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NONLINEAR HEAT TRANSFER IN RECTANGULAR FINS AND EXACT SOLUTIONS WITH TEMPERATURE DEPENDENT PROPERTIES

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Abstract: It is aimed to present general exact solution formulae valid for the nonlinear heat transfer of fin problems in this paper. Although some analytical solutions were previously worked out for temperature dependent thermal conductivity and heat transfer coefficients, they are shown to be inadequate in some cases. The presented formulas enable us to better understand the physical properties of interest in terms of the fin base heat transfer rate as well as fin efficiency and fin tip temperature.
Keywords: Heat transfer, Nonlinear fin problem, Exact solutions.

DİKDÖRTGENSEL FİNLERDE DOĞRUSAL OLMAYAN ISI TRANSFERİ VE SICAKLIĞA BAĞLI FİZİKSEL BÜYÜKLÜKLER İÇİN ANALİTİK ÇÖZÜMLER

Özet: Bu makalede fin problemlerinin doğrusal olmayan ısı transferi için geçerli olan genel çözüm formüllerinin ortaya konması hedeflenmektedir. Her nekadar sıcaklığa bağlı ısı geçirgenliği ve ısı katsayıları hakkında daha önceden bazı çalışmalar yapılmış olsa da, bazı hallerde bu çalışmaların yetersiz oldukları gösterilmiştir. Bu makalede sunulan yeni formüller fin taban ısı transferi, fin verimliliği ve fin bitim noktası sıcaklığı gibi ilgi çekici fiziksel özellikleri daha iyi anlamamıza ve analiz etmemize yardımcı olacaktır.
Anahtar Kelimeler: Isı transferi, Doğrusal olmayan fin problemi, Tam çözümler.

INTRODUCTION

A realistic modelling of heat transfer in fins (extended surfaces) results in a nonlinear problem to deal with, which has been the focus of recent research in industrial applications, like refrigeration, air-cooled craft engines and air conditioning (Kraus et al., 2002).

Due to the design simplicity in manufacturing processes, the rectangular fin is the most preferred (Lienard, 2011). A few experiments were performed over some fins to understand the impacts of heat transfer coefficient (Orzechowski, 2007; Castell et al., 2008). Popovych et al. (2008) constructed the general solutions for the unsteady state fin equation. Sharqawy and Zubair (2008) applied an analytical method for different profiles for straight fins with combined heat and mass transfer. Aziz and Bouaziz (2011) derived approximate expressions for fin related parameters. They also stated that some approximate methods produce results which are not easily accessible by the designers of fin. The exponential shape of fin was found better in terms of heating processes (Turkyilmazoglu, 2012). More recently, a series of work was implemented over different fin profiles to estimate some physical features (Turkyilmazoglu, 2014a; Turkyilmazoglu, 2014b; Turkyilmazoglu, 2014c).

Aziz and Na (1981) studied the effects variable thermal parameters on the periodic heat transfer in fins. Group classification of fin equation with variable thermal properties was then given (Pakdemirli and Sahin, 2004). Temperature-dependent thermal conductivity of some fin profiles was explored (Khani et al., 2009; Moradi and Ahmadikia, 2010). Some exact solutions of the fin problem with a power law temperature dependent thermal conductivity were presented (Moitsheki et al., 2010; Mhlongo and Moitsheki, 2014). However, it appears that there are some errors and the presentation by Moitsheki et al. (2010) is not adequate since no a general formulation for the active parameters of physical importance was given. Therefore, taking into account the comments made by Aziz and Bouaziz (2011) and the work by Moitsheki et al. (2010), the main objective of the present study is to search for closed-form solutions from which the temperature distribution over a straight rectangular fin, the efficiency of it and the rate of heat transfer may be easily computed. Therefore, an advance over the work of by Moitsheki et al. (2010) is achieved. Moreover, the present work differs from Turkyilmazoglu (2012) in general, since an exponential fin profile is adopted there. Furthermore, the given solutions could serve to validate other numerical scheme.

FORMULATION OF THE PROBLEM

Although the model was already given in by Moitsheki et al. (2010), for completeness, it is again presented here. A straight rectangular fin of length L with a uniform cross sectional area A is considered. The fin is attached to a base surface of temperature T_b and extends into a fluid of constant temperature T_a Moreover, the heat transfer through the tip end is not taken into account. The one-dimensional steady state heat balance equation in dimensional form may be written as

$$A\frac{d}{dX}(k(T)\frac{dT}{dX}) = ph(T)(T-T_a), \quad 0 < X < L, \tag{1}$$

where X measures the axial distance commencing from the tip of the fin, the perimeter is denoted by p and h is the heat transfer coefficient showing a nonlinear dependence on the temperature, an ideal case of which is studied in what follows. To put the governing equation (1) into the non dimensional form the following set of transformations are employed

$$k = k_a\theta^m, \quad N^2 = \frac{h_b pL^2}{k_a A}, \quad x = \frac{X}{L},$$
$$\theta = \frac{T-T_a}{T_b-T_a}, \quad h(T) = h_b\theta^n,$$

where h_b means the base heat transfer coefficient, N is the fin convective-conductive parameter, the thermal conductivity is k known to be decreasing sharply with increasing temperature for most of the engineering materials, k_a denotes the thermal conductivity relevant to the ambient temperature and the exponent n can take several values (Chowdhury and Hashim, 2008; Turkyilmazoglu, 2012). Taking into account that the fin tip is insulated, the following simplified model governs the problem under consideration

$$\frac{d}{dx}(\theta^m\frac{d\theta}{dx}) = N^2\theta^{n+1}, \quad 0 < x < 1$$
$$\theta'(0) = 0, \quad \theta(1) = 1. \tag{2}$$

In terms of engineering applications we need to take into account three physical parameters, the value of temperature at the tip of the fin ζ, the efficiency of fin η and the rate of heat transfer at the base of the fin φ, respectively, all calculated from

$$\zeta = \theta(0), \quad \eta = \int_0^1\theta(x)dx, \quad \varphi = \theta'(1). \tag{3}$$

PARTICULAR SOLUTIONS

For certain specific values of m and n, exact solutions are derived in the present section using state of the art program MATHEMATICA. The advantages over the existing results by Moitsheki et al. (2010) are discussed. Case by case study is pursued as in Turkyilmazoglu (2012) ending up with a general case implicit formulation.

CASE $m = n,\ n+1 > 0$

Equation (2) in this case can be linearized via the substitution $\theta^{n+1} = u$. After some manipulations the solution satisfying exactly (2) is expressed by

$$\theta = [\frac{\cosh\chi x}{\cosh\chi}]^{\frac{1}{n+1}}, \tag{4}$$

where $\chi^2 = N^2(n+1)$. Therefore, the solution depends on n and χ. It should be pointed out that this solution was also obtained in Moitsheki et al. (2010). However, besides this solution, another solution was also given in Moitsheki et al. (2010) for the interval $-1 < n < 0$ (see equation (9) in Moitsheki et al., 2010), which is certainly wrong since boundary conditions in (2) are violated.

The parameters of engineering interest are computed from (4)

$$\zeta = sech^{\frac{1}{1+n}}\chi,$$
$$\eta = \frac{(1+n)}{(2+n)\chi}(\cosh\chi F_1(\frac{1}{2},\frac{2+n}{2+2n},\frac{4+3n}{2+2n},\cosh^2\chi)$$
$$-sech^{\frac{1}{1+n}}\chi F(\frac{1}{2},\frac{2+n}{2+2n},\frac{4+3n}{2+2n},1)), \quad \varphi = \frac{\chi\tanh\chi}{1+n}, \tag{5}$$

where

$$F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt$$

and $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$

The hypergeometric function F_1 can alternatively be calculated from MacLaurin series expansion

$$F_1(a,b,c,z) = \sum_{k=0}^{\infty}(a)_k(b)_k/(c)_k z^k/k!,$$

that converges very slowly and becomes undefined when $|z| > 1$ and $c < 0$. Note also that Moitsheki et al. (2010) was able to give the fin efficiency parameter η only for $n = 0$. The symmetry with respect to χ, $\chi \geq 0$ should be anticipated in this case. Four distinct n are displayed in figure 1.

It can be inferred from the figure that ζ and η are decreasing functions χ, while φ is just opposite. As n gets higher, slight increase in ζ and η and slight decrease in φ are visualized.

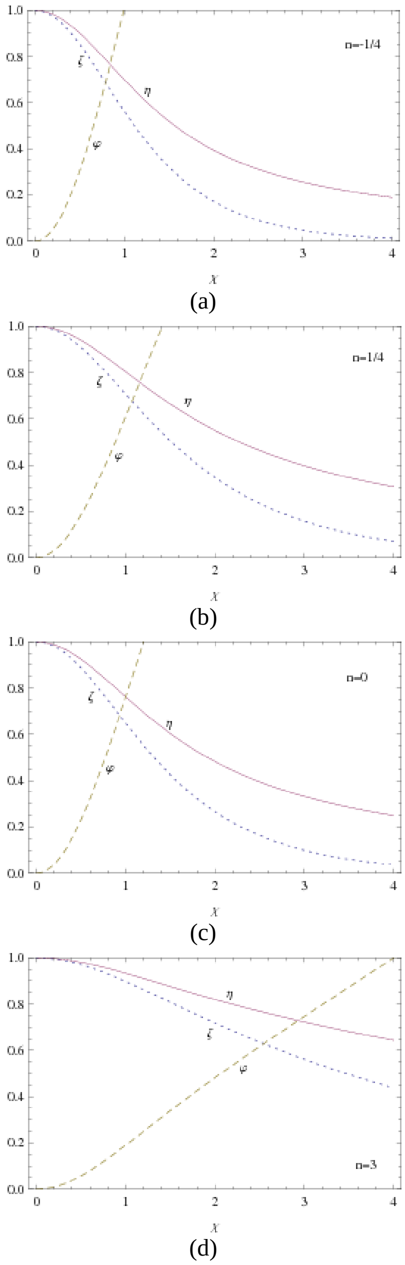


Figure 1. The physical parameters ζ , η and φ against χ .

CASE $m = n,\ n+1 < 0$

This particular constraint was surprisingly discarded by Moitsheki et al. (2010). In this case the temperature solution to (2)

$$\theta = [\frac{\cos \chi x}{\cos \chi}]^{\frac{1}{n+1}}, \tag{6}$$

where $\chi^2 = -N^2(n+1)$.

The physically admissible values lie in $0 \le \chi < \pi/2$ with symmetry again. The parameters ζ , η and φ in this case are computed from (6)

$$\zeta = \sec^{\frac{1}{1+n}} \chi,$$

$$\eta = \frac{(1+n)}{(2+n)\chi}(-\cos \chi F_1(\frac{1}{2}, \frac{2+n}{2+2n}, \frac{4+3n}{2+2n}, \cos^2 \chi)$$
$$+\sec^{\frac{1}{1+n}} \chi F(\frac{1}{2}, \frac{2+n}{2+2n}, \frac{4+3n}{2+2n}, 1)), \quad n \ne -2,$$
$$\eta = -\frac{1}{2\chi}\cos \chi(-\ln(4\sec^2 \chi) + F^{(0,0,1,0)}(\frac{1}{2}, 0, 1, \cos^2 \chi)$$
$$+F^{(0,1,0,0)}(\frac{1}{2}, 0, 1, \cos^2 \chi)), \quad n = -2, \ \varphi = -\frac{\chi \tan \chi}{1+n}, \tag{7}$$

where a superscript denotes differentiation with respect to its place.

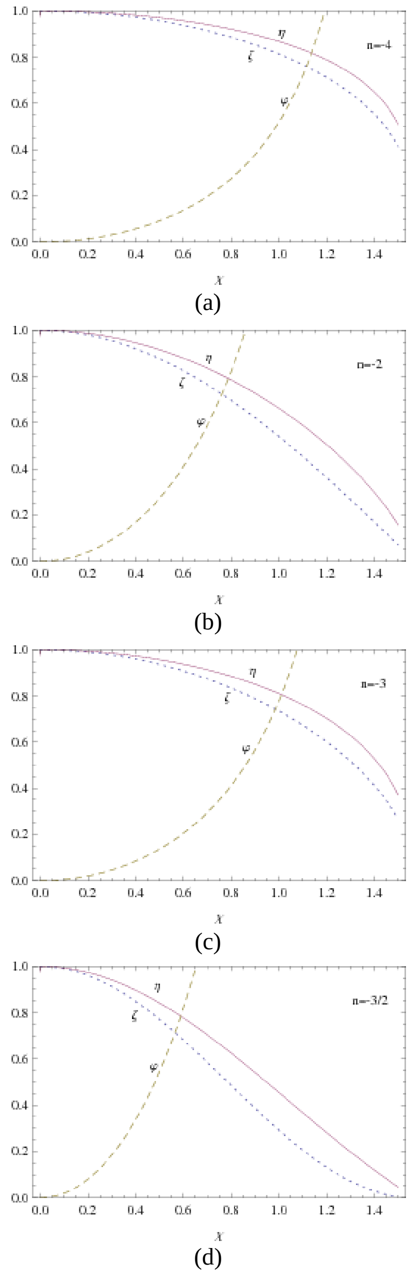


Figure 2. The physical parameters ζ , η and φ against χ

Figure 2 displays the parameters in (7) for four distinct negative n . The physical implication in this case is that increasing n gradually lowers the fin tip temperature and fin efficiency, but increases the base heat transfer rate.

CASE $m = n,\ n+1=0$

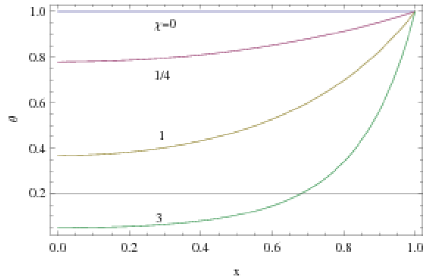
A direct integration of (2) yields the temperature solution in this case

$$\theta = e^{\chi(x^2-1)}, \tag{8}$$

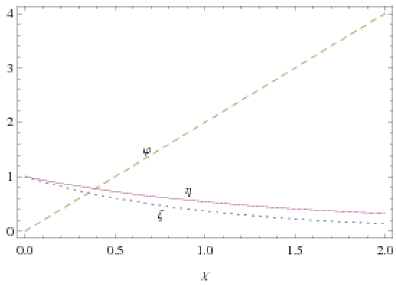
where $\chi^2 = N^2/2$. Symmetry regarding χ is absent, dissimilar to the above two situations. The physically relevant parameters are then computed from (8)

$$\begin{aligned} &\zeta = e^{-\chi}, \\ &\eta = \mathrm{D}awsonF(\sqrt{x})/\sqrt{x}, \\ &\varphi = 2\chi. \end{aligned} \tag{9}$$

Dawson's integral (Abramowitz, 1955), also sometimes called Dawson's function $\mathrm{D}awsonF(x)$, is the entire function defined by the integral $\frac{1}{i}e^{-x^2}\int_0^{ix} e^{-t^2}dt$ associated with the incomplete (imaginary) error function. In this case the physical parameters (9) perform similar to the first case for $\chi > 0$.



(a)



(b)

Figure 3. (a) Temperature profiles (b) ζ , η and φ against χ .

Figure 3 demonstrates temperature profiles and physical parameters in this case. The temperature is observed to be inversely proportional to the fin related parameter χ .

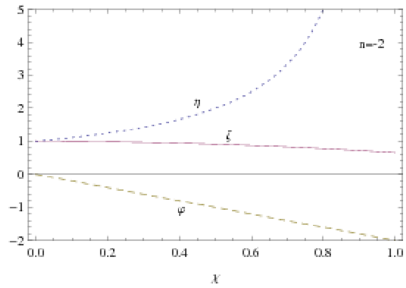
CASE $m \neq n,\ n+1=0$

Claiming a discontinuity at the fin tip, this case was not analyzed by Moitsheki et al. (2010). On the other hand, a straightforward integration of energy equation (2) results in

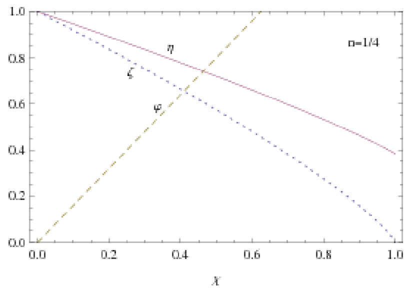
$$\theta = [\chi(x^2-1)+1]^{\frac{1}{m+1}}, \tag{10}$$

where $\chi^2 = N^2(m+1)/2$. Care should be given for $\chi = 1$, in which physical solutions occur only by the constraint $-1 < m < 1$, otherwise singularities arise. For $|\chi| > 1$, only $m = 0$ satisfies the physical requirement, otherwise $\theta(x)$ is not defined. For $|\chi| < 1$, true solutions can be found for all m. The important parameters in engineering are now

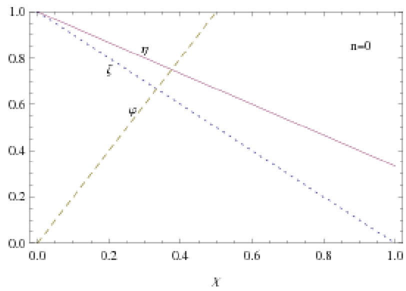
$$\begin{aligned} &\zeta = (1-\chi)^{\frac{1}{1+m}}, \\ &\eta = -\frac{F(1, \frac{3}{2}+\frac{1}{1+m}, \frac{3}{2}, \frac{\chi}{-1+\chi})}{-1+\chi}, \\ &\varphi = \frac{2\chi}{1+m}. \end{aligned} \tag{11}$$



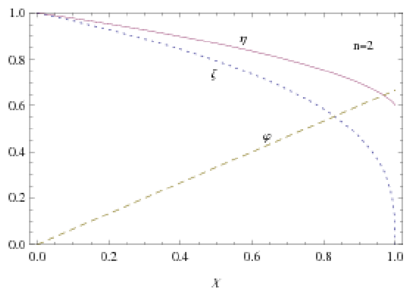
(a)



(b)



(c)



(d)

Figure 4. The physical parameters ζ , η and φ against χ .

The behaviors of physical parameters are captured in figure 4. In this case, increasing n degrades the base heat transfer while enhancing the fin tip temperature and efficiency.

GENERAL SOLUTIONS

Implicit form solutions in terms of non-algebraic functions, valid for the entire set of parameters m, n and N are presented in this section. To obtain the solutions in general case let us apply the change of variable $d\theta dx = u(\theta)$ in equation (2), which converts the equation into

$$\theta u u' + m u^2 = N^2 \theta^{n-m+2}, \quad u(\theta_0) = 0, \tag{12}$$

where $\theta_0 = \theta(0)$ is unknown (temperature at the tip of fin) to be determined by the second boundary condition in (2). Equation (12) is the first order Bernoulli type ordinary differential equation, whose solution is given by

$$u(\theta) = \sqrt{2}\sqrt{\frac{N^2\theta^{-2m}\left(\theta^{2+m+n} - \theta_0^{2+m+n}\right)}{2+m+n}}, \tag{13}$$

if $2+m+n \neq 0$. Otherwise, if $2+m+n = 0$ one gets

$$u(\theta) = \sqrt{2}\sqrt{N^2\theta^{-2m}\ln\frac{\theta}{\theta_0}}. \tag{14}$$

CASE $m+n+2 \neq 0$

In this case, direct integration of (13) gives rise to the implicit form solution

$$x(\theta) = \frac{x_1 + x_2(\theta)}{\sqrt{2}}, \tag{15}$$

where

$$x_1 = \frac{\sqrt{\pi}\,\Gamma\left[\dfrac{1+m}{2+m+n}\right]}{\sqrt{-(2+m+n)N^2\theta_0^{-m+n}}\,\Gamma\left[\dfrac{4+3m+n}{2(2+m+n)}\right]},$$

$$x_2(\theta) = \frac{1}{1+m}\sqrt{-(m+n+2)/N^2\theta^{n-m}\theta_0^{n+m+2}} \times$$
$$F\left(\frac{1}{2}, \frac{1+m}{2+m+n}, \frac{3+2m+n}{2+m+n}, \left(\frac{\theta}{\theta_0}\right)^{2+m+n}\right).$$

Enforcing the boundary condition at the right-hand location results in

$$\sqrt{2} = x_1 + x_2(1), \tag{16}$$

which enables us to determine the value of θ_0 for assigned parameters m, n and N. Eventually, substituting θ_0 found from (16) back into (15) yields the required temperature profiles in implicit form. Having found $\zeta = \theta(0)$ from (16), the other two physical parameters are evaluated as

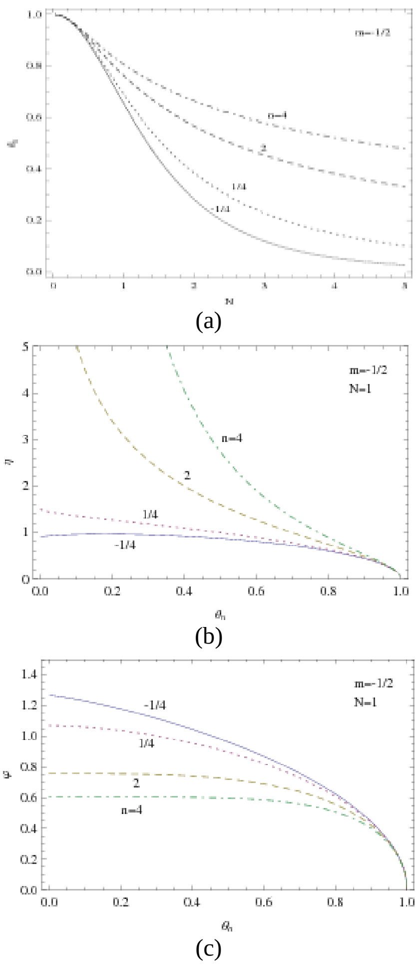
$$\eta = \int_0^1 \theta(x)dx = \int_{\theta_0}^1 \tau u(\tau)d\tau = \eta_1 + \eta_2\sqrt{2},$$
$$\varphi = u(1), \tag{17}$$

where

$$\eta_1 = -\frac{\sqrt{\pi}\theta_0\sqrt{-(2+m+n)\theta_0}\,\Gamma\left[\dfrac{4+2m+n}{2+m+n}\right]}{(2+m)\sqrt{N^2\theta_0^{1-m+n}}\,\Gamma\left[\dfrac{6+3m+n}{2(2+m+n)}\right]},$$

$$\eta_2 = \frac{1}{2+m}\sqrt{-(2+m+n)/N^2\theta_0^{2+m+n}} \times$$
$$F\left(\frac{1}{2}, \frac{2+m}{2+m+n}, \frac{4+2m+n}{2+m+n}, \theta_0^{-2-m-n}\right)$$

and the function u is meant to be taken from equation (13). Thus, unlike to the particular cases given in section 3, a general case covering the parameters $m+n+2 \neq 0$ and $N \in R$ is given by the formulae (15-17).



(a) (b) (c)

Figure 5. The physical parameters $\zeta = \theta_0$, η and φ.

Figure 5 depicts a sample for the physical parameters obtained from these formulas. For fixed $m = -1/4$, the best fin efficiency occurs for increasing n and for decreasing θ_0.

CASE m+n+2=0

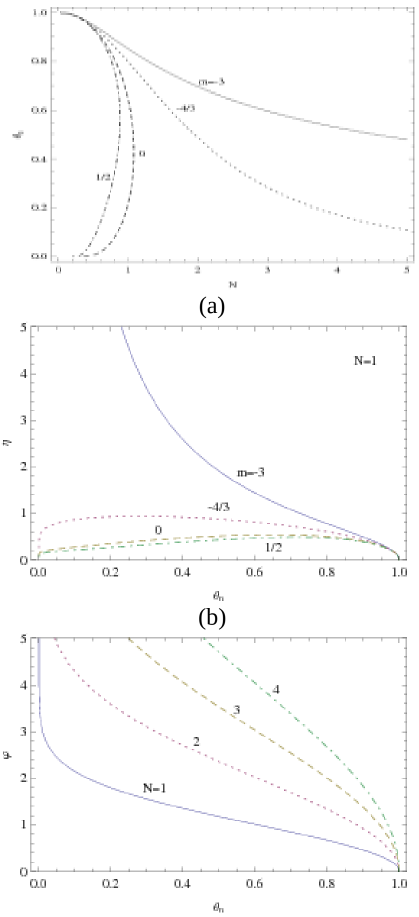
In this case, direct integration of (14) leads to the implicit form solution

$$x(\theta) = 2\theta^{1+m} / \sqrt{N^2(1+m)} \times \mathrm{D}awsonF(\sqrt{1+m}\sqrt{\ln\frac{\theta}{\theta_0}})\sqrt{\ln\frac{\theta}{\theta_0}}. \tag{18}$$

With the help of the boundary condition on the tip end, substituting $\theta(1) = 1$ in (18) results in the evaluation of θ_0. The rest of the physical parameters can be easily found as follows

$$\eta = \sqrt{2/N^2(2+m)}\mathrm{D}awsonF(\sqrt{-(2+m)\ln(\theta_0)}),$$
$$\varphi = u(1) = \sqrt{-2N^2\ln(\theta_0)}. \tag{19}$$

It should be remarked that in this case no an explicit dependence on the parameters m and n takes place for the base heat transfer rate.



(a) (b) (c)

Figure 6. The physical parameters $\zeta = \theta_0$, η and φ.

Figure 6 reveals a sample for the physical parameters obtained from these formulas. It is interesting to observe from figure 6(a) that double solutions are possible when N is small for positive values of m. The best fin efficiency occurs for smaller m and for decreasing θ_0. Moreover, the base transfer rate is seen to increase for increasing values of N.

CONCLUSIONS

The recent study of Moitsheki et al. (2010) has been greatly improved in this paper. By this mean, general closed-form formulae are provided for the straight fin of rectangular profiles and also implicit derivations are extracted valid for all physically interested parameters of the rectangular fins. The efficiency of fin, the temperature of fin tip and also the rate of heat transfer at the base of fin of engineering interest are easy to understand from the presented formulas.

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