-parameters is used in many studies as non-informative priors. However, we prefer to use the vague priors given in the previous Section 5.1. In this section, as informative hyper-parameters, we suggest using hyper-parameters to provide the actual values of the parameters. Since $\mathrm{E}(\psi_i) = \frac{a_i}{b_i}$ provides an average value for the gamma variables, available a_i and b_i values can be chosen to obtain actual parameter values of the parameters. Since the actual values of the parameters are unknown in real data studies, a_i and b_i can be chosen as providing MLEs of the parameters.

Under these assumptions, the joint posterior density function of α, λ,, β_1 and β_2 is obtained as

$$L(\theta|\mathbf{w},\ \mathbf{z}) \propto \alpha^{D_1+a_1-1}\lambda^{D_2+a_2-1}\beta_1^{D_1+a_3-1}\beta_2^{D_2+a_4-1}e^{\beta_1\left(\sum_{i=1}^{\eta} z_i \ln(w_i)-b_3\right)}, \\ \times\, e^{\beta_2\left(\sum_{i=1}^{\eta}(1-z_i)\ln(w_i)-b_4\right)}e^{-\alpha\left(\xi(\beta_1)+b_1\right)}e^{-\lambda\left(\xi(\beta_2)+b_2\right)}.$$

Then, the conditional posterior density functions of the parameters can be obtained easily as in the following

$$\pi(\alpha|\beta_1) \propto GA\Big(D_1+a_1, \xi(\beta_1)+b_1\Big),$$

$$\pi(\lambda|\beta_2) \propto GA\Big(D_2+a_2, \xi(\beta_2)+b_2\Big),$$

$$\pi(\beta_1|\alpha) \propto \beta_1^{D_1+a_3-1}\exp\Big\{\beta_1\big(\sum_{i=1}^{\eta} z_i\ln(w_i)-b_3\big)\Big\},$$

$$\pi(\beta_2|\lambda) \propto \beta_2^{D_2+a_4-1}\exp\Big\{\beta_2\big(\sum_{i=1}^{\eta}(1-z_i)\ln(w_i)-b_4\big)\Big\}.$$

Similarly to the posterior distributions given in Section 5.1 the samples for α and λ can be easily obtained by using gamma densities. However, the densities from $\pi(\beta_1|\alpha)$ and $\pi(\beta_2|\lambda)$ can not be reduced analytically to well-known distributions and therefore it is not possible to generate samples directly by standard methods. Their density curves are also observed like the Gaussian distribution as seen in Figure 3. Therefore, the M-H algorithm with normal proposal distribution as suggested by [19] can also be used for the Bayesian estimation of θ, in this case. The previous algorithm given Section 5.1 for Gibbs sampling with the M-H method can be also used in this case. In this case, we denote the estimations in the form of $(\hat{\theta}^{B_1})$.

6. Optimal Censoring Scheme

In practical applications based on censoring schemes, there are two options to construct available censoring plans. Firstly, experimenters determine the censoring plan before conducting experiments or collecting data. Alternatively, pre-tested censoring plans in a similar data environment can be used. For example, an experiment can be performed in multiple lines with similar components. The censoring plan of one production line can be a reference for other lines and help experimenters save time and cost. Consequently, experimental restrictions such as time and cost force researchers to seek to develop an optimal experiment plan. Therefore, determining the "optimum" censoring scheme is quite important, since it helps the experimenters to provide lower variance, minimum cost, etc. In reliability and life-testing studies, an optimal censoring plan is desired to get a sufficient amount of information about the unknown model parameters.

Optimal censoring schemes are handled by various authors. Recently, Lin et al. [27] studied optimum life-testing plans with joint progressively type-II censored Weibull populations. Abo-Kasem and Elshahhat [2] also obtained an optimal censoring plan for two Weibull populations under progressively hybrid progressively joint censoring. In this study, we consider three commonly used criteria based on the variance-covariance matrix (VCM) of the observed Fisher information matrix given in (4.1) corresponding to the MLEs of unknown parameters as given in the following.

- A-optimality goals minimum trace of $F^{-1}(\widehat{\theta})$
- D-optimality goals minimum determinant of $F^{-1}(\widehat{\theta})$

- F-optimality goals maximum trace of $F(\widehat{\theta})$

The A-optimality criterion is based on the trace of the first-order approximation of the variance-covariance matrix (VCM) of the MLEs. This criterion provides an overall measure of the average variability of the estimates under MLE. The trace of the VCM equals the sum of the diagonal elements of $F^{-1}(\widehat{\theta})$. The A- optimality criterion is defined as minimizing trace $F^{-1}(\widehat{\theta})$. The second criterion is D-optimality and is based on maximizing the determinant of the observed Fisher information matrix which is equivalent to minimizing the determinant of VCM. This criterion provides an overall measure of variability by taking into account the correlations between the estimates. It is known that the joint confidence region of θ is proportional to $\left|F^{-1}(\widehat{\theta})\right|^{1/2}$ under some fixed level of confidence. Thus, the smaller value of $\left|F^{-1}(\widehat{\theta})\right|$ provides a higher precision of the parameter estimators. It is defined as minimizing $|F^{-1}(\widehat{\theta})|$. The third and last criterion is the F-optimality and is based on the trace of the first-order approximation of the Fisher information matrix of the MLEs. The trace of $F(\widehat{\theta})$ is equal to the sum of the diagonal elements of $F(\widehat{\theta})$. The F- optimality criterion is defined as maximizing the trace of $F(\widehat{\theta})$.

7. Data Analysis

In this section, the theoretical findings are evaluated based on their performances via simulation studies and numerical data example.

7.1. Simulation Studies

In this section, we consider two different sets of the actual values of the parameters. In the first scenario, we take $(\alpha, \lambda, \beta_1, \beta_2)$ as $(1.25, 0.75, 2.00, 2.25)$ and $(\alpha, \lambda, \beta_1, \beta_2)$ as $(1.50, 1.50, 0.75, 0.75)$ in the second scenario. Based on these two sets of parameters, simulation studies are performed under different sample sizes as $(m, n) = (20, 24)$, $(35, 35)$ and $(75, 65)$. Then we determine the guaranteed failure number k and the eventual failure number r based on the provided at least 50% failures and the observed sample 70%, respectively. That is $k = (0.50) \times N$ where $(N = m + n)$ and $r = (0.70) \times N$. We also consider two different predetermined times T in each scenario. In the first scenario, we take predetermined times T as 1.25 and 1.75. In the other scenario, we take T as 3.50 and 5.00. We consider three types of progressive censoring schemes $(R_1, R_2, \cdots, R_r)$. First, we take $(0_{(r-1)}, N-r)$ as CS-I, $(0_{(\frac{r}{2}-1)}, N-r, 0_{(\frac{r}{2})})$ if r is even or $(0_{(\frac{r+1}{2}-1)}, N-r, 0_{(\frac{r+1}{2})})$ otherwise as CS-II and $(1_{(N-r)}, 0_{(2r-N)})$ as CS-III. Here, $0_{(k)}$ means repeated values 0, k times.

In the informative Bayesian inference, we determine the values of hyperparameters from gamma distributions by providing the actual values of the parameters as $a_1 = 1.25$, $a_2 = 0.75$, $a_3 = 2.00$, $a_4 = 2.25$ and $b_i = 1.00$ for $i = 1, 2, 3, 4$. for the first scenario and we determine $a_1 = 3.00$, $a_2 = 3.00$, $a_3 = 2.00$, $a_4 = 2.00$, $b_1 = b_2 = 2.00$ and $b_3 = b_4 = 1.50$ for the second scenario.

Thus, we repeat the iterations 1000 times and use 500 bootstrap samples for each replication. We used 3500 samples of the Markov chain and discarded the first 500 values as burn-in periods and took every third variate in the thinning procedure in the MCMC samples. Then, the algorithm is performed for 1000 replications. The significant level was taken as $\gamma = 0.05$.

After performing the simulations, we evaluated the estimations with their mean squared error (MSE) and we also compared the performances of the approximate confidence intervals with their average length (AL) and the coverage probabilities (CP). The point estimates with the corresponding MSEs are given in Tables 5 and 6. Then, the ALs with

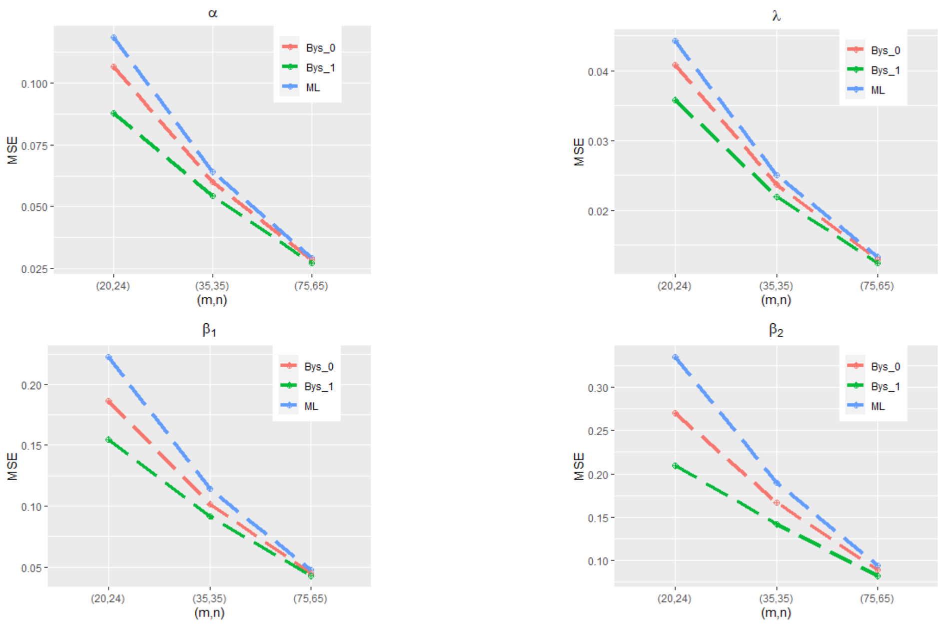

Figure 4. MSE plots of the estimates under CS-III, $\alpha = 1.25$, $\lambda = 0.75$, $\beta_1 = 2.00$, $\beta_2 = 2.25$ and $T = 1.75$.

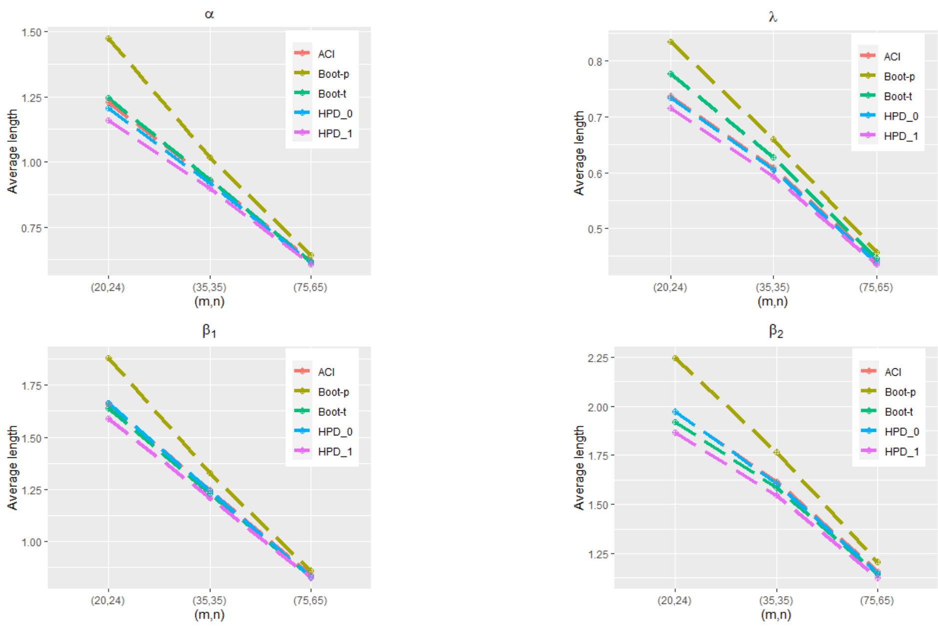

Figure 5. Average lengths for the approximate confidence intervals of the estimates under CS-III, $\alpha = 1.25$, $\lambda = 0.75$, $\beta_1 = 2.00$, $\beta_2 = 2.25$ and $T = 1.75$.

their corresponding CPs are given in Tables 7 and 8. Additionally, we provide a visual presentation of the performances of the MSEs of the estimations and ALs of the approximate confidence intervals. As an illustration, we provide these plots under CS-III in Figures 4 and 5. The R [14] software is used for all computations. A step by step flow chart of the procedure of the analysis of this work is presented in Figure 6. The observations about the simulation studies are listed as in the following.

- The MSE values decrease in parallel to the increasing sample sizes regardless of any censoring scheme used.
- The point estimations are getting closer to their actual values with the increasing sample sizes. That is, bias has been decreasing in parallel to the increasing sample sizes.
- Consistency of the estimation methods do not be affected from the censoring scheme. One can use the proposed methods in all cases of the J-GPHCS.
- The informative Bayesian method provide smallest MSE and non-informative Bayes follow it. Especially in small samples, MLE has the largest MSE values. However, performance differences between the methods are decreasing in parallel to increasing sample sizes.
- Approximate confidence intervals show good performances. ACI, boot-p, boot-t and HPD provided very close CPs to the their actual values 0.95.
- Similar to performances of the point estimations, informative Bayes method provide shortest ALs in all cases. The HPD based on the non-informative Bayesian method also give close results. Among the intervals based on the MLEs, the boot-t method gives smallest ALs and these ALs are obtained very close to the HPD intervals.

We can obtain consistent and acceptable results for different censoring plans at different values of the parameters. We also illustrate these findings in the next section using a real-data example.

7.2. Numerical Example

In this subsection, we consider progressively censored samples generated from the breaking strengths of jute fiber at different gauge lengths 5mm and 10mm for Sam-1 and Sam-2, respectively which is given by [41]. This data set has recently been used in some reliability problems by various authors such as [36], [24] and [10]. The sample size of each data set is equal and $m = n = 30$ ($N = 60$) in this example. We scale both samples by dividing by 1000 for better fitting to the reparameterized Weibull model. Then, we fit this dataset with the Weibull distributions by using the MLE method. We obtain the parameter estimations as $\hat{\alpha} = 6.358$, $\hat{\beta}_1 = 2.228$, $\hat{\lambda} = 4.279$, $\hat{\beta}_2 = 1.625$, respectively. Then, the KolmogorovSimirnov test statistic and associated p-value for Sam-I are obtained as 0.1333 and 0.9578, respectively. The same quantities for the data set Sam-II are 0.1667 and 0.8080, respectively. Thus, one cannot reject the null hypotheses that the data sets come from the Weibull distributions. The plots of the emprical and theoretical cdfs also support this observation (see Figure 7).

We first consider the following three schemes of removals, $(R_1, R_2, \cdots, R_r)$. We take $R = (0_{(r-1)}, N-r)$ and denoted it as Rmvl-I, then we take $R = (0_{(\frac{r}{2}-1)}, N-r, 0_{(\frac{r}{2})})$ if r is even or $R = (0_{(\frac{r+1}{2}-1)}, N-r, 0_{(\frac{r+1}{2})})$ if r is odd as Rmvl-II and we take $R = (1_{(N-r)}, 0_{(2r-N)})$ as Rmvl-III.

Further, we consider various values of k, r, and T and we created alternative censoring schemes as given in Table 1. Then, we determine the optimal plan among them for this data set according to the criteria given in Section 6. We obtained the optimality values and reported them in Table 2. The optimal scheme is determined as $k = 42$, $r = 52$, $T = 0.7$ with $(R_1, R_2, \cdots, R_r) = (8, 0_{(51)})$ according to the A-optimality and D-optimality. On the other hand, the optimal censoring scheme is determined as $k = 42$, $r = 52$, $T = 0.7$ with $(R_1, R_2, \cdots, R_r) = (0_{(51)}, 8)$ according to the F-optimality. Then, we use the first of these optimal plans to obtain the estimates based on these jute fiber data. In this numerical example, we get $k = 42$, $r = 52$, $T = 0.7$ with $(R_1, R_2, \cdots, R_r) = (8, 0_{(51)})$.

In the MCMC method, we use 100 000 iterations and take MLEs as initial values of the parameters so that we do not need to consider the burn-in period. However, since

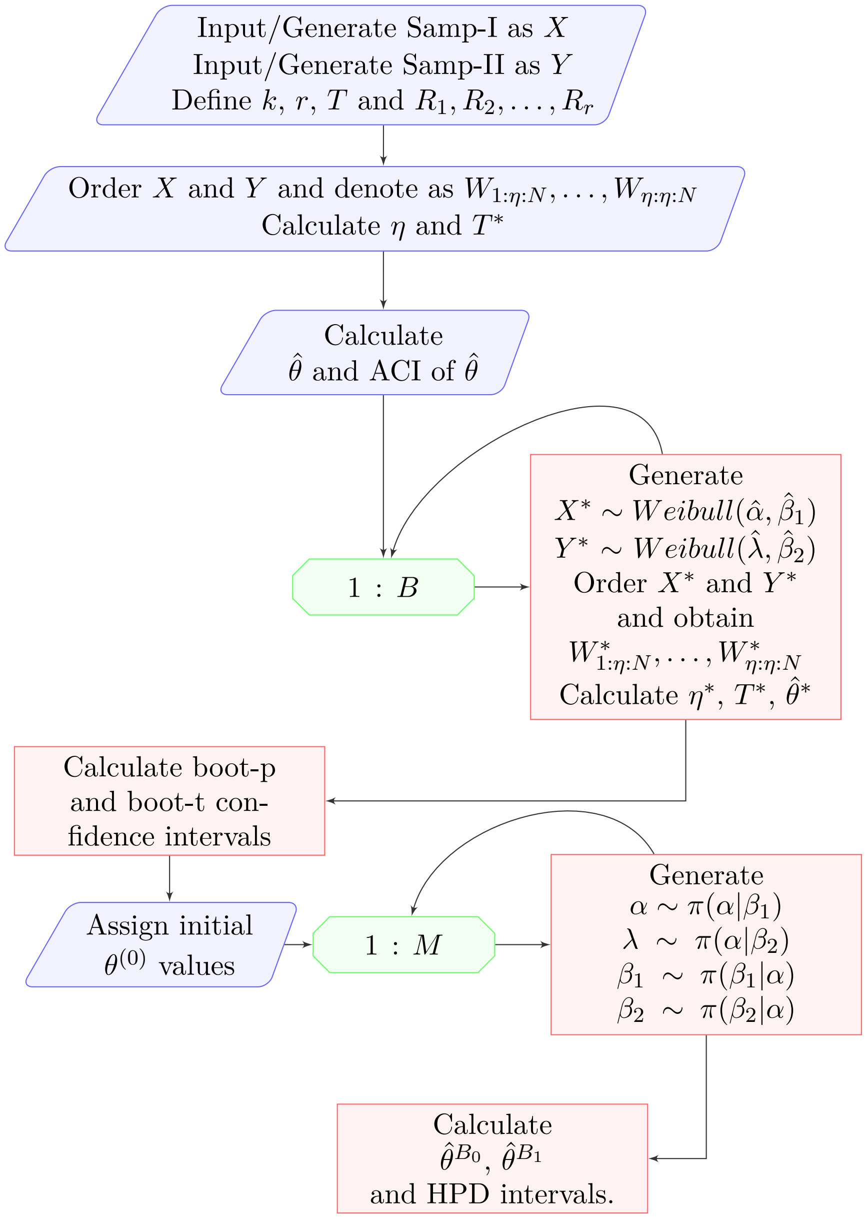

Figure 6. Flow chart of the analysis procesure for two Weibull populations under J-GPHCS.

the algorithm naturally generates an autocorrelated Markov chain, we use thinning and choose 10 for the thinning number to avoid losing much information from the data. Thus, the number of iterations becomes 10 000 after this thinning without not much lack of information. Further, we use 10 000 replications for the bootstrap confidence intervals. Thus, we obtain the estimations and their corresponding confidence intervals and report them in Table 3. We observe that point estimations are obtained very close to each other for all four parameters. We can conclude that the parameter values of these J-GPHCS data are observed as $\alpha \approx 6$, $\beta_1 \approx 2$, $\lambda \approx 4$, $\beta_2 \approx 1.5$. The values of the shape parameters (> 1) show that these data cannot be modeled with exponential distribution very well. In addition, the presence of the scale parameters makes the censoring model more flexible. On the other hand, the hazard of the Weibull model shows an increasing failure rate due to $\beta_1, \beta_2 > 1$. Since the exponential model has a constant hazard, these data cannot be evaluated with the exponential distribution. We can observe the data better in the

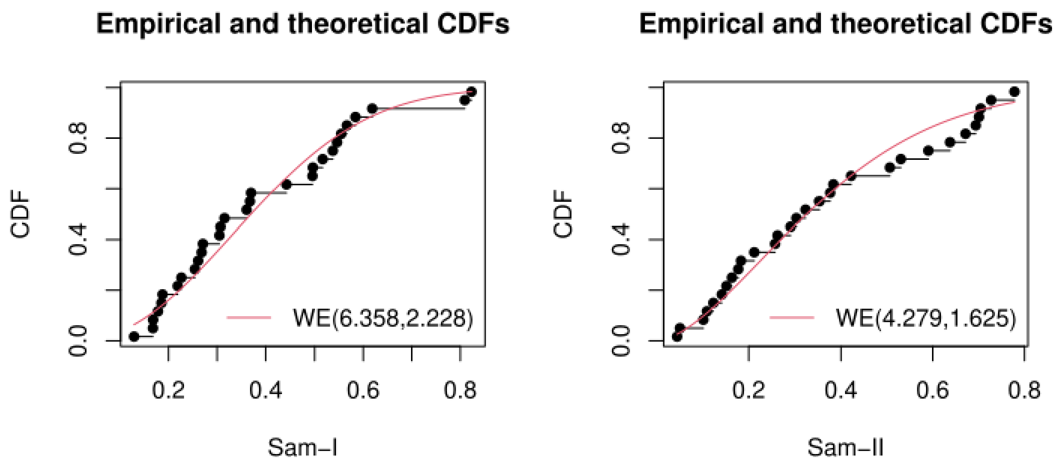

Figure 7. Emprical cdf plot for the fiber data fitted by the Weibull distribution

Table 1. Various censoring schemes based on the GPHCS for the fiber data by [41].

CS	k	r	T	R	CS	k	r	T	R
I	36	42	0.5	Rmvl-I	XIII	42	48	0.5	Rmvl-I
II	36	42	0.5	Rmvl-II	XIV	42	48	0.5	Rmvl-II
III	36	42	0.5	Rmvl-III	XV	42	48	0.5	Rmvl-III
IV	36	42	0.7	Rmvl-I	XVI	42	48	0.7	Rmvl-I
V	36	42	0.7	Rmvl-II	XVII	42	48	0.7	Rmvl-II
VI	36	42	0.7	Rmvl-III	XVIII	42	48	0.7	Rmvl-III
VII	36	48	0.5	Rmvl-I	XIX	42	52	0.5	Rmvl-I
VIII	36	48	0.5	Rmvl-II	XX	42	52	0.5	Rmvl-II
IX	36	48	0.5	Rmvl-III	XXI	42	52	0.5	Rmvl-III
X	36	48	0.7	Rmvl-I	XXII	42	52	0.7	Rmvl-I
XI	36	48	0.7	Rmvl-II	XXIII	42	52	0.7	Rmvl-II
XII	36	48	0.7	Rmvl-III	XIV	42	52	0.7	Rmvl-III

Table 2. Optimality values based on the J-GPHCS for the fiber data by [41].

CS	$A-$optimality	$D-$optimality	$F-$optimality	CS	$A-$optimality	$D-$optimality	$F-$optimality
I	6.9860	0.0190	41.893	XIII	5.1228	0.0058	49.7097
II	22.7487	0.0584	53.0491	XIV	10.3821	0.0226	54.7394
III	6.6500	0.0083	56.0939	XV	6.1973	0.0085	55.7345
IV	4.5585	0.0106	42.1091	XVI	4.2579	0.0067	46.9194
V	8.3509	0.0177	53.3666	XVII	5.0984	0.0088	54.0527
VI	6.1973	0.0085	55.7345	XVIII	7.2614	0.0077	56.4508
VII	15.6698	0.0458	47.2828	XIX	7.2614	0.0077	56.4508
VIII	16.9914	0.0698	54.1354	XX	10.3821	0.0226	54.7394
IX	6.6500	0.0083	56.0939	XXI	6.1973	0.0085	55.7345
X	4.2579	0.0067	46.9194	XXII	**4.2221**	**0.0052**	52.3428
XII	5.0984	0.0088	54.0527	XXIII	15.5187	0.0257	53.5388
XII	7.2614	0.0077	56.4508	XIV	6.3601	0.0058	**56.5618**

scope of the lifetime data analysis with the Weibull model. We also present a comparison of the exponential model given by [11] with the Weibull model to see the superiority of the Weibull model under J-GPHCS. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and the log-likelihood values are given in Table 4 and it is seen

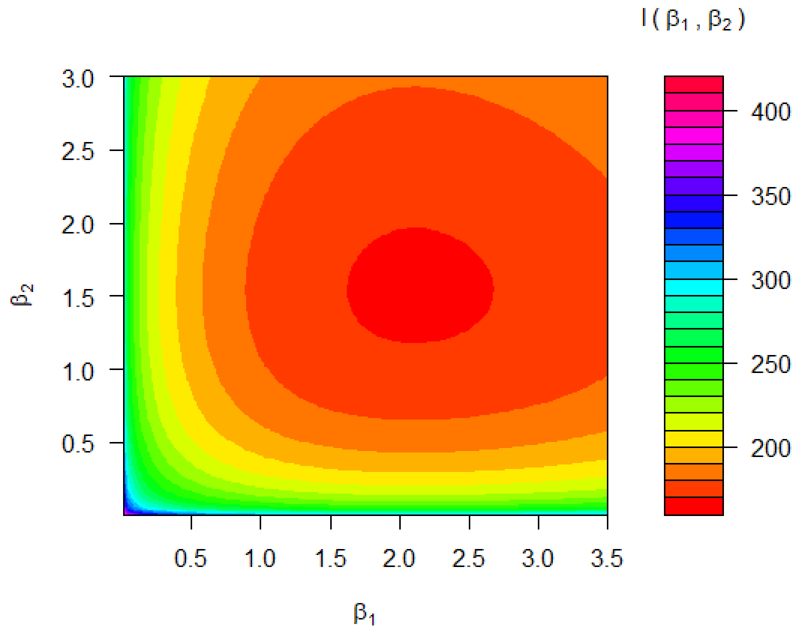

Figure 8. Contour plot for the profile log-likelihood function of β_1 and β_2 based on the fiber data

Table 3. Point estimations and lengths of approximate confidence intervals for the J-GPHCS jute fiber data.

	MLE	$Bayes_0$	$Bayes_1$	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1
α	6.1385	5.9037	5.9976	6.8297	10.3206	6.1483	6.5532	5.5899
β_1	2.1179	2.0408	2.0711	1.2353	1.4164	1.2137	1.1566	1.1395
λ	4.1207	3.9985	3.9771	3.9823	5.6709	3.6076	3.8525	3.5277
β_2	1.5420	1.4928	1.4931	0.9198	1.0197	0.8921	0.9298	0.8755

that the Weibull model has the smallest AIC and BIC values as well as maximum log-likelihood value. Thus, we can conclude that our model is a good fitting with the Weibull model under J-GPHCS. We also provide the contour plot of the profile log-likelihood plot of β_1 and β_2 in Figure 8 and it is observed that the log-likelihood plot has a unique solution at $\hat{\beta}_1$ and $\hat{\beta}_2$. The performances of the approximate confidence intervals are obtained the same as the simulation studies. In this numerical example, HPD credible intervals based on the informative Bayesian model have better performance since they have the smallest lengths. Among the approximate confidence intervals, Boot-t intervals show better performance than ACI and Boot-p. We observe these outcomes for all four parameters. Thus, the performances of the inference methods based on the numerical examples support the findings of the simulation studies in Section 7.1.

We further evaluate the convergence of the Markov chains by graphical and numerical methods effectively. For this purpose, the trace plot which is a plot of the iteration number, t, against the value of the parameters at each iteration, and the density plots of the posterior distribution of the parameters are used. Also, the running mean (ergodic average) plot which draws the mean of sampled values up to iteration t is used. We draw all graphics and present them in Figures 9,10, and 12. Further, we report Brooks-Gelman-Rubin (BGR) diagnostic in Figure 11 and it is seen that there is no substantial difference between the variance-within and variance-between the generated Markov chains.

Table 4. Comparisons of Weibull and exponential models for the fiber data under the J-GPHCS.

Models	Measurements		
	AIC	BIC	Log-likelihood
Weibull	-138.3613	-132.7565	73.18066
Exponential	-59.19467	-56.39228	31.59734

The acceptable limit of multivariate potential scale reduction factor and potential scale reduction factor is obtained around 1 as recommended by [19].

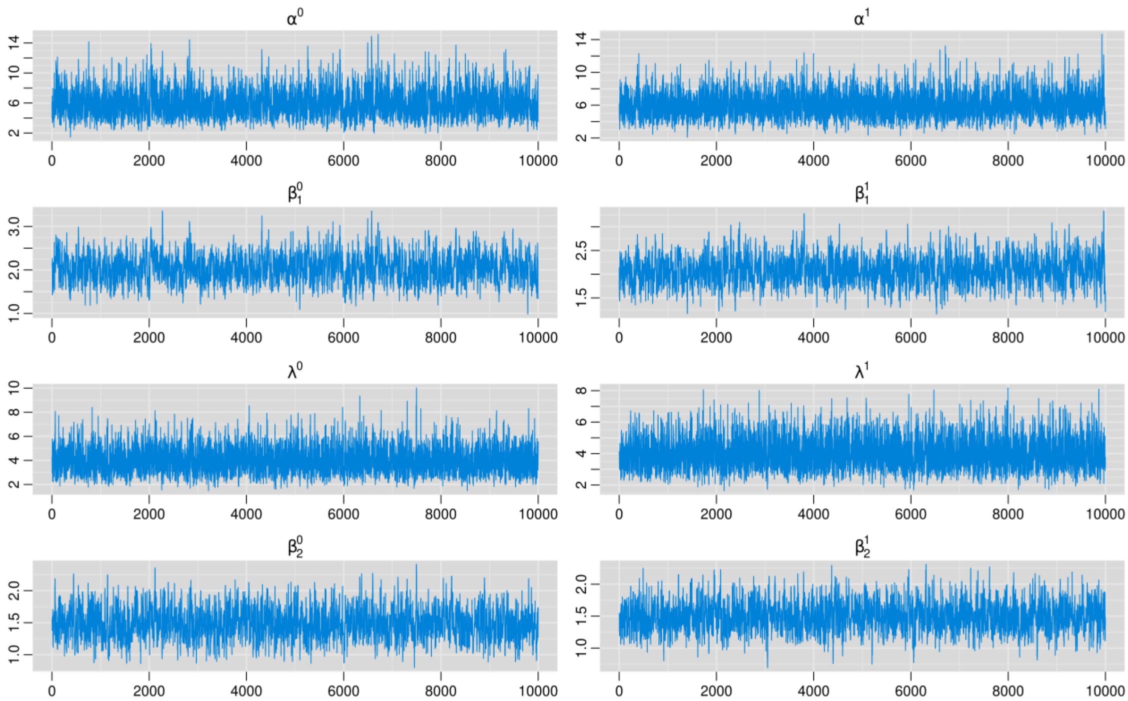

Figure 9. Trace plots of the posterior distributions of the parameters.

We observe that convergences of the Markov chains are completely satisfactory, and findings are obtained as expected. We draw all plots for the convergence by using the "mcmcplots" package defined by [15] in R [14] software and we draw BGR plots by using the "coda" package defined by [32]. We also assess the convergence by using the method introduced by [20]. Geweke's convergence diagnostic proposes a convergence diagnostic for Markov chains based on a test for equality of the means of the first and last part of a Markov chain. If the samples are drawn from the stationary distribution of the chain, the two means are equal and Geweke's statistic has an asymptotically standard normal distribution. We calculate the Geweke's statistic with the "geweke.diag" function in R [14] and we obtain the scores (Z-score) as (-0.3665), (-1.2530), (-0.2486), (-1.5710), (0.5732), (-1.5550), (0.5925) and (-1.4320) for $\hat{\alpha}^{B_0}$, $\hat{\alpha}^{B_1}$, $\hat{\beta}_1^{B_0}$, $\hat{\beta}_1^{B_1}$, $\hat{\lambda}^{B_0}$, $\hat{\lambda}^{B_1}$, $\hat{\beta}_2^{B_0}$ and $\hat{\beta}_2^{B_1}$, respectively. None of these Z-scores exceed the critical value,1.96, with the 0.05 significance level. Therefore, we can conclude that the convergences of the Markov chains are provided for all parameters.

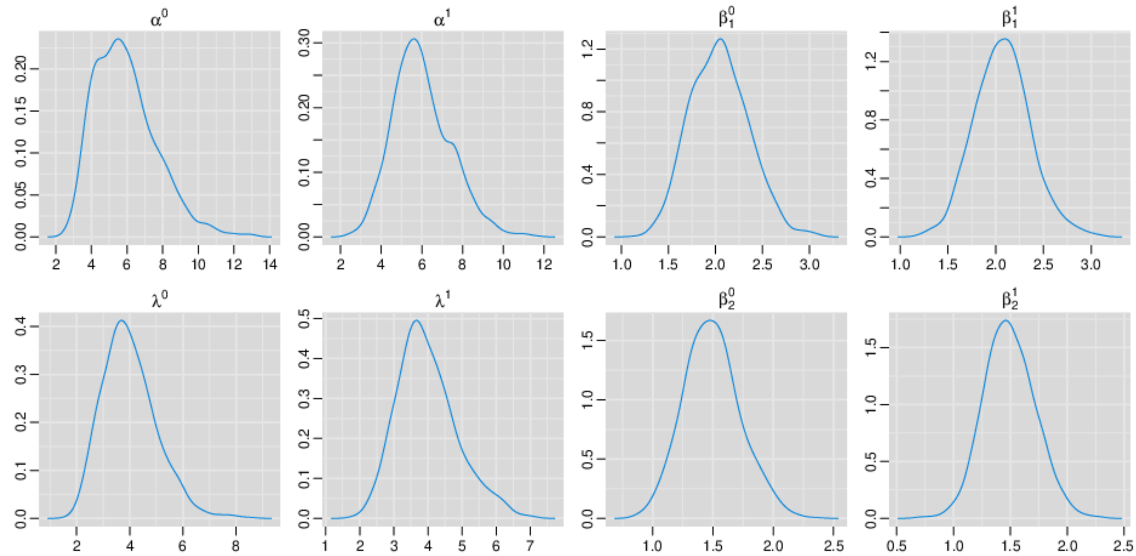

Figure 10. Density plots of the posterior distributions of the parameters.

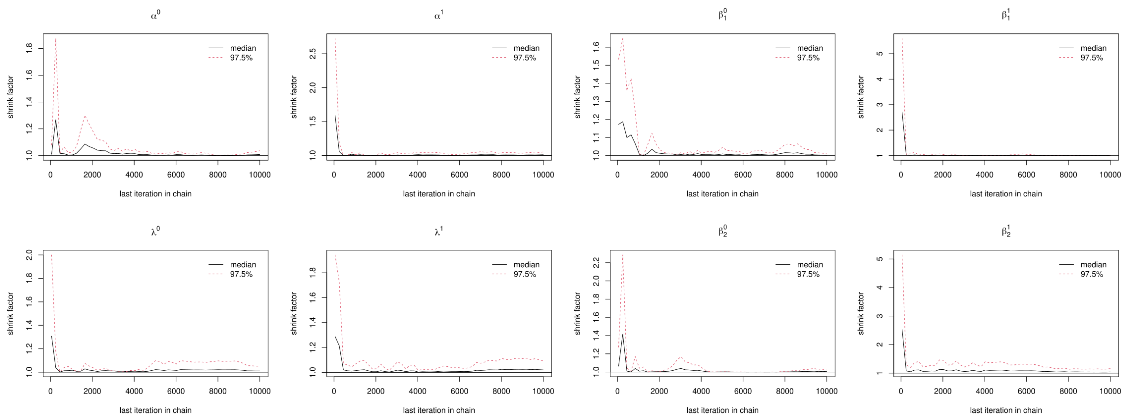

Figure 11. The BGR plots for 10 000 MCMC iterations in Monte Carlo simulations.

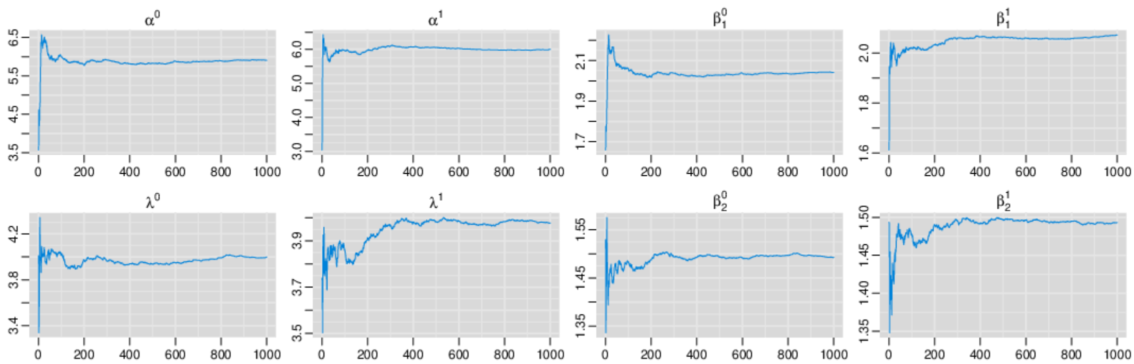

Figure 12. Running mean plots of the posterior distributions of the parameters.

8. Conclusions

In this paper, we studied inference procedures for two Weibull populations under J-GPHCS defined by [11]. Unlike this reference paper, we handle the Weibull distribution as a more flexible model. The outcomes of this study let the experimenters making life test studies under a generalized censoring scheme for the data sets with increasing, constant and decreasing hazard rates. Further, researches can use the findings of these work for the exponential and Rayleigh populations since they can be obtained as a special case of the Weibull distribution. The MLE and Bayesian estimations of the shape and scale parameters of two Weibull populations are obtained. In addition to point estimations, approximate confidence intervals by using the asymptotic properties of the MLEs and bootstrap confidence intervals are given for MLEs. Then, HPD credible intervals for Bayesian estimations are given. All theoretical findings are illustrated with simulation studies and a numerical example. It is observed that theoretical findings show good performances in all cases of the censoring scheme. Both Bayesian methods have superiority over MLEs in the small sample sizes, as expected. However, informative priors even more increase the performances of the Bayesian estimations, especially in small sizes. Then, we handle the problem of which censoring scheme is better than the others. For this purpose, the A-optimality, D-optimality, and F-optimality criteria are considered. Then, we applied these optimality criteria for numerical data and observed that the A-optimality and D-optimality criteria mark the same censoring scheme as optimal, and the F-optimality criterion suggests another scheme. However, the difference between optimality criteria differs only for the removal positioning. In contrast, all optimality criteria suggest the same predetermined minimum number of failures, k, a number of failures, r, and the end time of the experiment, T. Then, we used the optimal censoring plan for the numerical example. Consequently, this study proposes an extensive model of the J-GPHCS model based on the one-parameter exponential distribution using a more flexible two-parameter Weibull distribution. In the literature, there are still open problems in this regard. For example, various probability distributions can be handled in J-GPHCS. This study proposes two populations in its scheme, but the case of k-samples can be considered as an extensive study. The R codes for reproducing the results of the computations are available in the repository: https://github.com/cgtycetinkaya/JGPHCS-Weibull-Paper-Codes for reproducibility.

Table 5. Point estimates (first rows) and their MSEs (second rows) for the case of $\alpha = 1.25$, $\lambda = 0.75$, $\beta_1 = 2.00$, $\beta_2 = 2.25$ and $T = 1.25$.

(m,n)	CS	$\hat{\alpha}$	$\hat{\alpha}^{B_0}$	$\hat{\alpha}^{B_1}$	$\hat{\lambda}$	$\hat{\lambda}^{B_0}$	$\hat{\lambda}^{B_1}$	$\hat{\beta}_1$	$\hat{\beta}_1^{B_0}$	$\hat{\beta}_1^{B_1}$	$\hat{\beta}_2$	$\hat{\beta}_2^{B_0}$	$\hat{\beta}_2^{B_1}$
$T = 1.25$													
20,24	I	1.3614	1.3445	1.3289	0.8100	0.8211	0.8200	2.2196	2.1091	2.0872	2.5458	2.4285	2.4018
		0.1954	0.1731	0.1213	0.0450	0.0432	0.0378	0.2665	0.2100	0.1644	0.3761	0.2880	0.2196
	II	1.3612	1.3357	1.3236	0.8121	0.8141	0.8125	2.2328	2.1134	2.0955	2.5740	2.4459	2.4168
		0.1350	0.1170	0.0912	0.0471	0.0443	0.0390	0.2462	0.1865	0.1508	0.3896	0.2952	0.2226
	III	1.3083	1.2894	1.2823	0.7591	0.7603	0.7598	2.1402	2.0277	2.0162	2.3901	2.2515	2.2406
		0.1376	0.1252	0.1003	0.0420	0.0392	0.0350	0.2280	0.1926	0.1544	0.3302	0.2810	0.2161
35,35	I	1.3058	1.2994	1.2960	0.7764	0.7864	0.7873	2.1422	2.0842	2.0761	2.5334	2.4576	2.4365
		0.0709	0.0660	0.0583	0.0301	0.0291	0.0263	0.1293	0.1103	0.0970	0.2915	0.2395	0.1968
	II	1.3302	1.3171	1.3120	0.7925	0.7953	0.7947	2.1382	2.0758	2.0690	2.4978	2.4179	2.4032
		0.0733	0.0670	0.0590	0.0284	0.0272	0.0250	0.1194	0.0998	0.0906	0.2386	0.1937	0.1637
	III	1.2847	1.2744	1.2720	0.7676	0.7680	0.7672	2.0719	2.0118	2.0075	2.3396	2.2507	2.2440
		0.0711	0.0675	0.0603	0.0305	0.0291	0.0268	0.1116	0.1025	0.0911	0.2088	0.1901	0.1594
75,65	I	1.2865	1.2844	1.2837	0.7741	0.7798	0.7803	2.0830	2.0568	2.0556	2.4432	2.4053	2.3983
		0.0264	0.0255	0.0244	0.0147	0.0146	0.0138	0.0516	0.0465	0.0449	0.1460	0.1287	0.1173
	II	1.2928	1.2879	1.2862	0.7842	0.7861	0.7864	2.1043	2.0772	2.0745	2.4264	2.3873	2.3815
		0.0293	0.0281	0.0266	0.0131	0.0129	0.0124	0.0554	0.0495	0.0468	0.1292	0.1131	0.1052
	III	1.2658	1.2617	1.2610	0.7623	0.7624	0.7626	2.0445	2.0184	2.0168	2.3009	2.2566	2.2546
		0.0268	0.0262	0.0251	0.0142	0.0138	0.0134	0.0492	0.0469	0.0444	0.0948	0.0897	0.0823
$T = 1.75$													
20,24	I	1.3347	1.3227	1.3094	0.7633	0.7797	0.7782	2.1736	2.0712	2.0539	2.4307	2.3345	2.3187
		0.1729	0.1524	0.1100	0.0485	0.0454	0.0396	0.2320	0.1871	0.1508	0.2442	0.2011	0.1637
	II	1.3522	1.3312	1.3150	0.7718	0.7799	0.7783	2.2067	2.0953	2.0749	2.4122	2.3112	2.2983
		0.1613	0.1398	0.1038	0.0486	0.0450	0.0395	0.2864	0.2273	0.1778	0.2481	0.2064	0.1703
	III	1.3054	1.2896	1.2827	0.7786	0.7828	0.7812	2.1411	2.0343	2.0220	2.4620	2.3374	2.3158
		0.1184	0.1065	0.0877	0.0443	0.0408	0.0358	0.2227	0.1861	0.1548	0.3354	0.2700	0.2093
35,35	I	1.2938	1.2889	1.2853	0.7641	0.7755	0.7750	2.1029	2.0470	2.0416	2.3739	2.3093	2.3017
		0.0733	0.0689	0.0590	0.0286	0.0275	0.0250	0.1166	0.1026	0.0912	0.1560	0.1361	0.1187
	II	1.3126	1.3009	1.2961	0.7695	0.7748	0.7740	2.1085	2.0488	2.0429	2.3677	2.2992	2.2943
		0.0766	0.0708	0.0619	0.0290	0.0275	0.0255	0.1138	0.0984	0.0883	0.1499	0.1302	0.1169
	III	1.3008	1.2914	1.2886	0.7688	0.7704	0.7698	2.0955	2.0371	2.0316	2.3618	2.2775	2.2696
		0.0640	0.0599	0.0544	0.0250	0.0237	0.0219	0.1141	0.1016	0.0915	0.1897	0.1668	0.1413
75,65	I	1.2720	1.2707	1.2698	0.7536	0.7603	0.7602	2.0581	2.0330	2.0317	2.3061	2.2724	2.2706
		0.0288	0.0280	0.0266	0.0148	0.0145	0.0138	0.0505	0.0471	0.0450	0.0707	0.0665	0.0619
	II	1.2796	1.2751	1.2738	0.7610	0.7639	0.7640	2.0449	2.0192	2.0174	2.3261	2.2918	2.2893
		0.0295	0.0284	0.0271	0.0156	0.0152	0.0146	0.0428	0.0403	0.0385	0.0723	0.0667	0.0622
	III	1.2663	1.2628	1.2619	0.7592	0.7602	0.7600	2.0483	2.0224	2.0210	2.2964	2.2537	2.2520
		0.0292	0.0285	0.0272	0.0133	0.0129	0.0124	0.0472	0.0446	0.0424	0.0937	0.0888	0.0816

Table 6. Point estimates (first rows) and their MSEs (second rows) for the case of $\alpha = 1.50$, $\lambda = 1.50$, $\beta_1 = 0.75$, $\beta_2 = 0.75$.

(m,n)	CS	$\hat{\alpha}$	$\hat{\alpha}^{B_0}$	$\hat{\alpha}^{B_1}$	$\hat{\lambda}$	$\hat{\lambda}^{B_0}$	$\hat{\lambda}^{B_1}$	$\hat{\beta}_1$	$\hat{\beta}_1^{B_0}$	$\hat{\beta}_1^{B_1}$	$\hat{\beta}_2$	$\hat{\beta}_2^{B_0}$	$\hat{\beta}_2^{B_1}$
$T = 3.50$													
20,24	I	1.6203	1.5920	1.6314	1.6367	1.6126	1.6459	0.8181	0.7793	0.7868	0.8092	0.7782	0.7849
		0.2486	0.2165	0.1694	0.2200	0.1954	0.1576	0.0335	0.0269	0.0241	0.0287	0.0241	0.0221
	II	1.6711	1.6268	1.6590	1.6734	1.6369	1.6699	0.8182	0.7762	0.7860	0.8066	0.7730	0.7822
		0.3219	0.2737	0.1883	0.2342	0.2015	0.1650	0.0364	0.0292	0.0259	0.0258	0.0213	0.0196
	III	1.6173	1.5753	1.6249	1.6088	1.5743	1.6167	0.8013	0.7572	0.7739	0.8053	0.7685	0.7832
		0.2333	0.2043	0.1526	0.2231	0.2008	0.1603	0.0355	0.0300	0.0257	0.0273	0.0227	0.0208
35,35	I	1.5666	1.5530	1.5840	1.5836	1.5694	1.5998	0.7868	0.7657	0.7718	0.7895	0.7686	0.7747
		0.1200	0.1109	0.1014	0.1160	0.1068	0.0991	0.0155	0.0137	0.0130	0.0158	0.0138	0.0134
	II	1.6097	1.5864	1.6160	1.5796	1.5573	1.5911	0.7901	0.7675	0.7749	0.7920	0.7694	0.7775
		0.1541	0.1395	0.1247	0.1120	0.1016	0.0942	0.0178	0.0156	0.0149	0.0155	0.0134	0.0130
	III	1.5843	1.5606	1.5972	1.5902	1.5666	1.6040	0.7839	0.7594	0.7715	0.7917	0.7670	0.7791
		0.1306	0.1211	0.1074	0.1180	0.1085	0.1009	0.0185	0.0167	0.0158	0.0209	0.0185	0.0176
75,65	I	1.5274	1.5222	1.5390	1.5303	1.5241	1.5432	0.7717	0.7622	0.7656	0.7689	0.7580	0.7615
		0.0433	0.0419	0.0408	0.0536	0.0516	0.0496	0.0068	0.0063	0.0063	0.0073	0.0069	0.0067
	II	1.5358	1.5267	1.5447	1.5358	1.5249	1.5462	0.7666	0.7568	0.7608	0.7675	0.7560	0.7607
		0.0450	0.0433	0.0423	0.0532	0.0505	0.0490	0.0065	0.0061	0.0060	0.0069	0.0064	0.0063
	III	1.5281	1.5186	1.5392	1.5366	1.5246	1.5492	0.7707	0.7599	0.7663	0.7717	0.7591	0.7664
		0.0436	0.0423	0.0412	0.0571	0.0545	0.0532	0.0086	0.0081	0.0081	0.0100	0.0094	0.0092
$T = 5.00$													
20,24	I	1.6203	1.5920	1.6314	1.6367	1.6126	1.6459	0.8181	0.7793	0.7868	0.8092	0.7782	0.7849
		0.2486	0.2165	0.1694	0.2200	0.1954	0.1576	0.0335	0.0269	0.0241	0.0287	0.0241	0.0221
	II	1.6711	1.6268	1.6590	1.6734	1.6369	1.6699	0.8182	0.7762	0.7860	0.8066	0.7730	0.7822
		0.3219	0.2737	0.1883	0.2342	0.2015	0.1650	0.0364	0.0292	0.0259	0.0258	0.0213	0.0196
	III	1.6173	1.5753	1.6249	1.6088	1.5743	1.6167	0.8013	0.7572	0.7739	0.8053	0.7685	0.7832
		0.2333	0.2043	0.1526	0.2231	0.2008	0.1603	0.0355	0.0300	0.0257	0.0273	0.0227	0.0208
35,35	I	1.5666	1.5530	1.5840	1.5836	1.5694	1.5998	0.7868	0.7657	0.7718	0.7895	0.7686	0.7747
		0.1200	0.1109	0.1014	0.1160	0.1068	0.0991	0.0155	0.0137	0.0130	0.0158	0.0138	0.0134
	II	1.6097	1.5864	1.6160	1.5796	1.5573	1.5911	0.7901	0.7675	0.7749	0.7920	0.7694	0.7775
		0.1541	0.1395	0.1247	0.1120	0.1016	0.0942	0.0178	0.0156	0.0149	0.0155	0.0134	0.0130
	III	1.5843	1.5606	1.5972	1.5902	1.5666	1.6040	0.7839	0.7594	0.7715	0.7917	0.7670	0.7791
		0.1306	0.1211	0.1074	0.1180	0.1085	0.1009	0.0185	0.0167	0.0158	0.0209	0.0185	0.0176
75,65	I	1.5274	1.5222	1.5390	1.5303	1.5241	1.5432	0.7717	0.7622	0.7656	0.7689	0.7580	0.7615
		0.0433	0.0419	0.0408	0.0536	0.0516	0.0496	0.0068	0.0063	0.0063	0.0073	0.0069	0.0067
	II	1.5358	1.5267	1.5447	1.5358	1.5249	1.5462	0.7666	0.7568	0.7608	0.7675	0.7560	0.7607
		0.0450	0.0433	0.0423	0.0532	0.0505	0.0490	0.0065	0.0061	0.0060	0.0069	0.0064	0.0063
	III	1.5281	1.5186	1.5392	1.5366	1.5246	1.5492	0.7707	0.7599	0.7663	0.7717	0.7591	0.7664
		0.0436	0.0423	0.0412	0.0571	0.0545	0.0532	0.0086	0.0081	0.0081	0.0100	0.0094	0.0092

Table 7. Lengths (first rows) and coverage probabilities (second rows) of the approximate confidence intervals for the case of $\alpha = 1.25$, $\lambda = 1.25$, $\beta_1 = 2.00$, $\beta_2 = 2.25$.

		α					λ					β_1					β_2				
m,n	CS	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1
$T = 1.25$																					
20,24	I	1.3741	1.7719	1.3380	1.3479	1.2738	0.8052	0.8575	0.7818	0.8040	0.7804	1.7319	2.0575	1.6618	1.7357	1.6463	1.9127	2.1955	1.8134	1.9047	1.8012
		0.9580	0.9140	0.9420	0.9480	0.9590	0.9690	0.9320	0.9640	0.9620	0.9730	0.9530	0.8540	0.9420	0.9570	0.9720	0.9530	0.7990	0.9540	0.9560	0.9670
	II	1.3436	1.7485	1.2866	1.3093	1.2470	0.7876	0.8651	0.7656	0.7810	0.7593	1.7236	2.0275	1.6548	1.7310	1.6452	1.9284	2.1764	1.8290	1.9390	1.8282
		0.9710	0.9230	0.9600	0.9790	0.9850	0.9570	0.9160	0.9470	0.9630	0.9660	0.9640	0.8550	0.9580	0.9650	0.9710	0.9510	0.7850	0.9580	0.9610	0.9720
	III	1.2542	1.5062	1.2805	1.2275	1.1770	0.7355	0.8409	0.7891	0.7293	0.7111	1.7160	1.9418	1.7081	1.7208	1.6359	2.0363	2.3082	1.9089	2.0360	1.9218
		0.9340	0.9250	0.9490	0.9370	0.9460	0.9370	0.9370	0.9560	0.9530	0.9530	0.9530	0.9160	0.9520	0.9580	0.9640	0.9520	0.8890	0.9400	0.9470	0.9630
35,35	I	0.9891	1.1042	0.9589	0.9772	0.9513	0.6523	0.6586	0.6178	0.6522	0.6392	1.2551	1.3791	1.2081	1.2498	1.2157	1.5594	1.7169	1.5082	1.5504	1.4900
		0.9560	0.9280	0.9470	0.9590	0.9640	0.9450	0.9310	0.9330	0.9550	0.9660	0.9460	0.8870	0.9440	0.9460	0.9570	0.9250	0.7760	0.9320	0.9380	0.9420
	II	0.9897	1.1192	0.9452	0.9750	0.9490	0.6408	0.6636	0.6059	0.6366	0.6255	1.2339	1.3534	1.1923	1.2314	1.2002	1.5061	1.6320	1.4537	1.5075	1.4519
		0.9660	0.9140	0.9460	0.9640	0.9640	0.9620	0.9400	0.9460	0.9600	0.9640	0.9590	0.8700	0.9520	0.9590	0.9660	0.9450	0.7910	0.9460	0.9540	0.9570
	III	0.9310	1.0188	0.9393	0.9196	0.8973	0.6149	0.6706	0.6465	0.6096	0.5995	1.2610	1.3489	1.2506	1.2557	1.2193	1.6485	1.8030	1.5678	1.6456	1.5832
		0.9430	0.9270	0.9490	0.9420	0.9500	0.9390	0.9340	0.9540	0.9440	0.9520	0.9440	0.9290	0.9440	0.9480	0.9460	0.9460	0.9100	0.9310	0.9430	0.9450
75,65	I	0.6608	0.6733	0.6285	0.6564	0.6485	0.4775	0.4576	0.4387	0.4764	0.4715	0.8305	0.8653	0.8035	0.8242	0.8162	1.0901	1.1493	1.0706	1.0817	1.0613
		0.9700	0.9460	0.9530	0.9680	0.9710	0.9580	0.9410	0.9300	0.9600	0.9610	0.9460	0.8820	0.9390	0.9470	0.9470	0.8990	0.7750	0.9120	0.8980	0.9050
	II	0.6511	0.6667	0.6136	0.6456	0.6377	0.4636	0.4511	0.4251	0.4616	0.4572	0.8214	0.8511	0.7964	0.8172	0.8086	1.0507	1.0940	1.0267	1.0465	1.0274
		0.9560	0.9260	0.9370	0.9550	0.9570	0.9710	0.9470	0.9390	0.9640	0.9680	0.9440	0.8580	0.9410	0.9510	0.9550	0.9050	0.8040	0.9170	0.9090	0.9120
	III	0.6233	0.6488	0.6248	0.6192	0.6105	0.4467	0.4672	0.4596	0.4435	0.4391	0.8487	0.8718	0.8412	0.8420	0.8335	1.1829	1.2309	1.1334	1.1744	1.1513
		0.9500	0.9360	0.9470	0.9460	0.9450	0.9460	0.9340	0.9480	0.9510	0.9470	0.9520	0.9310	0.9450	0.9550	0.9560	0.9470	0.9250	0.9360	0.9430	0.9490
$T = 1.75$																					
20,24	I	1.3292	1.7374	1.3487	1.3074	1.2399	0.7669	0.8469	0.8290	0.7703	0.7455	1.6306	1.9270	1.5890	1.6347	1.5561	1.6830	1.9347	1.6429	1.6760	1.6044
		0.9390	0.9170	0.9490	0.9590	0.9700	0.9200	0.9190	0.9530	0.9390	0.9520	0.9510	0.8720	0.9390	0.9450	0.9570	0.9550	0.8800	0.9510	0.9580	0.9620
	II	1.3245	1.7551	1.3178	1.2947	1.2263	0.7472	0.8473	0.7911	0.7453	0.7241	1.6386	1.9419	1.5998	1.6450	1.5668	1.6304	1.8815	1.5932	1.6373	1.5715
		0.9560	0.9060	0.9340	0.9520	0.9630	0.9140	0.8980	0.9420	0.9340	0.9400	0.9440	0.8580	0.9400	0.9450	0.9520	0.9470	0.8760	0.9460	0.9520	0.9650
	III	1.2291	1.4738	1.2454	1.2063	1.1589	0.7376	0.8343	0.7768	0.7344	0.7156	1.6602	1.8813	1.6384	1.6635	1.5887	1.9702	2.2458	1.9200	1.9733	1.8643
		0.9450	0.9190	0.9460	0.9440	0.9620	0.9300	0.9260	0.9500	0.9490	0.9560	0.9490	0.9000	0.9450	0.9440	0.9520	0.9540	0.8860	0.9490	0.9620	0.9730
35,35	I	0.9750	1.1064	0.9850	0.9654	0.9378	0.6424	0.6857	0.6785	0.6424	0.6289	1.2012	1.3150	1.1793	1.1957	1.1645	1.3693	1.5061	1.3419	1.3617	1.3187
		0.9470	0.9410	0.9600	0.9590	0.9610	0.9470	0.9470	0.9680	0.9600	0.9660	0.9540	0.9010	0.9520	0.9570	0.9570	0.9450	0.8920	0.9430	0.9480	0.9500
	II	0.9694	1.1177	0.9667	0.9550	0.9309	0.6205	0.6730	0.6453	0.6189	0.6058	1.1894	1.3025	1.1689	1.1889	1.1557	1.3343	1.4709	1.3116	1.3305	1.2932
		0.9520	0.9190	0.9470	0.9450	0.9550	0.9370	0.9170	0.9540	0.9440	0.9500	0.9550	0.9030	0.9450	0.9520	0.9570	0.9440	0.8940	0.9410	0.9430	0.9440
	III	0.9303	1.0164	0.9313	0.9179	0.8971	0.6092	0.6602	0.6275	0.6052	0.5937	1.2465	1.3278	1.2277	1.2396	1.2071	1.6136	1.7665	1.5831	1.6079	1.5463
		0.9520	0.9280	0.9500	0.9520	0.9490	0.9460	0.9440	0.9570	0.9550	0.9550	0.9570	0.9140	0.9510	0.9550	0.9610	0.9590	0.9220	0.9510	0.9560	0.9590
75,65	I	0.6521	0.6858	0.6545	0.6479	0.6399	0.4668	0.4783	0.4776	0.4657	0.4607	0.8035	0.8320	0.7904	0.7978	0.7895	0.9748	1.0224	0.9613	0.9677	0.9540
		0.9530	0.9370	0.9510	0.9590	0.9560	0.9460	0.9410	0.9510	0.9500	0.9490	0.9500	0.9130	0.9460	0.9450	0.9460	0.9480	0.9160	0.9410	0.9470	0.9480
	II	0.6408	0.6767	0.6386	0.6361	0.6284	0.4507	0.4679	0.4590	0.4484	0.4440	0.7825	0.8133	0.7731	0.7782	0.7719	0.9547	1.0011	0.9406	0.9512	0.9359
		0.9460	0.9300	0.9490	0.9510	0.9530	0.9390	0.9360	0.9450	0.9420	0.9370	0.9500	0.9270	0.9430	0.9450	0.9540	0.9380	0.9160	0.9390	0.9430	0.9440
	III	0.6182	0.6419	0.6174	0.6140	0.6062	0.4416	0.4574	0.4468	0.4390	0.4352	0.8372	0.8589	0.8279	0.8304	0.8235	1.1569	1.2072	1.1423	1.1486	1.1258
		0.9340	0.9240	0.9250	0.9260	0.9350	0.9500	0.9380	0.9470	0.9500	0.9520	0.9610	0.9340	0.9580	0.9590	0.9590	0.9510	0.9390	0.9480	0.9480	0.9530

Table 8. Lengths (first rows) and coverage probabilities (second rows) of the approximate confidence intervals for the case of $\alpha = 1.50$, $\lambda = 1.50$, $\beta_1 = 0.75$, $\beta_2 = 0.75$.

		α					λ					β_1					β_2				
m,n	CS	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1	ACI	$Boot-p$	$Boot-t$	HPD_0	HPD_1
$T = 3.50$																					
20,24	I	1.6234	2.2357	1.5995	1.5856	1.4806	1.4849	1.9203	1.4661	1.4556	1.3759	0.6170	0.7354	0.6027	0.6185	0.5925	0.5531	0.6373	0.5402	0.5519	0.5348
		0.9540	0.8980	0.9330	0.9470	0.9540	0.9540	0.9020	0.9250	0.9400	0.9440	0.9460	0.8850	0.9470	0.9540	0.9600	0.9460	0.8660	0.9470	0.9480	0.9490
	II	1.7270	2.5091	1.6519	1.6798	1.5355	1.5528	2.0805	1.4963	1.5141	1.4179	0.6098	0.7259	0.5960	0.6132	0.5860	0.5454	0.6292	0.5341	0.5470	0.5279
		0.9570	0.8920	0.9260	0.9460	0.9600	0.9680	0.8900	0.9350	0.9560	0.9590	0.9460	0.8750	0.9460	0.9470	0.9530	0.9540	0.8720	0.9510	0.9520	0.9580
	III	1.6747	2.1417	1.6219	1.6308	1.5160	1.5192	1.8890	1.4694	1.4847	1.3969	0.6633	0.7604	0.6504	0.6653	0.6396	0.6095	0.6842	0.5967	0.6102	0.5903
		0.9550	0.9250	0.9480	0.9520	0.9670	0.9420	0.9050	0.9240	0.9300	0.9450	0.9570	0.9060	0.9530	0.9510	0.9620	0.9710	0.9130	0.9630	0.9710	0.9720
35,35	I	1.1802	1.3792	1.1750	1.1655	1.1252	1.1911	1.3995	1.1871	1.1715	1.1334	0.4510	0.4970	0.4437	0.4488	0.4391	0.4521	0.4978	0.4441	0.4500	0.4404
		0.9420	0.9050	0.9410	0.9410	0.9450	0.9540	0.9030	0.9420	0.9420	0.9470	0.9520	0.9140	0.9480	0.9480	0.9470	0.9510	0.9040	0.9510	0.9540	0.9530
	II	1.2389	1.4938	1.2120	1.2159	1.1699	1.2096	1.4529	1.1871	1.1875	1.1455	0.4445	0.4893	0.4371	0.4438	0.4332	0.4448	0.4899	0.4365	0.4443	0.4341
		0.9390	0.8750	0.9200	0.9370	0.9410	0.9640	0.9110	0.9570	0.9630	0.9660	0.9400	0.8860	0.9350	0.9380	0.9360	0.9540	0.8970	0.9530	0.9540	0.9550
	III	1.2542	1.4360	1.2253	1.2299	1.1876	1.2558	1.4400	1.2254	1.2326	1.1902	0.5007	0.5401	0.4934	0.4976	0.4882	0.5045	0.5438	0.4965	0.5030	0.4920
		0.9570	0.9270	0.9400	0.9470	0.9540	0.9600	0.9250	0.9510	0.9560	0.9600	0.9530	0.9270	0.9520	0.9560	0.9610	0.9390	0.9030	0.9390	0.9400	0.9480
75,65	I	0.7783	0.8288	0.7765	0.7717	0.7615	0.8380	0.9050	0.8371	0.8301	0.8158	0.3024	0.3146	0.2985	0.3006	0.2972	0.3239	0.3392	0.3197	0.3212	0.3182
		0.9470	0.9270	0.9420	0.9450	0.9420	0.9490	0.9280	0.9420	0.9470	0.9410	0.9450	0.9140	0.9460	0.9460	0.9440	0.9570	0.9190	0.9500	0.9510	0.9560
	II	0.7922	0.8535	0.7845	0.7840	0.7743	0.8540	0.9272	0.8423	0.8449	0.8296	0.2933	0.3051	0.2900	0.2919	0.2898	0.3155	0.3305	0.3115	0.3139	0.3100
		0.9480	0.9330	0.9380	0.9470	0.9490	0.9520	0.9360	0.9480	0.9570	0.9600	0.9480	0.9230	0.9520	0.9450	0.9500	0.9650	0.9380	0.9610	0.9610	0.9650
	III	0.8143	0.8626	0.8026	0.8074	0.7935	0.8812	0.9353	0.8690	0.8703	0.8569	0.3359	0.3465	0.3319	0.3334	0.3312	0.3612	0.3738	0.3574	0.3585	0.3550
		0.9630	0.9380	0.9560	0.9570	0.9560	0.9520	0.9360	0.9520	0.9540	0.9570	0.9350	0.9270	0.9350	0.9390	0.9400	0.9380	0.9150	0.9330	0.9300	0.9340
$T = 5.00$																					
20,24	I	1.6234	2.2357	1.5995	1.5856	1.4806	1.4849	1.9203	1.4662	1.4556	1.3759	0.6170	0.7354	0.6027	0.6185	0.5925	0.5531	0.6373	0.5403	0.5519	0.5348
		0.9540	0.8980	0.9330	0.9470	0.9540	0.9540	0.9020	0.9250	0.9400	0.9440	0.9460	0.8850	0.9470	0.9540	0.9600	0.9460	0.8670	0.9470	0.9480	0.9490
	II	1.7270	2.5091	1.6519	1.6798	1.5355	1.5528	2.0805	1.4963	1.5141	1.4179	0.6098	0.7259	0.5960	0.6132	0.5860	0.5454	0.6292	0.5341	0.5470	0.5279
		0.9570	0.8920	0.9260	0.9460	0.9600	0.9680	0.8900	0.9360	0.9560	0.9590	0.9460	0.8750	0.9460	0.9470	0.9530	0.9540	0.8720	0.9510	0.9520	0.9580
	III	1.6747	2.1417	1.6219	1.6308	1.5160	1.5192	1.8888	1.4691	1.4847	1.3969	0.6633	0.7604	0.6504	0.6653	0.6396	0.6095	0.6842	0.5968	0.6102	0.5903
		0.9550	0.9250	0.9480	0.9520	0.9670	0.9420	0.9050	0.9230	0.9300	0.9450	0.9570	0.9060	0.9530	0.9510	0.9620	0.9710	0.9130	0.9630	0.9710	0.9720
35,35	I	1.1802	1.3792	1.1750	1.1655	1.1252	1.1911	1.3995	1.1871	1.1715	1.1334	0.4510	0.4970	0.4437	0.4488	0.4391	0.4521	0.4978	0.4441	0.4500	0.4404
		0.9420	0.9050	0.9410	0.9410	0.9450	0.9540	0.9030	0.9420	0.9420	0.9470	0.9520	0.9140	0.9480	0.9480	0.9470	0.9510	0.9040	0.9510	0.9540	0.9530
	II	1.2389	1.4939	1.2120	1.2159	1.1699	1.2096	1.4529	1.1871	1.1875	1.1455	0.4445	0.4893	0.4371	0.4438	0.4332	0.4448	0.4899	0.4365	0.4443	0.4341
		0.9390	0.8750	0.9200	0.9370	0.9410	0.9640	0.9110	0.9570	0.9630	0.9660	0.9400	0.8860	0.9350	0.9380	0.9360	0.9540	0.8970	0.9530	0.9540	0.9550
	III	1.2542	1.4360	1.2253	1.2299	1.1876	1.2558	1.4400	1.2254	1.2326	1.1902	0.5007	0.5401	0.4934	0.4976	0.4882	0.5045	0.5438	0.4965	0.5030	0.4920
		0.9570	0.9270	0.9400	0.9470	0.9540	0.9600	0.9250	0.9510	0.9560	0.9600	0.9530	0.9270	0.9520	0.9560	0.9610	0.9390	0.9030	0.9390	0.9400	0.9480
75,65	I	0.7783	0.8288	0.7765	0.7717	0.7615	0.8380	0.9050	0.8371	0.8301	0.8158	0.3024	0.3146	0.2985	0.3006	0.2972	0.3239	0.3392	0.3197	0.3212	0.3182
		0.9470	0.9270	0.9420	0.9450	0.9420	0.9490	0.9280	0.9420	0.9470	0.9410	0.9450	0.9140	0.9460	0.9460	0.9440	0.9570	0.9190	0.9500	0.9510	0.9560
	II	0.7922	0.8535	0.7845	0.7840	0.7743	0.8540	0.9272	0.8423	0.8449	0.8296	0.2933	0.3051	0.2900	0.2919	0.2898	0.3155	0.3305	0.3115	0.3139	0.3100
		0.9480	0.9330	0.9380	0.9470	0.9490	0.9520	0.9360	0.9480	0.9570	0.9600	0.9480	0.9230	0.9520	0.9450	0.9500	0.9650	0.9380	0.9610	0.9610	0.9650
	III	0.8143	0.8626	0.8026	0.8074	0.7935	0.8812	0.9353	0.8690	0.8703	0.8569	0.3359	0.3465	0.3319	0.3334	0.3312	0.3612	0.3738	0.3574	0.3585	0.3550
		0.9630	0.9380	0.9560	0.9570	0.9560	0.9520	0.9360	0.9520	0.9540	0.9570	0.9350	0.9270	0.9350	0.9390	0.9400	0.9380	0.9150	0.9330	0.9300	0.9340

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References

[1] O. Abo-Kasem and A. Elshahhat, *A new two sample generalized type-II hybrid censoring scheme*, Am. J. Math. Manag. Sci. **41** (2), 170184, 2022.

[2] O. E. Abo-Kasem and A. Elshahhat, *Analysis of two Weibull populations under joint progressively hybrid censoring*, Commun. Stat. Simul. Comput. **52** (9), 4469-4490, 2021.

[3] I. Alam, M. Kamal, A. Rahman, T. Agarwal and A. Mishra, *Statistical analysis using multiple censoring scheme under partially accelerated life tests for the power Lindley distribution*, Int. J. Syst. Assur. Eng. Manag. **15**, 34243436, 2024.

[4] F. E. Almuhayfith, *Statistical inference of comparative generalized inverted exponential populations under joint adaptive progressive type-II censored samples*, Alex. Eng. J. **95**, 262271, 2024.

[5] R. Alotaibi, A. Elshahhat and M. Nassar, *Analysis of muth parameters using generalized progressive hybrid censoring with application to sodium sulfur battery*, J. Radiat. Res. Appl. Sci. **16** (3), 100624, 2023.

[6] S. K. Ashour and O. E. Abo-Kasem, *Statistical inference for two exponential populations under joint progressive type-I censored scheme*, Commun. Stat. Theory Methods **46** (7), 34793488, 2017.

[7] N. Balakrishnan and E. Cramer, *The art of progressive censoring*, Springer, Birkhäuser, New York, 2014.

[8] N. Balakrishnan and A. Rasouli, *Exact likelihood inference for two exponential populations under joint type-II censoring*, Comput. Stat. Data Anal. **52** (5), 27252738, 2008.

[9] N. Balakrishnan and F. Su, *Exact likelihood inference for k exponential populations under joint type-II censoring*, Commun. Stat. Simul. Comput. **44** (3), 591613, 2015.

[10] Ç. Çetinkaya, *Reliability estimation of a stress-strength model with non-identical component strengths under generalized progressive hybrid censoring scheme*, Stat. **55** (2), 250275, 2021.

[11] Ç. Çetinkaya, F. Sultana and D. Kundu, *Exact likelihood inference for two exponential populations under jointly generalized progressive hybrid censoring*, J. Stat. Comput. Simul. **92** (17), 36053629, 2022.

[12] Y. Cho, H. Sun and K. Lee, *Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme*, Stat. Methodol. **23**, 1834, 2015.

[13] P. Congdon, *Bayesian statistical modelling*, John Wiley Sons, 2007.

[14] R Core Team et al., *R: A language and environment for statistical computing*, R Found. Stat. Comput., Vienna, 2013.

[15] S. M. Curtis, I. Goldin and E. Evangelou, *Package mcmcplots: Create Plots from MCMC Output*, CRAN Repository, 2018.

[16] M. Doostparast, M. V. Ahmadi and J. Ahmadi, *Bayes estimation based on joint progressive type-II censored data under Linex loss function*, Commun. Stat. Simul. Comput. **42** (8), 18651886, 2013.

[17] A. Elshahhat, *Parameters estimation for the exponentiated Weibull distribution based on generalized progressive hybrid censoring schemes*, Am. J. Appl. Math. Stat. **5** (2), 3348, 2017.

[18] A. Elshahhat, H. H. Ahmad, A. Rabaiah and O. E. Abo-Kasem, *Analysis of a new jointly hybrid censored Rayleigh populations*, AIMS Math. **9** (2), 37403762, 2024.

[19] A. Gelman, J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari and D. B. Rubin, *Bayesian data analysis*, 3rd ed., Chapman Hall/CRC, 2004.

[20] J. Geweke, *Evaluating the accuracy of sampling-based approaches to the calculations of posterior moments*, Bayesian Stat. **4**, 641649, 1992.

[21] A. S. Hassan, R. M. Mousa, and M. H. Abu-Moussa, *Bayesian analysis of generalized inverted exponential distribution based on generalized progressive hybrid censoring competing risks data*, Ann. Data Sci. **11** (4), 12251264, 2024.

[22] H. Jeffreys, *Theory of probability (3rd ed.)*, Oxford: Clarendon Press, 1961.

[23] A. Koley and D. Kundu, *On generalized progressive hybrid censoring in presence of competing risks*, Metrika **80**, 401426, 2017.

[24] H. Krishna, M. Dube and R. Garg, *Estimation of stress-strength reliability of inverse Weibull distribution under progressive first failure censoring*, Aust. J. Stat. **48** (1), 1437, 2019.

[25] H. Krishna and R. Goel, *Jointly type-II censored Lindley distributions*, Commun. Stat. Theory Methods **51** (1), 135149, 2022.

[26] D. Kundu and A. Joarder, *Analysis of type-II progressively hybrid censored data*, Comput. Stat. Data Anal. **50** (10), 25092528, 2006.

[27] C. T. Lin, Y. C. Chen, T. C. Yeh and H. K. T. Ng, *Statistical inference and optimum life-testing plans with joint progressively type-II censoring scheme*, Qual. Technol. Quant. Manag. **20** (3), 279306, 2023.

[28] S. M. Lynch, *Introduction to applied Bayesian statistics and estimation for social scientists*, Springer, New York, 2007.

[29] M. Maswadah, *Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme*, J. Appl. Stat. **49** (11), 28252844, 2022.

[30] S. Mondal and D. Kundu, *Point and interval estimation of Weibull parameters based on joint progressively censored data*, Sankhya B **81** (1), 125, 2019.

[31] S. Mondal and D. Kundu, *Bayesian inference for Weibull distribution under the balanced joint type-II progressive censoring scheme*, Am. J. Math. Manag. Sci. **39** (1), 5674, 2020.

[32] M. Plummer, N. Best, K. Cowles and K. Vines, *CODA: Convergence Diagnosis and Output Analysis for MCMC*, R News **6**, 711, 2006.

[33] A. Rahman, M. Kamal, S. Khan, M. F. Khan, M. S. Mustafa, E. Hussam, M. N. Atchadé and A. Al Mutairi, *Statistical inferences under step stress partially accelerated life testing based on multiple censoring approaches using simulated and real-life engineering data*, Sci. Rep. **13**, 12452, 2023.

[34] A. Rasouli and N. Balakrishnan, *Exact likelihood inference for two exponential populations under joint progressive type-II censoring*, Commun. Stat. Theory Methods **39** (12), 21722191, 2010.

[35] S. Salem, O. E. Abo-Kasem and A. Hussien, *On joint type-II generalized progressive hybrid censoring scheme*, Comput. J. Math. Stat. Sci. **2** (1), 123158, 2023.

[36] B. Saraçolu, . Kinaci and D. Kundu, *On estimation of $R = P(Y < X)$ for exponential distribution under progressive type-II censoring*, J. Stat. Comput. Simul. **82** (5), 729744, 2012.

[37] W. Shi and W. Gui, *Estimation for two Gompertz populations under a balanced joint progressive type-II censoring scheme*, J. Appl. Stat. **51** (8), 14701496, 2024.

[38] F. Su and X. Zhu, *Exact likelihood inference for two exponential populations based on a joint generalized type-I hybrid censored sample*, J. Stat. Comput. Simul. **86** (7), 13421362, 2016.

[39] F. Sultana, A. Koley, A. Pal and D. Kundu, *On two exponential populations under a joint adaptive type-II progressive censoring*, Stat. **55** (6), 13281355, 2021.

[40] R. J. Tibshirani and B. Efron, *An introduction to the bootstrap*, Monogr. Stat. Appl. Probab., London: CRC Press, **57**, 1436, 1993.

[41] Z. Xia, J. Yu, L. Cheng, L. Liu and W. Wang, *Study on the breaking strength of jute fibres using modified Weibull distribution*, Compos. Part A Appl. Sci. Manuf. **40** (1), 5459, 2009.

[42] T. Zhu, *Reliability inference for multicomponent stressstrength model under generalized progressive hybrid censoring*, J. Comput. Appl. Math. **451**, 116015, 2024.

APPENDIX

It can be shown that the MLEs of α and λ maximize the log-likelihood function $\ell(w, z, \alpha, \lambda, \beta_1, \beta_2)$ for given β_1 and β_2. For this purpose, let $\Phi(\theta) = \Phi(\alpha, \lambda)$ be the Hessian matrix of $\ell(w, z, \alpha, \lambda, \beta_1, \beta_2)$ at $(\hat{\alpha}, \hat{\lambda})$. Thus,

$$\phi_{ii}(\theta) = \frac{\partial^2 \ell}{\partial \theta_i^2} = -\frac{D_i}{\theta_i^2}, \quad i = 1, 2 \quad \text{and} \quad \phi_{12}(\theta) = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = 0$$

then, the determinant of the Hessian matrix is obtained as

$$\det[\phi(\hat{\theta})] = \phi_{11}(\hat{\alpha}, \hat{\lambda})\phi_{22}(\hat{\alpha}, \hat{\lambda}) - \left[\phi_{12}(\hat{\alpha}, \hat{\lambda})\right]^2 = \frac{D_1 D_2}{\hat{\alpha}^2 \hat{\lambda}^2} > 0.$$

It is clearly seen that $(\hat{\alpha}, \hat{\lambda})$ is the local maximum of $\ell(\mathbf{w{,}z}, \alpha, \lambda, \beta_1, \beta_2)$ for given β_1 and β_2. Since there is no singular point of $\ell(\mathbf{w{,}z}, \alpha, \lambda, \beta_1, \beta_2)$ and it has a single critical point, $\hat{\alpha}$ and $\hat{\lambda}$ are the absolute maximum of the log-likelihood function. We can further express the equations (3.4) as

$$\frac{D_1}{\beta_1} = \hat{\alpha}\xi'(\beta_1) - \sum_{i=1}^{\eta} z_i \ln(w_i) \qquad \text{and} \qquad \frac{D_2}{\beta_2} = \hat{\lambda}\xi'(\beta_2) - \sum_{i=1}^{\eta}(1 - z_i)\ln(w_i) \tag{8.1}$$

Let denote the left and the right-hand sides of the equations in (8.1) by $\psi_{1_a}(\beta_1, \mathbf{w{,}z})$, $\psi_{1_b}(\beta_1, \mathbf{w{,}z})$, $\psi_{2_a}(\beta_2, \mathbf{w{,}z})$, $\psi_{2_b}(\beta_2, \mathbf{w{,}z})$ respectively as given in the following

$$\psi_{1_a}(\beta_1, \mathbf{w{,}z}) = \frac{D_1}{\beta_1} \quad \text{and} \quad \psi_{1_b}(\beta_1, \mathbf{w{,}z}) = \hat{\alpha}\xi'(\beta_1) - \sum_{i=1}^{\eta} z_i \ln(w_i)$$

$$\psi_{2_a}(\beta_2, \mathbf{w{,}z}) = \frac{D_2}{\beta_2} \quad \text{and} \quad \psi_{2_b}(\beta_2, \mathbf{w{,}z}) = \hat{\lambda}\xi'(\beta_2) - \sum_{i=1}^{\eta}(1 - z_i)\ln(w_i)$$

For a given sample of $\mathbf{w}$, it can be shown that $\psi_{1_a}(\beta_1, \mathbf{w{,}z})$ and $\psi_{2_a}(\beta_2, \mathbf{w{,}z})$ are the increasing monotonic functions of β_1 and β_2 with finite and positive limits such as $\beta_1 \to \infty$, $\beta_2 \to \infty$. The plots of $\psi_{1_a}(\beta_1, \mathbf{w{,}z})$ and $\psi_{1_b}(\beta_1, \mathbf{w{,}z})$ would intersect exactly once at $\hat{\beta_1}$ since $\frac{D_1}{\beta_1}$ is strictly decreasing with the right limit ∞ at 0. A similar intersection would also be observed for β_2. The proof of the β_1 case can be shown that $\frac{\partial \psi_{1_b}(\beta_1, \mathbf{w{,}z})}{\partial \beta_1} \geq 0$. It is seen that

$$\frac{\partial \psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z})}{\partial \beta_1} = \hat{\alpha}\xi^{''}(\beta_1) = \hat{\alpha}\left[\sum_{i=1}^{\eta}(s_i + z_i)w_i^{\beta_1}\ln^2(w_i) + \delta T^{\beta_1}\ln^2(T)R_S^*\right] \geq 0$$

The same case can be easily seen for β_2, also. It is seen that $\psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z})$ is a monotone increasing function of β_1. Furthermore, $\lim_{\beta_! \to 0}\psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z}) > 0$ and $\lim_{\beta_1 \to \infty}\psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z}) \to \infty$ indicates that the curves of $\psi_{1_a}(\beta_1, \mathbf{w},\mathbf{z})$ and $\psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z})$ have a unique intersection point. Therefore, $\hat{\beta}_1$, as the root of equation $\psi_{1_a}(\beta_1, \mathbf{w},\mathbf{z}) = \psi_{1_b}(\beta_1, \mathbf{w},\mathbf{z})$ exist and unique. Similarly, $\hat{\beta}_2$, as the root of equation $\psi_{2_a}(\beta_2, \mathbf{w},\mathbf{z}) = \psi_{2_b}(\beta_2, \mathbf{w},\mathbf{z})$ exist and unique.

Following the obtaining unique MLEs of β_1 and β_2, the MLEs of α and λ can be obtained uniquely as $\hat{\alpha} = \hat{\alpha}(\hat{\beta}_1)$ and $\hat{\lambda} = \hat{\lambda}(\hat{\beta}_2)$, respectively. Thus, the existence and uniqueness of MLEs are proved.