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On the Generalization of Pentanacci Sequences

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Abstract

In this article, a generalizations of the Pentanacci sequence, which is generated by the fifth-order recurrence relation $V_n\left(a_j,p_j\right)=\sum_{j=1}^{5}p_jV_{n-j}$, $n>5$, with the initial terms $V_j=a_j$, where $a_j,p_j,\ j=1,2,3,4,5$ are any non–zero real numbers is studied. Generating function and Binet's formula are established for this sequence in the denotative form. Noted sequences generated by the recurrence relations of lower orders are contained implicitly in this generalization and are discussed as special cases. A graphical representation is presented to exhibit the relations how the terms of these sequences are related and varies with different a_j,p_j. Pentanacci constant are also studied and represented in the tabular form, it is shown that it depends on the coefficients of the recurrence relations only and has no effect of the initial terms.

Keywords: *Pentanacci sequence; Generating function; Binet formula; Pentanacci constant; Pentanacci Identity*

2010 Mathematics Subject Classification: *11B39, 11C08, 33F05, 65D20.*

1. Introduction

In recursive sequence, terms are obtained by a recurrence process where a term is the sum of preceding terms. This process requires the computation of all of its predecessors to get any term, so require a lot of computation. the generating functions and the Binet's formulas are the alternative definitions of any term of a recursive sequence in the indexical form. Since generating functions are normally used with constant coefficients of linear recurrence relations, so to study the sequences generated, employng linear homogeneous recurrence relations such functions are the appropriate technique. In this article, we have considered both the recurrence relations and initial conditions for the pentanacci sequences in the most generalized form and the generating function, Binet's formula are also obtained in the general form. A generalized Pentanacci sequences, $V_n(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)$, are the consequences of the linear recurrence relations with five arbitrary constant coefficients p_j, and the first five terms $V_j=a_j$, where $p_j,\ j=1,2,3,4,5$ are any non–zero real numbers. Grpahical and tabular representations are displayed to support the pentanacci sequences progression and comparison. Pentanacci numbers mentioned in [1] using 5th order recurrence relation and relation between sequence and matrices is studied. Yüksel Soykan et al [6] established the generating functions, Binet's and summation formulas for the generalized Pentanacci quaternions. Generalization of Pentanacci sequences have been considered and examined by many authors (see literature [2, 3, 4, 7]).

Definition 1.1. *We define the Generalized Pentanacci Sequence $\{V_n\}$ by the following linear recurrence relation:*

$$V_n(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)=p_1V_{n-1}+p_2V_{n-2}+p_3V_{n-3}+p_4V_{n-4}+p_5V_{n-5},n>5, \tag{1.1}$$

with the initial terms $V_j=a_j$, where a_j and p_j ($j=1,2,3,4,5$) are any non–zero real numbers.

1.1. Terms of the Generalized Pentanacci Sequence

The first few terms of the Pentanacci sequence defined in (1.1) are:

$$V_n=\left\{\begin{array}{l} a_1,a_2,a_3,a_4,a_5,p_1a_5,p_2a_4,p_3a_3,p_4a_2,p_5a_1,\\ \left(p_1^2+p_2\right)a_5+(p_1p_2+p_3)a_4+(p_1p_3+p_4)a_3+(p_1p_4+p_5)a_2+p_1p_5a_1,\\ \left[p_1\left(p_1^2+p_2\right)+p_1p_2+p_3\right]a_5+\left(p_1(p_1p_2+p_3)+p_4+p_2^2\right)a_4\\ +(p_2p_3+p_1(p_1p_3+p_4)+p_5)a_3+\left(p_2p_5+p_5p_1^2\right)a_1,\cdots \end{array}\right\}.$$

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1.2. Pentanacci Sequences pictorial representations

Some of Pentanacci sequences [5, 8]are taken as representative in the following figure.

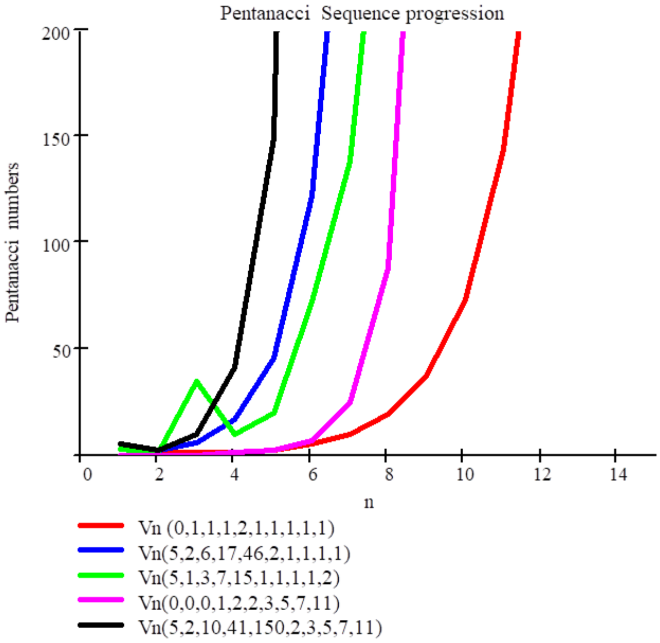

Figure 1.1: Pentanacci sequences progression and comparison

1.3. Special Cases

Remark 1.2. *With initial conditions* $V_1 = 0,\ V_2 = 1,\ V_3 = 1,\ V_4 = 1,\ V_5 = 2$ *and* $p_1 = p_2 = p_3 = p_4 = p_5 = 1$, *see [4], recurrence relation (1.1) is known as the Pentanacci sequences. The first few terms of this sequence deduced from the above generalization are:*

$$\{V_n\}_{n\geq 0} = 0,1,1,1,2,5,10,19,37,73,144,283,556,1093,2149,4225,\cdots.$$

Remark 1.3. *If we substitute the initial conditions* $V_1 = 0,\ V_2 = 1,\ V_3 = 1,\ V_4 = 3, V_5 = 8$,*and* $p_1 = 2, p_2 = p_3 = p_4 = p_5 = 1$ *in (1.1) , it reduces to Pentanacci sequence which is also discussed in [4].The first few terms of the sequence are:*

$$\{V_n\}_{n\geq 0} = 0,1,1,3,8,21,55,143,373,973,2538,6620,17267,45038,117474,\cdots.$$

Remark 1.4. *If we substitute the initial conditions* $V_1 = 0, V_2 = 0,\ V_3 = 0, V_4 = 1, V_5 = 1$,*and* $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$ *in (1.1) , it reduces to Pentanacci sequence which is also discussed in [5].The first few terms of the sequence are:*

$$\{V_n\}_{n\geq 0} = 0,0,0,1,1,5,18,63,223,771,2707,9481,33192,116212,406835,\cdots.$$

2. Generating function

The explicit generalized generating function for the Pentanacci sequence

Theorem 2.1. *Generating Function*
The generalized generating function of the sequence satisfies the recursion in (1.1) is $V(x)$, *then*

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4 - p_5x^5}$$

where

$$f(x) = V_1 + (V_2 - p_1V_1)x + (V_3 - p_1V_2 - p_2V_1)x^2 + (V_4 - p_1V_3 - p_2V_2 - p_3V_1)x^3 + (V_5 - p_1V_4 - p_2V_3 - p_3V_2 - p_4V_1)x^4.$$

Proof. If the generating function of (1.1) is $V(x)$, then we have

$$V(x) = \sum_{n=0}^{\infty} V_nx^n, \tag{2.1}$$

and

$$p_jx^jV(x) = \ p_j\sum_{n=j}^{\infty} V_{n-j}x^j,\ j = 1,2,3,4.5. \tag{2.2}$$

Now employing (2.1) and (2.2), we obtain generating function $V(x)\left[1-p_1x-p_2x^2-p_3x^3-p_4x^4-p_5x^5\right]=f(x)$ for Pentanacci sequence in the rational form:

$$V(x)=\frac{f(x)}{1-p_1x-p_2x^2-p_3x^3-p_4x^4-p_5x^5}, \tag{2.3}$$

where $f(x)=V_1+(V_2-p_1V_1)x+(V_3-p_1V_2-p_2V_1)x^2+(V_4-p_1V_3-p_2V_2-p_3V_1)x^3+(V_5-p_1V_4-p_2V_3-p_3V_2-p_4V_1)x^4$ is a polynomial.
Hence $V(x)$ is the generating function of the sequence $\{V_n\}$. □

2.1. Special cases

Remark 2.2. *If we substitute $V_1=0,\ V_2=1,\ V_3=1,\ V_4=2,\ V_5=4$ and $p_1=p_2=p_3=p_4=p_5=1$ in the result obtained in (2.3), it reduces to the generating function*

$$V(x)=\frac{x}{1-x-x^2-x^3-x^4-x^5}.$$

Remark 2.3. *If we substitute $V_1=0,\ V_2=1,\ V_3=2,\ V_4=5,\ V_5=13$ and $p_1=2,\ p_2=1,\ p_3=1,\ p_4=1,\ p_5=1$ in the result obtained in (2.3), it reduces to the generating function*

$$V(x)=\frac{x}{1-2x-x^2-x^3-x^4-x^5}.$$

Remark 2.4. *If we substitute $V_1=0,\ V_2=1,\ V_3=1,\ V_4=2,\ V_5=4,$ and $p_1=1,\ p_2=1,\ p_3=1,\ p_4=1,\ p_5=2$ in the result obtained in (2.3), it reduces to the generating function*

$$V(x)=\frac{x^2+x^3+x^4}{1-x-x^2-x^3-x^4-2x^5}.$$

Remark 2.5. *If we substitute $V_1=0,\ V_2=0,\ V_3=0,\ V_4=1,\ V_5=1$ and $p_1=2,\ p_2=3,\ p_3=5,\ p_4=7,\ p_5=11$ in the result obtained in (2.3), it reduces to the generating function*

$$V(x)=\frac{x^3-x^4}{1-2x-3x^2-5x^3-7x^4-11x^5}.$$

2.2. Even and odd terms of Generating Functions

Theorem 2.6. *The generating functions of even $V_{2n}(x)$ and odd $V_{2n+1}(x)$ terms of the generalized pentanacci Sequence are*

$$V_{2n}(x)=\frac{N_1}{D_1} \tag{2.4}$$

and

$$V_{2n+1}(x)=\frac{N_2}{D_1} \tag{2.5}$$

where

$$\begin{aligned}N_1=&V_1+\Big(V_3-\Big(p_1^2+2p_2\Big)V_1\Big)x+\Big(V_5-\Big(p_1^2+2p_2\Big)V_3+\Big(p_2^2-2p_4-2p_1p_3\Big)V_1\Big)x^2\\&+\Big(-p_2V_5+(p_1p_2+p_3)V_4+\Big(p_2^2-p_4-p_1p_3\Big)V_3+(p_5+p_1p_4)V_2+\Big(2p_2p_4-p_1p_5-p_3^2\Big)V_1\Big)x^3,\\&+\Big(-p_4V_5+(p_1p_4+p_5)V_4+(p_2p-p_1p_5)V_3+(p_3p_4-p_2p_5)V_2+\Big(p_4^2-p_3p_5\Big)V_1\Big)x^4\end{aligned}$$

$$\begin{aligned}N_2=&V_2+\Big(V_4-\Big(p_1^2+2p_2\Big)V_2\Big)x+\Big(p_1V_5-\Big(p_1^2+p_2\Big)V_2+p_3V_3+\Big(p_2^2-p_4-2p_1p_3\Big)V_2+p_5V_1\Big)x^2\\&+\Big(p_3V_5-(p_1p_3+p_4)V_4+(p_5+p_1p_4-p_2p_3)V_3+\Big(p_2p_4-p_1p_5-p_3^2\Big)V_2-p_2p_5V_1\Big)x^3\\&+(p_5V_5-p_1p_5V_4-p_2p_5V_3-p_3p_5V_2-p_4p_5V_1)x^4\end{aligned}$$

and

$$D_1=1-(p_1^2+2p_2)x-(2p_1p_3-p_2^2+2p_4)x^2-\Big(p_3^2+2p_1p_5-2p_2p_4\Big)x^3+\Big(p_4^2-2p_3p_5\Big)x^4.$$

Proof. Using the definition of generating functions of the even $V_{2n}(x)=\frac{V_n(\sqrt{x})+V_n(-\sqrt{x})}{2}$ and odd $V_{2n+1}(x)=\frac{V_n(\sqrt{x})-V_n(-\sqrt{x})}{2\sqrt{x}}$ sequences and generalized generating function of entanacci sequence obtained in the above theorem.
On simplification we obtained the Ggneralized generating function of even and odd terms of Pentanacci sequence

$$V_{2n}(x)=\frac{V_1-\left[(2p_2+p_1^2)V_1-V_3\right]x-\left[(p_1p_3-p_2^2)V_1-(p_1p_2+p_3)V_2+p_2V_3\right]x^2}{1-(p_1^2+2p_2)x-(2p_1p_3-p_2^2)x^2-p_3^2x^3}, \tag{2.6}$$

and

$$V_{2n+1}(x)=\frac{V_2-\left[V_1p_3-(p_1^2+p_2)V_2-p_1V_3\right]x-[p_3V_3-p_1p_3V_2-p_2p_3V_1]x^2}{1-(p_1^2+2p_2)x-(2p_1p_3-p_2^2)x^2-p_3^2x^3}. \tag{2.7}$$

□

3. Binet's formula

Theorem 3.1. *Generalized form of the Binet's formula for the generalized sequence defined (1.1) is If the sequence $\{V_n\}$ satisfies the recursion $V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + p_4V_{n-4} + p_5V_{n-5}$ is*

$$V_n = \sum_{j=1}^{5} \left[\frac{A_1\alpha_j^4 + A_2\alpha_j^3 + A_3\alpha_j^2 + A_4\alpha_j + A_5}{\prod\limits_{\substack{1\le i\le k \\ i\neq j}} (\alpha_j - \alpha_i)} \right] \alpha_j^n.$$

Proof. Consider partial fraction decomposition of the right-hand side of the generating function (2.3) of the sequence we have

$$\begin{aligned} V(x) &= \frac{A_1 + A_2x + A_3x^2 + A_4x^3 + A_5x^4}{1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4 - p_5x^5} \\ &= \frac{A_1 + A_2x + A_3x^2 + A_4x^3 + A_5x^4}{(1-\alpha_1x)(1-\alpha_2x)(1-\alpha_3x)(1-\alpha_4x)(1-\alpha_5x)} \end{aligned}$$

,

$$\begin{aligned} &A_1 = V_1,\ A_2 = V_2 - p_1V_1,\ A_3 = V_3 - p_1V_2 - p_2V_1,\ A_4 = V_4 - p_1V_3 - p_2V_2 - p_3V_1, \\ &A_5 = V_5 - p_1V_4 - p_2V_3 - p_3V_2 - p_4V_1 \end{aligned}, \tag{3.1}$$

where $\alpha_i,\ i = 1,2,3,4,5.$ are roots of the equation $1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4 - p_5x^5 = 0$. The equation (3.1) simplifies and generalize, we obtained the following formula

$$V_n = \sum_{j=1}^{5} \left[\frac{A_1\alpha_j^4 + A_2\alpha_j^3 + A_3\alpha_j^2 + A_4\alpha_j + A_5}{\prod\limits_{\substack{1\le i\le k \\ i\neq j}} (\alpha_j - \alpha_i)} \right] \alpha_j^n. \tag{3.2}$$

□

Since the generalized Pentanacci sequence is a fifth-order recurrence relation, the generalized Binet's formula may also be written alternately as in the following theorem.

Alternate Generalized Binet's formula for the Tetranacci sequence

Theorem 3.2 (Alternate Generalized Binet's formula for the Tetranacci sequence)**.** *Generalized form of the Binet's formula for the generalized sequence defined (1.1) is If the sequence $\{V_n\}$ satisfies the recursion The Generalized Binet's formula for the Tetranacci sequence may written as*

$$V_n = \sum_{j=1}^{n} P_j\alpha_j^n,\ j = 1,2,3,4,5,$$

P_j and α_j are constants and are roots of the polynomial equation .

$$V_n(x) = \sum_{j=1}^{5} \left[\frac{\left[\alpha_j^3a_2 + (a_3 - p_1a_2)\alpha_j^2 + (a_4 - p_1a_3 - p_2a_2)\alpha_j + (a_5 - p_1a_4 - p_2a_3 - p_3a_2 + p_5a_1)\right]}{\prod\limits_{\substack{i=1 \\ i\neq j}}^{4} (\alpha_j - \alpha_i)} \right] \alpha_j^n.$$

Proof. Solving the five expression

$$V_n = \sum_{j=1}^{n} P_j\alpha_j^n,\ j = 1,2,3,4,5$$

for P_j, we obtain P_j as

$$P_j = \frac{\left[\alpha_j^3a_2 + (a_3 - p_1a_2)\alpha_j^2 + (a_4 - p_1a_3 - p_2a_2)\alpha_j + (a_5 - p_1a_4 - p_2a_3 - p_3a_2 + p_5a_1)\right]}{\prod\limits_{\substack{i=1 \\ i\neq j}}^{4} (\alpha_j - \alpha_i)}.$$

Substituting these P_j in the first relation we obtain

$$\sum_{j=0}^{4} \left[\frac{p_4V_1 + \left(\alpha_j^2 - p_1\alpha_j - p_2\right)\alpha_jV_2 + (\alpha_j - p_1)\alpha_jV_3 + \alpha_jV_4}{\prod\limits_{\substack{1\le j\le 4 \\ i\neq j}} (\alpha_j - \alpha_i)} \right] \alpha_j^{n-1}. \tag{3.3}$$

□

3.1. Special cases

Remark 3.3. *If we consider* $V_1 = 0,\ V_2 = 1,\ V_3 = 1,\ V_4 = 2,\ V_5 = 4$ *and* $p_1 = 1,\ p_2 = 1,\ p_3 = 1,\ p_4 = 1,\ p_5 = 1$ *in the result obtained in the above Theorem and (2.2), it reduces to*

$$V_n(x) = \sum_{j=1}^{5}\left[\frac{\alpha_j^{n+3}}{\prod\limits_{\substack{1\le i\le 5\\ i\ne j}}(\alpha_j-\alpha_i)}\right]. \tag{3.4}$$

which is the same as obtained by [2, 3, 4]..

Remark 3.4. *If we consider* $V_1 = 0,\ V_2 = 0,\ V_3 = 1,\ V_4 = 2,\ V_5 = 4$ *and* $p_1 = 1,\ p_2 = 1,\ p_3 = 1,\ p_4 = 1,\ p_5 = 2$ *in the result obtained in the above Theorem and (2.2), it reduces to*

$$V_n(x) = \sum_{j=1}^{5}\left[\frac{\alpha_j^{n+2}+\alpha_j^{n+1}+\alpha_j^{n}}{\prod\limits_{\substack{1\le i\le 5\\ i\ne j}}(\alpha_j-\alpha_i)}\right]. \tag{3.5}$$

which is the same as obtained by [2, 3, 4].

Remark 3.5. *If we consider* $V_1 = 0,\ V_2 = 0,\ V_3 = 0,\ V_4 = 1,\ V_5 = 1$ *and* $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11$ *in the result obtained in the above Theorem and (2.2), it reduces to*

$$V_n(x) = \sum_{j=1}^{5}\left[\frac{\alpha_j^{n+1}-\alpha_j^{n}}{\prod\limits_{\substack{1\le i\le 5\\ i\ne j}}(\alpha_j-\alpha_i)}\right] \tag{3.6}$$

which is the same as obtained by [2, 3, 4].

3.2. Pentanacci Constant pictorial representations

A few values [5] of Pentanacci sequences represented in the following figure.

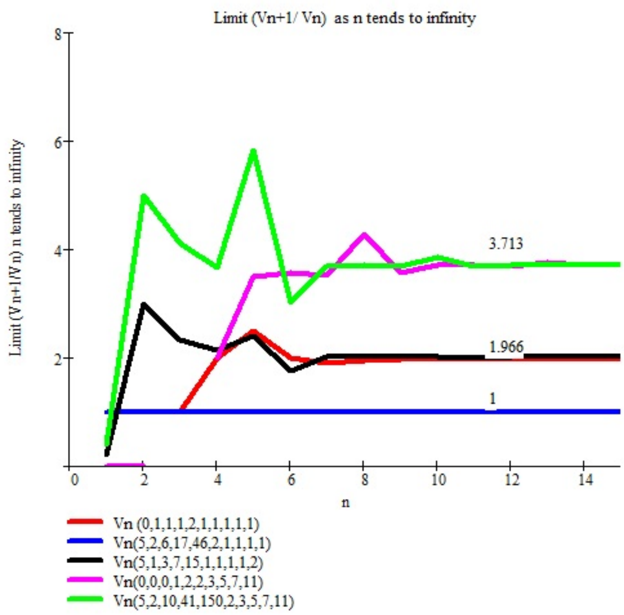

Figure 3.1: Pentanacci sequences progression and comparison

Pentanacci Constant

Theorem 3.6.

$$\lim_{n\to\infty}\frac{V_{n+1}(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)}{V_n(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)}\to\begin{cases}\alpha\text{, largest root among the real roots } f(x)=x^5-p_1x^4-p_2x^3-p_3x^2-p_4x-p_5=0, p_i>0\\ 3.713,\ \text{if } p_i=2,3,5,7 \text{and } 11\ (j=1,2,3,4,5) \text{ and } a_j \text{ are any real numbers.}\\ 1.966,\ \text{if } p_i=1,1,1,1 \text{ and } 1(or2)\ (j=1,2,3,4,5) \text{ and } a_j \text{ are any real numbers.}\\ 1,\ \text{if } p_i=2,1,1,1 \text{ and } 1\ (j=1,2,3,4,5) \text{ and } a_j \text{ are any real numbers.}\\ \beta\text{, largest root among the real roots } f(x)=x^4-p_1x^3-p_2x^2-p_3x-p_4=0, p_i>0,\\ 1.928,\ \text{if } p_i=1 \text{ and } a_j\ (j=1,2,3) \text{ are any real numbers.}\\ 1.618,\ \text{if } p_2=p_3=0, a_2=a_3=0, \text{ and } p_j, a_j, (j=1,2) \text{ are are any real numbers.}\end{cases} \tag{3.7}$$

Pentanacci constant satisfies the identity

Theorem 3.7. *If*

$$T_n(p_1,p_2,p_3,p_4,p_5)=\frac{V_{n+1}(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)}{V_n(a_1,a_2,a_3,a_4,a_5,p_1,p_2,p_3,p_4,p_5)},$$

then

$$T_n(p_1,p_2,p_3,p_4,p_5)+T_n(p_1,p_2,p_3,p_4,p_5)^{-5}=\beta(p_1,p_2,p_3,p_4,p_5)\ \text{a constant.} \tag{3.8}$$

Sr. No. 5th order Sequences	OEIS ([8]) Order of recurrence relations	Pentanacci Sequences 4th order recurrence Initial terms (a1, a2, a3, a4,a5)	(p1, p2, p3, p4, p5)	Pentanacci Identity $T+T^{-5}$	Ratio of adjacent numbers as $n\to\infty$
Fibonacci	A000045, 2nd	(0,1,0,0,0)	(1,1,0,0,0)	1.927	1.839
Tribonacci	A000073,3rd	(0,0,1,0,0)	(1,1,1,0,0)	1.764	1.618
Tetranacci	A000078, 4th	(0,1,1,2,0)	(1,1,1,1,0)	1	1.928
Pentanacci	A001591,5th	(0,1,1,2,4)	(1,1,1,1,1)	2	1.966
Pent-Lucas	A074048	(5,1,3,7,15)	(1,1,1,1,1)	2	1.966
Pell	A141448	(0,1,2,5,13)	(2,1,1,1,1)	2	1
Pell-Lucas		(5,2,6,17,46)	(2,1,1,1,1)	2	1
Mod. Pell		(0,1,1,3,8)	(2,1,1,1,1)	2	1
Jacobsthal	A226310	(0,1,1,1,1)	(1,1,1,1,2)	2.031	1.966
Jacobsthal-Lucas	A226311	(2,1,5,10,20)	(1,1,1,1,2)	2.031	1.966
M. Jacobsthal		(3,1,3,10,20)	(1,1,1,1,2))	2.031	1.966
J-Perrin		(3,0,2,8,16)	(1,1,1,1,2)	2.031	1.966
A. Jacobsthal		(0,1,1,2,4)	(1,1,1,1,2)	2.031	1.966
M. Jacobsthal-Lucas		(5,1,3,3,7)	(1,1,1,1,2)	2.031	1.966
5-primes		(0,0,0,1,2)	(2,3,5,7,11)	3.714	3.713
Lucas-5 primes		(5,2,10,41)	(2,3,5,7,11)	3.714	3.713
M. 5-primes		(0,0,0,1,1)	(2,3,5,7,11)	3.714	3.713

Table 1: Fibonacci,Tribonacci Tetranacci, Pentanacci constants of Generalized sequence (see [8, 4])

4. Conclusion

Pentanacci sequence, a fifth-order recurrence relation in the most generalized form is considered and investigated. Foremost some terms of the sequence in general form are explored and the terms of the known sequences are written. A pictorial representation is presented (see Figure (1.1)) for Pentanacci numbers are displayed using the known Pentanacci sequences. Generating functions and the Binet formula are derived in the comprehensive form which include sequences of lower orders recurrence relations as special cases.The generating functions of even and odd terms for are also obtained explicitly. The ratio of (n+1)th term to nth when $n\to\infty$, is also exhibited in Figure (3.1). Employing the obtained results, 2nd to 4th orders recurrence relations become the special cases of this generalization. Pentanacci constant and identity are also discussed and is represented in the Table (1), varying both the coefficients of the recurrence relations and initial terms of the available sequences in the literature. It is observed that both vary only with the coefficients of the recurrence relations and has no influence of initial terms. In future kth order $(k\geq 6)$ generalized sequence could obviously be considered for more thoroughly and the study can further be extended considering alternate approaches of number theory to have added identities and theorems.

Article Information

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