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ON G-(n, d)-RINGS AND n-COHERENT RINGS

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Abstract. Let n and d be non-negative integers. We introduce the concept of *strongly* (n, d)-*injective* modules to characterize n-coherent rings. For a right perfect ring R, it is shown that R is right n-coherent if and only if every right R-module has a strongly (n, d)-injective (pre)cover for some non-negative integer $d \leq n$. We also provide equivalent conditions for an (n, d)-ring being n-coherent. Then we investigate the so-called *right* G-(n, d)-*rings*, over which every n-presented right module has Gorenstein projective dimension at most d. Finally, we prove a Gorenstein analogue of Costa's first conjecture.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary R-modules.

Let n and d be non-negative integers. Following Costa [16], Chen and Ding [14], a right R-module M is called *n-presented* if there exists an exact sequence of right R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where F_i is finitely generated and free for every $i = 0, 1, \cdots, n$. M is said to be of *type* FP_∞ if it is n-presented for any non-negative integer n. A ring R is called *right n-coherent* ([14,16]) in case every n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent (1-coherent) if and only if R is right Noetherian (coherent). According to Costa [16] and Zhou [50], R is said to be a *right* (n, d)-*ring* (resp. *right weak* (n, d)-*ring*) if every n-presented right R-module has projective (resp. flat) dimension at most d. Thus, right $(0, d)$-rings are exactly the rings of right global dimension at most d, and right weak $(1, d)$-rings are exactly the rings of weak global dimension at most d. n-coherent rings and (weak) (n, d)-rings have

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been extensively studied in the existing literature (see, for instance, [1, 7, 8, 11-14, 16, 32-35, 46, 48-50]).

In this paper, we introduce and study the concepts of *strongly* (n,d)-*injective* modules and *strongly* (n,d)-$flat$ modules (see Definition 3.1), and use these classes of modules, among others, to give new characterizations for n-coherent rings and (n,d)-rings. We also provide equivalent conditions for an (n,d)-ring being n-coherent. Another goal of this paper is to extend the idea of Costa and introduce a doubly indexed set of classes of rings called *right* G-(n,d)-*rings* (Section 6).

This paper is organized as follows.

In Section 2, we collect some known definitions and notions.

In Section 3, we introduce the concepts of strongly (n,d)-injective right R-modules and strongly (n,d)-flat left R-modules (these classes of modules are denoted by $\mathcal{SI}_{n,d}$ and $\mathcal{SF}_{n,d}$, respectively). For any ring R, we prove that $({}^{\perp}\mathcal{SI}_{n,d}$, $\mathcal{SI}_{n,d})$ is a hereditary complete cotorsion theory, and $(\mathcal{SF}_{n,d}$, $\mathcal{SF}_{n,d}^{\perp})$ is a hereditary perfect cotorsion theory.

Section 4 is devoted to study the classes of modules of finite weak injective (flat) dimension. As in [26, Definition 3.6], we set $r.sp.gldim(R) = \sup\{\mathrm{pd}(M) \mid M$ is a right R-module of type $FP_\infty\}$, where $\mathrm{pd}(M)$ is the projective dimension of M. We provide examples to show that rings R with $r.sp.gldim(R) \leq d$ may fail to be right (n,d)-rings (see Example 4.8), and in particular, answers affirmatively a problem posted by Bravo and Parra in [11].

In Section 5, we explore some applications of our previous results. We first give some new characterizations for right n-coherent rings (see Theorem 5.6); several interesting corollaries are obtained, allowing us to provide new counterexamples to an open problem posed by Gillespie in [27] (see Example 5.10). For a right perfect ring R, we show in Theorem 5.14 that R is right n-coherent if and only if $\mathcal{SI}_{n,t}$ is (pre)covering for some non-negative integer $t \leq n$. We also provide equivalent conditions for a right (n,d)-ring being right n-coherent (see Theorem 5.20).

Costa's paper [16] concludes with a number of open problems for commutative rings, including his first conjecture: given non-negative integers n and d, there is an (n,d)-ring which is neither an $(n,d-1)$-ring nor an $(n-1,d)$-ring. The final section is devoted to prove a Gorenstein analogue of Costa's first conjecture.

2. Preliminaries

In this section, we shall recall some known definitions and notions needed in the sequel.

For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is denoted by M^+. $\mathrm{Hom}(M,N)$ (resp. $\mathrm{Ext}^d(M,N)$) means $\mathrm{Hom}_R(M,N)$ (resp. $\mathrm{Ext}^d_R(M,N)$), and similarly $M\otimes N$ (resp. $\mathrm{Tor}_d(M,N)$) denotes $M\otimes_R N$ (resp. $\mathrm{Tor}^R_d(M,N)$). The symbol $\mathrm{rD}(R)$ (resp. $\mathrm{wD}(R)$) stands for the usual right (resp. weak) global dimension of R.

We denote by $\mathcal{P}_m$ the class of all right R-modules of projective dimension at most m. For a class of right R-modules $\mathscr{C}$, we put

$$\mathscr{C}^{<\infty}=\{C\mid C\in\mathscr{C}\text{ and }C\text{ is of type }FP_\infty\}.$$

2.1. Ext and Tor orthogonal classes. Let $\mathscr{C}$ be a class of right R-modules and $\mathscr{D}$ a class of left R-modules. We will use the following notation:

$$\begin{aligned}
\mathscr{C}^{\perp}&=\{X\text{ is a right }R\text{-module}\mid \mathrm{Ext}^1(C,X)=0\text{ for all }C\in\mathscr{C}\},\\
{}^{\perp}\mathscr{C}&=\{X\text{ is a right }R\text{-module}\mid \mathrm{Ext}^1(X,C)=0\text{ for all }C\in\mathscr{C}\},\\
\mathscr{C}^{\top}&=\{Y\text{ is a left }R\text{-module}\mid \mathrm{Tor}_1(C,Y)=0\text{ for all }C\in\mathscr{C}\},\\
{}^{\top}\mathscr{D}&=\{X\text{ is a right }R\text{-module}\mid \mathrm{Tor}_1(X,D)=0\text{ for all }D\in\mathscr{D}\},\\
\mathscr{C}^{\perp\infty}&=\{X\text{ is a right }R\text{-module}\mid \mathrm{Ext}^i(C,X)=0\text{ for all }C\in\mathscr{C}\text{ and any }i\geq 1\},\\
{}^{\perp\infty}\mathscr{C}&=\{X\text{ is a right }R\text{-module}\mid \mathrm{Ext}^i(X,C)=0\text{ for all }C\in\mathscr{C}\text{ and any }i\geq 1\},\\
\mathscr{C}^{\top\infty}&=\{Y\text{ is a left }R\text{-module}\mid \mathrm{Tor}_i(C,Y)=0\text{ for all }C\in\mathscr{C}\text{ and any }i\geq 1\}.
\end{aligned}$$

2.2. Precover and preenvelope. Let $\mathscr{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi: C\to M$ with $C\in\mathscr{C}$ is called a $\mathscr{C}$-*precover* [19] of M if for any homomorphism $f: C'\to M$ with $C'\in\mathscr{C}$, there is a homomorphism $g: C'\to C$ such that $\phi g=f$. Moreover, if the only such g are automorphisms of C when $C'=C$ and $f=\phi$, then the $\mathscr{C}$-precover ϕ is called a $\mathscr{C}$-*cover*. An epimorphic $\mathscr{C}$-precover $\phi: C\to M$ is said to be *special* in case $\ker(\phi)\in\mathscr{C}^{\perp}$. Dually, we have the definitions of a (special) $\mathscr{C}$-preenvelope and a $\mathscr{C}$-envelope. We say that $\mathscr{C}$ is *(pre)covering* (resp. *(pre)enveloping*) in case every right R-module has a $\mathscr{C}$-(pre)cover (resp. $\mathscr{C}$-(pre)envelope).

2.3. Cotorsion theory. A pair ($\mathscr{C}$, $\mathscr{D}$) of classes of right R-modules is called a *cotorsion theory* [23] if $\mathscr{C}^{\perp}=\mathscr{D}$ and ${}^{\perp}\mathscr{D}=\mathscr{C}$. A cotorsion theory ($\mathscr{C}$, $\mathscr{D}$) is called *complete* if every right R-module has a special $\mathscr{C}$-precover and a special $\mathscr{D}$-preenvelope. A cotorsion theory ($\mathscr{C}$, $\mathscr{D}$) is called *perfect* if every right R-module has a $\mathscr{C}$-cover and a $\mathscr{D}$-envelope. A cotorsion theory ($\mathscr{C}$, $\mathscr{D}$) is said to be *hereditary* if whenever $0\to C'\to C\to C''\to 0$ is exact with C, $C''\in\mathscr{C}$, then $C'\in\mathscr{C}$.

3. Strongly (n,d)-injective and strongly (n,d)-flat modules

Let n and d be non-negative integers and R a ring. Recall that a right R-module M (resp. left R-module N) is called (n,d)-*injective* (resp. (n,d)-*flat*) if $\mathrm{Ext}^{d+1}(P,M)=0$ (resp. $\mathrm{Tor}_{d+1}(P,N)=0$) for any n-presented right R-module P [50].

Definition 3.1. Let n, d be non-negative integers. A right R-module M is called *strongly* (n,d)-*injective* if $\mathrm{Ext}^{d+j}(P,M)=0$ for any n-presented right R-module P and all $j\geq 1$.

A left R-module N is called *strongly* (n,d)-*flat* if $\mathrm{Tor}_{d+j}(P,N)=0$ for any n-presented right R-module P and all $j\geq 1$.

We write:

$$\begin{aligned}\mathcal{I}_{n,d} &= \{(n,d)\text{-injective right } R\text{-modules}\},\\ \mathcal{F}_{n,d} &= \{(n,d)\text{-flat left } R\text{-modules}\},\\ \mathcal{SI}_{n,d} &= \{\text{strongly } (n,d)\text{-injective right } R\text{-modules}\},\\ \mathcal{SF}_{n,d} &= \{\text{strongly } (n,d)\text{-flat left } R\text{-modules}\}.\end{aligned}$$

It is clear that $\mathcal{SI}_{n,d}\subseteq\mathcal{I}_{n,d}$ and $\mathcal{SF}_{n,d}\subseteq\mathcal{F}_{n,d}$. For the other direction, we have:

Proposition 3.2. *Let R be a right n-coherent ring. Then $\mathcal{I}_{n,d}=\mathcal{SI}_{n,d}$ and $\mathcal{F}_{n,d}=\mathcal{SF}_{n,d}$.*

Proof. Since R is right n-coherent, we deduce from [12, Corollary 2.6] that every n-presented right R-module G admits a projective resolution

$$\cdots\to P_m\xrightarrow{f_m}P_{m-1}\xrightarrow{f_{m-1}}\cdots\xrightarrow{f_1}P_0\xrightarrow{f_0}G\xrightarrow{f_{-1}}0$$

with $\ker(f_m)$ $(m\geq -1)$ n-presented. Hence $\mathcal{I}_{n,d}\subseteq\mathcal{SI}_{n,d}$. But it is obvious that $\mathcal{SI}_{n,d}\subseteq\mathcal{I}_{n,d}$. So $\mathcal{I}_{n,d}=\mathcal{SI}_{n,d}$. The second identity can be proved similarly. □

We say that a class $\mathscr{C}$ of modules is *definable* provided that $\mathscr{C}$ is closed under direct limits, direct products and pure submodules.

Proposition 3.3. *Let R be a ring.*

(1) *If $n\geq d+1$, then $\mathcal{I}_{n,d}$ is closed under pure submodules.*
(2) *$\mathcal{F}_{n,d}$ is closed under direct limits, extensions and pure submodules. A left R-module N is (n,d)-flat if and only if N^+ is (n,d)-injective.*
(3) *If either one of the following two conditions holds, then $\mathcal{I}_{n,d}$ is definable and closed under pure quotients, and a right R-module M is (n,d)-injective if and only if M^+ is (n,d)-flat:*

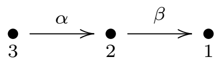
(I) $n \leq d+1$ *and* R *is right* n*-coherent;*

(II) $n > d+1$.

Proof. It is clear that $\mathcal{I}_{n,d}$ is closed under direct products (see [50, Proposition 2.2(2)]).
(1) This is [50, Proposition 2.4(1)].
(2) It is clear that $\mathcal{F}_{n,d}$ is closed under direct limits and extensions. In addition, $\mathcal{F}_{n,d}$ is closed under pure submodules by [50, Proposition 2.4(2)]. The final assertion follows from [50, Proposition 2.3].
(3) Assume that R satisfies one of the conditions (I) or (II). Then $\mathcal{I}_{n,d}$ is closed under pure submodules and direct limits by [48, Lemma 2.1] and [50, Proposition 3.1], respectively. We also see from [50, Proposition 3.1] that a right R-module M is (n,d)-injective if and only if M^+ is (n,d)-flat.

Now let $0 \to C \to B \to A \to 0$ be a pure short exact sequence of right R-modules with B (n,d)-injective. Then B^+ is (n,d)-flat, and we have a split exact sequence of left R-modules $0 \to A^+ \to B^+ \to C^+ \to 0$ by [28, Lemma 1.2.13(e)]. Thus both A^+ and C^+ are (n,d)-flat. Hence A and C are (n,d)-injective by what we have proved. This proves (3). □

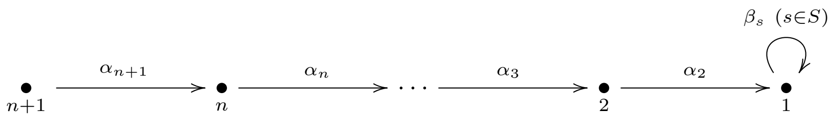
In what follows, the composition



of two paths α and β in a quiver is denoted by $\alpha\beta$.

The following example tells us that (n,d)-injective modules may fail to be strongly (n,d)-injective.

Example 3.4. Let n be a fixed non-negative integer. Let Q be the following quiver



with $n+1$ vertices, one arrow α_{i+1} from vertex $i+1$ to vertex i for each $i \in \{1, 2, \cdots, n\}$, and infinitely many loops $\{\beta_s \mid s \in \mathcal{S}\}$ at the vertex 1.

Let R be the quotient of the path algebra of Q over an algebraically closed field k by the ideal generated by the set of all paths of length $\ell \geq 2$.

For any $s \in \mathcal{S}$, let E_s be the injective envelope of the right ideal $\overline{\beta_s}R$. Write $M := \bigoplus_{s\in S} E_s$. Then $M \in \mathcal{I}_{n,t}$ for $t < n$, but $M \notin \mathcal{SI}_{n,d}$ for any d.

Proof. It is clear that $M \in \mathcal{I}_{n,t}$ for $t < n$ (see Proposition 3.3).

Let P_i be the indecomposable projective right R-module corresponding to the vertex $i \in \{\ 1,\ 2,\ \cdots,\ n+1\}$, and let S_{n+1} be the simple right R-module corresponding to the vertex $n+1$. Write $N_s = \overline{\beta_s} R$. We have naturally the following exact sequences of right R-modules

$$0 \longrightarrow \text{rad } P_1 = \bigoplus_{\gamma \in S} N_\gamma \longrightarrow P_1 \longrightarrow N_s \longrightarrow 0, \qquad (\zeta_0)$$

$$0 \to \text{rad } P_1 \to P_1 \to P_2 \to \cdots \to P_n \to P_{n+1} \to S_{n+1} \to 0. \qquad (\zeta_1)$$

By using the exact sequence (ζ_0) and mimicking the proof of [32, Example 1], one can show that $\text{Ext}^1_R(N_s, M) \neq 0$. So $\text{Ext}^1_R(\bigoplus_{\gamma \in S} N_\gamma, M) \cong \prod_{\gamma \in S} \text{Ext}^1_R(N_\gamma, M) \neq 0$, and hence $\text{Ext}^2_R(N_s, M) \neq 0$ again by (ζ_0). Continuing this way, we see that $\text{Ext}^m_R(N_s, M) \neq 0$ for any $m \geq 1$. It follows from the exact sequence (ζ_1) that $\text{Ext}^{n+m}_R(S_{n+1}, M) \neq 0$ for any $m \geq 1$. Note that S_{n+1} is n-presented. Therefore, $M \notin \mathcal{SI}_{n,d}$ for any d. □

Remark 3.5. For an arbitrary ring R, it is known that $\mathcal{I}_{n,d}$ is covering if $n \geq d+2$ [48, Lemma 2.4]. Note that for any family $\{M_j\}_{j \in J}$ of R-modules, $\bigoplus_{j \in J} M_j$ is pure in $\prod_{j \in J} M_j$. Hence, for $n \geq d+1$, one can deduce from Proposition 3.3(1) that $\mathcal{I}_{n,d}$ is closed under direct sums. However, for $n \leq d$, both classes $\mathcal{SI}_{n,d}$ and $\mathcal{I}_{n,d}$ given in Example 3.4 are not closed under direct sums, so, they are not precovering by [34, Proposition 2.6].

Lemma 3.6. *Let R be a ring.*

(1) *$\mathcal{SI}_{n,d}$ is closed under extensions, products and cokernels of monomorphisms.*
(2) *$\mathcal{SF}_{n,d}$ is closed under direct limits, extensions, pure submodules and kernels of epimorphisms. A left R-module M is strongly (n,d)-flat if and only if M^+ is strongly (n,d)-injective.*

Proof. The proof of part (1) is straightforward.

Clearly, we have that $\mathcal{SF}_{n,d}$ is closed under kernels of epimorphisms. Note that an R-module is strongly (n,d)-flat (resp. strongly (n,d)-injective) if and only if it is $(n,d+j)$-flat (resp. (n,d)-injective) for all $j \geq 0$. This observation together with Proposition 3.3(2) give part (2). □

Following [30], a *duality pair* over a ring R is a pair $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is a class of left R-modules and $\mathcal{C}$ is a class of right R-modules, subject to the following conditions:

(1) for a left R-module M, one has $M \in \mathcal{M}$ if and only if $M^+ \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{C})$ is called *perfect* if $\mathcal{M}$ is closed under extensions and direct sums in the category of all left R-modules, and if R belongs to $\mathcal{M}$.

I. Bican, R. El Bashir, and E. E. Enochs proved that $(\mathcal{SF}_{1,0}, \mathcal{SF}_{1,0}^{\perp})$ is a perfect cotorsion theory, thus proving the celebrated Flat Cover Conjecture: every module over any ring has a flat cover (see [9]). More generally, we have:

Theorem 3.7. *For any ring R, $(\mathcal{SF}_{n,d}, \mathcal{SF}_{n,d}^{\perp})$ is a hereditary perfect cotorsion theory.*

Proof. By Lemma 3.6(2), $(\mathcal{SF}_{n,d}, \mathcal{SI}_{n,d})$ is a perfect duality pair. It follows from [30, Theorem 3.1(c)] that $(\mathcal{SF}_{n,d}, \mathcal{SF}_{n,d}^{\perp})$ is a perfect cotorsion theory. Moreover, $(\mathcal{SF}_{n,d}, \mathcal{SF}_{n,d}^{\perp})$ is hereditary again by Lemma 3.6(2). □

The following result is a generalization of [28, Theorem 4.1.7] and [32, Theorem 3.4].

Theorem 3.8. *For any ring R, $({}^{\perp}\mathcal{SI}_{n,d}, \mathcal{SI}_{n,d})$ is a hereditary complete cotorsion theory.*

Proof. The proof is similar to that of [32, Theorem 3.4]. Let M be a right R-module. Then $M \in \mathcal{SI}_{n,d}$ if and only if $\mathrm{Ext}^{d+j}(P, M) = 0$ for every $j \geq 1$ and $P \in \mathcal{FP}_n$, where $\mathcal{FP}_n$ is the class of n-presented right R-modules. Since $\mathcal{FP}_n$ is skeletally small, we can choose a set $\mathcal{S}$ of representatives for $\mathcal{FP}_n$. Let X_i be a set of representatives of ith syzygy modules of modules in $\mathcal{S}$. Then $\mathcal{X} = \bigcup_{t=0}^{\infty} X_{d+t}$ is also a set. Note that $\mathrm{Ext}^1(\bigoplus_{X\in\mathcal{X}} X, M) \cong \prod_{X\in\mathcal{X}} \mathrm{Ext}^1(X, M)$. Hence $\mathcal{SI}_{n,d} = \mathcal{X}^{\perp}$. So $({}^{\perp}\mathcal{SI}_{n,d}, \mathcal{SI}_{n,d})$ is a complete cotorsion theory by [18, Theorem 10], and $({}^{\perp}\mathcal{SI}_{n,d}, \mathcal{SI}_{n,d})$ is hereditary by Lemma 3.6(1). □

Corollary 3.9. *The following are equivalent for a right R-module M.*

(1) $M \in \mathcal{SI}_{n,d+m}$.
(2) *There is an exact sequence* $0 \to M \to A^0 \to A^1 \to \cdots \to A^{m-1} \to A^m \to 0$ *with each* $A^i \in \mathcal{SI}_{n,d}$, *for* $i = 0, 1, \cdots, m$.
(3) *If the sequence* $0 \to M \to A^0 \to A^1 \to \cdots \to A^{m-1} \to A^m \to 0$ *is exact with each* $A^i \in \mathcal{SI}_{n,d}$, *for* $i = 0, 1, \cdots, m-1$, *then* A^m *also belongs to* $\mathcal{SI}_{n,d}$.

Proof. Using Theorem 3.8 and dimension shifting. □

Following [16] and [50], R is said to be a *right (n,d)-ring* (resp. *right weak (n,d)-ring*) if every n-presented right R-module has projective (resp. flat) dimension at most d.

Remark 3.10. Let R be a ring. We see from the definitions that: R is a right (n,d)-ring if and only if every right R-module is strongly (n,d)-injective; R is a right weak (n,d)-ring if and only if every left R-module is strongly (n,d)-flat.

Let $R[x]$ denote the polynomial ring in one variable x with coefficients in a ring R, where x commutes with each element of R. Richman [42, Corollary 8] proved the flat Hilbert syzygy theorem: $\mathrm{wD}(R[x]) = \mathrm{wD}(R)+1$. This allows us to give the following proposition which will be used in Section 5.

Proposition 3.11. *Let R be a non-right-coherent ring with $\mathrm{wD}(R) = 1$, and let $S := R[x_1, x_2, \dots, x_m]$ be the polynomial ring in m indeterminates over R, where every x_i commutes with each element of R. Then S is non-right-coherent with* $\mathrm{wD}(S) = m+1$.

Proof. By [42, Corollary 8], we have that $\mathrm{wD}(S) = m+1$, i.e., S is a right weak $(1, m+1)$-ring. Next we show that S is non-right-coherent. Suppose the contrary that S is right coherent. Then S is a right $(1, m+1)$-ring by [50, Proposition 2.6(3)]. Thus, by [16, Theorem 6.3], R is a right $(1,1)$-ring, i.e., R is right semihereditary. This contradicts the condition that R is non-right-coherent. Hence S is non-right-coherent. □

Remark 3.12. We do not know whether there is a "syzygy theorem" to the effect that if R is a right (resp. weak) (n,d)-ring, then $R[x]$ is a right (resp. weak) $(n, d+1)$-ring; we know that this is true for $n = 0$.

4. Modules of finite weak injective (flat) dimension

Recall that a right R-module M (resp. left R-module N) is called *weak injective* (resp. *weak flat*) [26] if $\mathrm{Ext}^1(G, M) = 0$ (resp. $\mathrm{Tor}_1(G, N) = 0$) for any right R-module G of type FP_∞. Weak injective (resp. weak flat) modules coincide with absolutely clean (resp. level) modules in the sense of [10].

We let $\mathcal{WI}_d$ denote the class of right R-modules M such that $\mathrm{Ext}^{d+1}(G, M) = 0$ for any right R-module G of type FP_∞. Similarly, $\mathcal{WF}_d$ denotes the class of left R-modules N such that $\mathrm{Tor}_{d+1}(G, N) = 0$ for any right R-module G of type FP_∞. Note that $\mathcal{WI}_d$ ($\mathcal{WF}_d$) is just the class of right (left) R-modules of weak injective (weak flat) dimension at most d (see [26]).

It is clear that the following inclusions hold:

$$\mathcal{SI}_{n,d} \subseteq \mathcal{I}_{n,d} \subseteq \mathcal{WI}_d \text{ and } \mathcal{SF}_{n,d} \subseteq \mathcal{F}_{n,d} \subseteq \mathcal{WF}_d.$$

Proposition 4.1. *For any ring R, the following statements hold.*

(1) *A right R-module M belongs to $\mathcal{WI}_d$ if and only if $M^+ \in \mathcal{WF}_d$.*
(2) *A left R-module N belongs to $\mathcal{WF}_d$ if and only if $N^+ \in \mathcal{WI}_d$.*
(3) *$\mathcal{WI}_d$ is definable and closed under cokernels of monomorphisms.*
(4) *$\mathcal{WF}_d$ is definable and closed under kernels of epimorphisms.*
(5) *Both $\mathcal{WI}_d$ and $\mathcal{WF}_d$ are covering and preenveloping.*

Proof. Parts (1) and (2) hold by [49, Propositions 4.6 and 4.2], respectively.
The proofs of (3) and (4) are straightforward.
Part (5) follows from [49, Theorems 4.4, 4.5, 4.8 and 4.9]. □

We notice that Theorem 4.2(2) below is a generalization of [10, Theorem 2.14].

Theorem 4.2. *The following are true for any ring R.*

(1) *$({}^{\perp}\mathcal{WI}_d, \mathcal{WI}_d)$ is a hereditary complete cotorsion theory.*
(2) *$(\mathcal{WF}_d, \mathcal{WF}_d^{\perp})$ is a hereditary perfect cotorsion theory.*

Proof. The proof of (1) is similar to the proof of Theorem 3.8, and (2) follows from [49, Proposition 4.18]. □

Following [10], a short exact sequence $0 \to A \to B \to C \to 0$ is said to be *clean* if the sequence $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact for any M of type FP_∞.

To give a new characterization of weak injective modules, we introduce the following definition.

Definition 4.3. A right R-module M is called *clean injective* if for any clean exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, the induced sequence

$$\mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow 0$$

is exact.

A left R-module N is called *clean flat* if for any clean exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, the induced sequence $0 \to A \otimes N \to B \otimes N$ is exact.

Remark 4.4. (1) It is easy to see that every pure exact sequence is clean. Hence every clean injective module is pure injective.

(2) We have that every right R-module has a clean injective envelope by [47, Theorem 3.8], and every left R-module has a clean flat cover by [47, Corollary 2.3].

(3) By [47, Lemma 2.2], we get that a left R-module M is clean flat if and only if M^+ is clean injective.

Now we are in a position to give the following characterization of weak injective modules by clean injective modules.

Proposition 4.5. *A right R-module M is weak injective if and only if every homomorphism $f : M \to C$ with C clean injective factors through an injective right R-module.*

Proof. "$\Rightarrow$" The canonical exact sequence $0 \to M \to E(M) \to L \to 0$, with $E(M)$ the injective envelope of M, is clean because M is weak injective. Hence f factors through $E(M)$, as desired.

"$\Leftarrow$" By Theorem 4.2(1), there is an exact sequence $0 \to M \overset{i}{\to} A \to B \to 0$ with A weak injective. It is enough to show that this sequence is clean. From [47, Corollary 2.5], we only need to check that the canonical sequence $\mathrm{Hom}(A, C) \overset{i^*}{\longrightarrow} \mathrm{Hom}(M, C) \to 0$ is exact, for all clean injective right R-module C. Indeed, let $f : M \to C$ be any homomorphism with C clean injective. By hypothesis, there exist $g : M \to E$ with E injective and $h : E \to C$ such that $f = hg$. Hence there is $\theta : A \to E$ such that $g = \theta i$. So $f = h\theta i$. This shows that i^* is epic, completing the proof. □

Corollary 4.6. *Let R be a ring.*

(1) *For any clean injective right R-module M, there exists a weak injective cover $A \to M$ with A injective.*
(3) *If $N \in \mathcal{WI}_d^{\perp}$, then there exists a $\mathcal{WI}_d$-cover $A \to N$ with A injective.*

Proof. We only prove (1); the proof of (2) is similar. By Proposition 4.1(5), M has a weak injective cover $f : A \to M$. Then there exists $g : E \to M$ with E injective and $i : A \to E$ such that $f = gi$ by Proposition 4.5. So we get $h : E \to A$ such that $g = fh$ since f is a weak injective cover. So $f = gi = fhi$, and hence hi is an isomorphism. Therefore A is isomorphic to a direct summand of the injective module E, as desired. □

As in [26, Definition 3.6], we set $r.sp.gldim(R) = \sup\{\mathrm{pd}(M) \mid M$ is a right R-module of type $FP_\infty\}$, where $\mathrm{pd}(M)$ is the projective dimension of M. Now we give some characterizations of those rings over which all modules are weak injective (cf. [26, Corollary 3.10]).

Corollary 4.7. *The following are equivalent for any ring R.*

(1) $r.sp.gldim(R) = 0$.
(2) *Every right R-module is weak injective.*
(3) *Every left R-module is weak flat.*

(4) *Every right R-module of type FP_∞ is projective.*
(5) *Every clean injective right R-module is injective.*
(6) *Every short exact sequence of right R-modules is clean.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) holds by [26, Corollary 3.10].
(2) $\Leftrightarrow$ (6) is easy and (2) $\Leftrightarrow$ (5) follows from Proposition 4.5. □

It is obvious that if R is a right (n,d)-ring, then $r.sp.gldim(R) \leq d$. However, a ring R with $r.sp.gldim(R) \leq d$ may fail to be a right (n,d)-ring, as shown in the following example.

Example 4.8. For any fixed integers $m \geq 2$ and $d \geq 0$, by [33, Theorem 2.1], there exists a ring R_m such that:

(1) R_m is a right (m,d)-ring;
(2) R_m is not a right $(m-1,t)$-ring for each non-negative integer t;
(3) R_m is not a right $(n,d-1)$-ring (for $d \geq 1$) for each non-negative integer n.

Let $R = \prod_{m=2}^{\infty} R_m$. Then $r.sp.gldim(R) \leq d$; but R is not a right (n,d)-ring for each non-negative integer n.

Proof. By [33, Corollary 2.2], R is not a right (n,d)-ring for each $n \geq 0$.

Next we prove that $r.sp.gldim(R) \leq d$; it is enough to show that every right R-module M belongs to $\mathcal{WI}_d$. Note that M is a direct limit of a direct system of finitely presented right R-modules. In addition, $\mathcal{WI}_d$ is closed under direct limits by Proposition 4.1(3). So we need only to show that every finitely presented right R-module P lies in $\mathcal{WI}_d$.

By [23, Theorem 3.2.22], we have

$$P \cong P \otimes_R R \cong P \otimes_R \prod_{m=2}^{\infty} R_m \cong \prod_{m=2}^{\infty} (P \otimes_R R_m).$$

Then each right R_m-module $P \otimes_R R_m$ is (m,d)-injective as each R_m is a right (m,d)-ring. Thus each $P \otimes_R R_m$ is also an (m,d)-injective right R-module by [40, Lemma 3.3(1)]. On the other hand, each class $\mathcal{I}_{m,d}$ is contained in $\mathcal{WI}_d$, and $\mathcal{WI}_d$ is closed under products by Proposition 4.1(3). It follows that the right R-module P lies in $\mathcal{WI}_d$, as desired. □

In [11], Bravo and Parra called right $(n,1)$-rings *right n-hereditary*, while a ring R was said to be *right ∞-hereditary* provided that $r.sp.gldim(R) \leq 1$.

Remark 4.9. In [11, Example 3.6], the authors wondered whether there is an example of a right ∞-hereditary ring that is not right n-hereditary for any $n \geq 0$. The example above gives a positive answer to this question.

Zhao proved in [49, Proposition 4.17] that the class $\mathcal{WI}_d^{\perp}$ is enveloping under the condition that $R_R \in \mathcal{WI}_d$. We will show that $\mathcal{WI}_d^{\perp}$ is enveloping for any ring R. But to do that we need the following lemma.

Lemma 4.10. *For any ring R, there exists a set $\mathcal{X}$ such that $\mathcal{WI}_d^{\perp} = \mathcal{X}^{\perp}$.*

Proof. The proof is inspired by that of [25, Corollary 3.3.4].

Let Card$(R) = \kappa$. Let $A \in \mathcal{WI}_d$ and choose any $x \in A$. By [23, Lemma 5.3.12], there is a pure submodule A_0 of A with $x \in A_0$ such that Card$(A_0) \leq \kappa$ (simply $N = Rx$, $M = A$ and f the inclusion map from N to M in the lemma). We see that both A_0 and A/A_0 are in $\mathcal{WI}_d$ by Proposition 4.1(3).

For any $x_1 \in A/A_0$, again by [23, Lemma 5.3.12], there is a pure submodule A_1/A_0 of A/A_0 such that $x_1 \in A_1/A_0$ and Card$(A_1/A_0) \leq \kappa$. Since A_0 is pure in A and A_1/A_0 is pure in A/A_0, A_1 is pure in A by [28, Lemma 1.2.17]. Thus we obtain that A_1/A_0, A_1 and A/A_1 all lie in $\mathcal{WI}_d$ again by Proposition 4.1(3).

Note that $\mathcal{WI}_d$ is closed under direct limits (see Proposition 4.1(3)). Proceeding by transfinite induction we can write A as a union of a continuous chain $(A_\alpha)_{\alpha<\lambda}$ of pure submodules of A, such that $A_0 \in \mathcal{WI}_d$, $A_{\alpha+1}/A_\alpha \in \mathcal{WI}_d$ and card$(A_{\alpha+1}/A_\alpha) \leq \kappa$ whenever $\alpha + 1 < \lambda$.

Let $\mathcal{X}$ be a set of representatives of modules $A \in \mathcal{WI}_d$ with Card$(A) \leq \kappa$. By [23, Theorem 7.3.4], we have that for any right R-module M, $M \in \mathcal{WI}_d^{\perp}$ if and only if $M \in \mathcal{X}^{\perp}$. This means that $\mathcal{WI}_d^{\perp} = \mathcal{X}^{\perp}$. □

The following corollaries 4.11 and 4.12 were proved in [1, Corollary 2.7] and [28, Theorem 4.1.13] when the ring is right Noetherian, respectively.

Corollary 4.11. *For any ring R, $\mathcal{WI}_d^{\perp}$ is enveloping.*

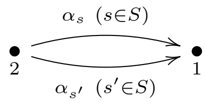
Proof. Follows from Proposition 4.1(3), Lemma 4.10 and [25, Corollary 3.1.10]. □

Corollary 4.12. *For any ring R, $({}^{\perp}(\mathcal{WI}_d^{\perp}), \mathcal{WI}_d^{\perp})$ is a complete cotorsion theory.*

Proof. Combine Lemma 4.10 with [18, Theorem 10]. □

It is clear that $\mathcal{WI}_d^{\perp}$ is closed under direct products. However, the following example shows that $\mathcal{WI}_d^{\perp}$ is not closed under direct sums (hence not precovering) in general.

Example 4.13. Let Q be the quiver



consisting of two points and infinitely many arrows $\{\alpha_s \mid s \in \mathcal{S}\}$, and let R be the path algebra of Q over an algebraically closed field k. For any $s \in \mathcal{S}$, let E_s be the injective envelope of $\overline{\alpha_s}R$. Then $\bigoplus_{s\in S} E_s \notin \mathcal{I}_{0,0}^{\perp}$. Thus $\bigoplus_{s\in S} E_s \notin \mathcal{WI}_d^{\perp}$ since $\mathcal{I}_{0,0} \subseteq \mathcal{WI}_d$.

Proof. A similar argument to that of Example 3.4 shows that $\mathrm{Ext}_R^1(S_2, \bigoplus_{s\in S} E_s) \neq 0$, where S_2 is the simple right R-module corresponding to the vertex 2. Then $\bigoplus_{s\in S} E_s \notin \mathcal{I}_{0,0}^{\perp}$ because S_2 is injective by [6, p. 81, Lemma 2.6]. □

Remark 4.14. The modules in $\mathcal{I}_{0,0}^{\perp}$ are just the so-called *copure injective* modules (see [21]). We see from Example 4.13 that the class of copure injective modules is not closed under direct sums in general.

Recall that R is said to be a QF ring if R is right Noetherian and R_R is injective.

Proposition 4.15. *R is a QF ring if and only if every right R-module belongs to $\mathcal{WI}_d^{\perp}$.*

Proof. Note that every injective right R-module belongs to $\mathcal{WI}_d$. In addition, we know that R is a QF ring if and only if every injective right R-module is projective (cf. [2, Theorem 31.9]). It follows that R is a QF ring if and only if every right R-module contained in $\mathcal{WI}_d$ is projective. Thus, R is a QF ring if and only if every right R-module belongs to $\mathcal{WI}_d^{\perp}$. □

5. Applications

In 1981, Enochs proved that a ring R is right Noetherian if and only if $\mathcal{I}_{0,0}$ is (pre)covering (see [19, Sec. 2]). Recently, Dai and Ding [17, Corollary 3.5] showed that a ring R is right coherent if and only if $\mathcal{I}_{1,0}$ is (pre)covering. In 1996, for a positive integer n, Chen and Ding [14, Theorem 3.1] obtained that R is a right n-coherent ring if and only if $\mathcal{I}_{n,n-1}$ is closed under direct limits. In 2004, Zhou [50, Theorem 3.4] proved that R is a right n-coherent ring if and only if $\mathcal{I}_{n,0} = \mathcal{I}_{n+1,0}$ if and only if $\mathcal{F}_{n,0} = \mathcal{F}_{n+1,0}$ $(n \geq 1)$. More characterizations for right n-coherent rings can be found in [11,12,14,16,32,34,39,40,48,50].

To present some new characterizations for right n-coherent rings, we need several lemmas.

Lemma 5.1. *The following statements hold for a ring R.*

(1) $\mathcal{I}_{n,d} = \mathcal{SI}_{n,d}$ *if and only if every (n,d)-injective right R-module is $(n,d{+}1)$-injective.*

(2) $\mathcal{I}_{j,0} \subseteq \mathcal{I}_{n,n-j}$ *and* $\mathcal{F}_{j,0} \subseteq \mathcal{F}_{n,n-j}$ *for* $0 \leq j \leq n$.

Proof. By dimension shifting. □

The following result is a refinement of [12, Lemmas 5.2, 5.3].

Lemma 5.2. *The following are equivalent for a right R-module M:*

(1) M *is n-presented.*

(2) M *is finitely generated and* $M \in {}^{\perp}\mathcal{I}_{n,0}$.

If $n \geq 2$, then the above conditions are also equivalent to:

(3) M *is finitely presented and* $M \in {}^{\top}\mathcal{F}_{n,0}$.

Proof. (1) $\Leftrightarrow$ (2) has been proved in [40, Theorem 2.1], and (1) $\Rightarrow$ (3) is trivial. The proof of (3) $\Rightarrow$ (2) is analogous to that of [12, Lemma 5.3]. □

The following result can also be proved using the technique of [10, Proposition 2.4].

Corollary 5.3. *R is a right coherent ring if and only if every right R-module is a direct limit of n-presented right R-modules for some $n > 1$.*

Proof. We only need to prove the sufficiency part. Suppose that every finitely presented right R-module M can be written as a direct limit $\varinjlim M_j$ of n-presented right R-modules with $n > 1$. Since the Tor-functor commutes with $\varinjlim$, we have that

$$\mathrm{Tor}_1(M, F) \cong \mathrm{Tor}_1(\varinjlim M_j, F) \cong \varinjlim \mathrm{Tor}_1(M_j, F) = 0$$

for any $F \in \mathcal{F}_{n,0}$. So M is n-presented by Lemma 5.2, and hence R is right coherent. □

Let $\mathscr{Y}$ be a class of right R-modules. We denote by $\overline{\mathscr{Y}}$ the smallest definable class containing $\mathscr{Y}$. Šaroch and Šťovíček [43, Theorem 2.8] recently proved that ${}^{\perp}\mathscr{Y} = {}^{\perp}\overline{\mathscr{Y}}$ provided that $\mathscr{Y}$ is closed under direct limits and products. There is more to say in case $\mathscr{Y}$ is the right part of a cotorsion theory.

Lemma 5.4. *Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion theory. If $\mathscr{Y}$ is closed under direct limits, then $\mathscr{Y}$ is definable.*

Proof. Note that the right part of a cotorsion theory is always closed under products. It follows from [43, Theorem 2.8] that ${}^{\perp}\mathscr{Y} = {}^{\perp}\overline{\mathscr{Y}}$ since $\mathscr{Y}$ is closed under direct limits. This yields the inclusion $\overline{\mathscr{Y}} \subseteq \mathscr{Y}$ because $({}^{\perp}\mathscr{Y},\ \mathscr{Y})$ is a cotorsion theory. But then $\overline{\mathscr{Y}} = \mathscr{Y}$, i.e., $\mathscr{Y}$ is definable. □

Recall that a ring R is said to be *von Neumann regular* if every short exact sequence of right R-modules is pure exact. We now give a characterization of the right global dimension of von Neumann regular rings, which is far from obvious.

Corollary 5.5. *Let R be a von Neumann regular ring. Then $\mathrm{rD}(R) \leq d$ if and only if $\mathcal{SI}_{0,d}$ is closed under direct limits.*

Proof. We only need to prove the sufficiency part. Assume that $\mathcal{SI}_{0,d}$ is closed under direct limits. Then $\mathcal{SI}_{0,d}$ is closed under pure submodules by Theorem 3.8 and Lemma 5.4. But every submodule of an R-module is pure since R is von Neumann regular. Hence every right R-module belongs to $\mathcal{SI}_{0,d}$, i.e., $\mathrm{rD}(R) \leq d$. □

Theorem 5.6. *The following are equivalent for a ring R and a positive integer n.*

(1) *R is a right n-coherent ring.*
(2) *$\mathcal{I}_{n,n-1}$ is (pre)covering.*
(3) *$\mathcal{I}_{n,n}$ is closed under direct limits.*
(4) *$\mathcal{SI}_{n,t}$ is closed under direct limits for some non-negative integer $t \leq n$.*
(5) *There exist a non-negative integer $m \leq n$ and an integer $j \geq n-m+1$ such that $\mathcal{I}_{j,0} \subseteq \mathcal{I}_{n,m}$.*
(6) *$\mathcal{WI}_m = \mathcal{SI}_{n,m}$ for some non-negative integer $m \leq n$.*
(7) *$\mathcal{WI}_m = \mathcal{I}_{n,m}$ for some non-negative integer $m \leq n$.*
(8) *$\mathcal{I}_{n,t} = \mathcal{SI}_{n,t}$ for some non-negative integer $t \leq n-1$.*
(9) *$\mathcal{WF}_t = \mathcal{SF}_{n,t}$ for some non-negative integer $t \leq n-1$.*
(10) *$\mathcal{WF}_t = \mathcal{F}_{n,t}$ for some non-negative integer $t \leq n-1$.*
(11) *There exist a non-negative integer $m \leq n-1$ and an integer $j \geq n-m+1$ such that $\mathcal{F}_{j,0} \subseteq \mathcal{F}_{n,m}$.*

If $n \geq 2$, then the above conditions are also equivalent to:

(12) *$\mathcal{F}_{n,t} = \mathcal{SF}_{n,t}$ for some non-negative integer $t \leq n-2$.*

Proof. (1) ⇒ (2) See [34, Theorem 3.6].
(2) ⇒ (1) It is obvious that $\mathcal{I}_{n,n-1}$ is closed under direct products. In addition, $\mathcal{I}_{n,n-1}$ is closed under pure submodules by Proposition 3.3(1). Now suppose $\mathcal{I}_{n,n-1}$ is precovering. Then $\mathcal{I}_{n,n-1}$ is closed under direct limits by [17, Theorem 3.4]. Thus

R is a right n-coherent ring by [14, Theorem 3.1].

(1) $\Rightarrow$ (6) Let R be a right n-coherent ring. Then every n-presented right R-module is of type FP_∞. So $\mathcal{WI}_m = \mathcal{I}_{n,m}$. Thus (6) is true since $\mathcal{I}_{n,m} = \mathcal{SI}_{n,m}$ by Proposition 3.2.

(6) $\Rightarrow$ (7) is clear.

(7) $\Rightarrow$ (5) Let m be the integer described in (7) and let $j \geq n-m+1$. It is clear that $\mathcal{I}_{j,0} \subseteq \mathcal{WI}_0 \subseteq \mathcal{WI}_m$. But $\mathcal{WI}_m = \mathcal{I}_{n,m}$ by (7). Hence $\mathcal{I}_{j,0} \subseteq \mathcal{I}_{n,m}$.

(5) $\Rightarrow$ (1) Let m and j be the integers described in (5). We must prove that any n-presented right R-module P is $(n+1)$-presented. Consider a projective resolution

$$F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} P \xrightarrow{f_{-1}} 0$$

of P with each F_i finitely generated. We need to prove that $K_{m-1} = \ker(f_{m-1})$ is $(n-m+1)$-presented. Let E be any $(j,0)$-injective right R-module. Then E is (n,m)-injective by (5). Whence $\mathrm{Ext}^1(K_{m-1}, E) \cong \mathrm{Ext}^{m+1}(P, E) = 0$. So $K_{m-1} \in {}^{\perp}\mathcal{I}_{j,0}$. Clearly, K_{m-1} is finitely generated. Thus K_{m-1} is j-presented by Lemma 5.2. Note that $j \geq n-m+1$. Hence P is $(n+1)$-presented, as desired.

(1) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (11) is similar to that of (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (5).

(11) $\Rightarrow$ (1) This is the same as that of (5) $\Rightarrow$ (1) (by replacing injective and Ext with flat and Tor, but using the equivalence of (1) and (3) in Lemma 5.2).

(1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (12) See Proposition 3.2.

(8) $\Rightarrow$ (5) Let t be as in (8). Then $\mathcal{I}_{n-t,0} \subseteq \mathcal{I}_{n,t}$ by Lemma 5.1. But $\mathcal{I}_{n,t} = \mathcal{SI}_{n,t}$ by (8), hence $\mathcal{I}_{n-t,0} \subseteq \mathcal{I}_{n,t} = \mathcal{SI}_{n,t} \subseteq \mathcal{I}_{n,t+1}$. So (5) follows by letting $j = n-t$ and $m = t+1$.

(12) $\Rightarrow$ (11) is analogous to that of (8) $\Rightarrow$ (5).

(1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) See Proposition 3.3(3) and Proposition 3.2.

(4) $\Rightarrow$ (5) Assume that $\mathcal{SI}_{n,t}$ is closed under direct limits for some non-negative integer $t \leq n$. Then $\mathcal{SI}_{n,t}$ is closed under pure submodules by Theorem 3.8 and Lemma 5.4. But it is clear that every $(1,0)$-injective module is a pure submodule in every module that contains it. So $\mathcal{I}_{1,0} \subseteq \mathcal{SI}_{n,t}$. On the other hand, it is clear from the definition of strongly (n,t)-injective modules that $\mathcal{SI}_{n,t} \subseteq \mathcal{I}_{n,n}$ for $t \leq n$. Hence $\mathcal{I}_{1,0} \subseteq \mathcal{I}_{n,n}$ and (5) follows.

(3) $\Rightarrow$ (5) is similar to that of (4) $\Rightarrow$ (5) (using [39, Theorem 3.9] and Lemma 5.4). The proof is finished. $\square$

Immediately we get the following corollary which was proved by Costa in [16, Theorem 2.2].

Corollary 5.7. *Let R be a right (n,d)-ring. Then R is a right max$\{n,\ d\}$-coherent ring.*

Proof. Noting that a right (n,d)-ring is a right (max$\{n,\ d\}$, d)-ring, the conclusion follows from the equivalence of (1) and (4) in Theorem 5.6. □

Corollary 5.8. *Let R be a right weak (n,d)-ring. Then R is a right max$\{n,\ d+1\}$-coherent ring.*

Proof. Holds by the equivalence of (1) and (10) in Theorem 5.6 and the fact that a right weak (n,d)-ring is a right weak (max$\{n,\ d+1\}$, d)-ring. □

Recall that a chain complex I of injective right R-modules is said to be *AC-injective* (see [27, Definition 5.1]), if each chain map $A \to I$ is null homotopic whenever A is an exact complex with each cycle $Z_i(A) \in \mathcal{WI}_0$.

Let $\mathrm{K}(Inj)$ be the chain homotopy category of all complexes of injective right R-modules, and let $\mathrm{K}(\mathcal{AC})$ denote the chain homotopy category of all AC-injective complexes. Surprisingly, Šťovíček [44] showed that $\mathrm{K}(Inj) = \mathrm{K}(\mathcal{AC})$ whenever R is just a right coherent ring. Gillespie asked in [27] that whether the ring R is necessary right coherent in order that $\mathrm{K}(Inj) = \mathrm{K}(\mathcal{AC})$. Later, a counterexample to the problem was presented in [46, Example 5.4]. To give new counterexamples to Gillespie's question, we need the following proposition.

Proposition 5.9. *Let R be a ring and n a non-negative integer. Then $\mathrm{K}(Inj) = \mathrm{K}(\mathcal{AC})$ provided that the following three conditions are satisfied:*

(1) *R is left and right n-coherent;*
(2) *every $(n,0)$-injective right R-module has flat dimension less than or equal to n;*
(3) *every $(n,0)$-injective left R-module has flat dimension less than or equal to n.*

Proof. This is due to [46, Theorem 5.3]. □

Now we are able to give new counterexamples to Gillespie's question.

Example 5.10. Let $S = (\prod_1^\infty(\mathbb{Z}/2\mathbb{Z}))/(\bigoplus_1^\infty(\mathbb{Z}/2\mathbb{Z}))$, and let $R_0 = S[[X]]$ be the power series ring. Then $\mathrm{wD}(R_0) = 1$, and R_0 is not semihereditary (see [13, Example 2]). So R_0 is a weak $(1,1)$-ring, and R_0 is not a $(1,1)$-ring (see [50, Corollary 2.7(5,6)]). Thus R_0 is not coherent by [50, Proposition 2.6(3)]. Denote by $R_m := R_0[x_1, x_2, \ldots, x_m]$ the polynomial ring in m indeterminates over R_0. Then R_i is not coherent with $\mathrm{wD}(R_i) = i+1$ (see Proposition 3.11) for $0 \le i \le m$;

but R_i is $(i+2)$-coherent by Corollary 5.8 since R_i is a weak $(1, i+1)$-ring, and thus $\mathrm{K}(Inj) = \mathrm{K}(\mathcal{AC})$ by Proposition 5.9.

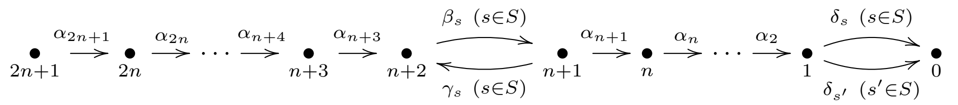
Costa [16, Theorem 4.5] proved that, if R is a commutative weak $(1, d)$-ring, then R is a $(d+1, d)$-ring. We generalize this result as follows.

Corollary 5.11. *Let R be a right weak (n, d)-ring. Then R is a right (t, d)-ring where $t = \max\{n,\ d+1\}$.*

Proof. Combine Corollary 5.8 with [50, Proposition 2.6(3)]. □

Next we give examples to show the sharpness of Theorem 5.6.

Example 5.12. Let $n \geq 2$ be a fixed integer. Let Q be the quiver with $2n+2$ vertices, one arrow α_{i+1} from vertex $i+1$ to vertex i for each $i \in \{1,\ 2,\ \cdots,\ 2n\} \setminus \{n+1\}$, infinitely many arrows $\{\beta_s \mid s \in \mathcal{S}\}$ from vertex $n+2$ to vertex $n+1$, infinitely many arrows $\{\gamma_s \mid s \in \mathcal{S}\}$ from vertex $n+1$ to vertex $n+2$, and infinitely many arrows $\{\delta_s \mid s \in \mathcal{S}\}$ from vertex 1 to vertex 0.



Let R be the quotient of the path algebra of Q over an algebraically closed field k by the ideal generated by the set of all paths of length $\ell \geq 2$. Then the following are true for R.

(1) R is a right $(n, n+1)$-ring.
(2) R is not a right (m, n)-ring for $0 \leq m \leq n$.
(3) R is not a right $(n-1, t)$-ring for each non-negative integer t.
(4) R is not a right n-coherent ring.
(5) R is a right $(n+1, 1)$-ring.

Proof. We only prove (4); the proof of the remainder is similar to that of [33, Theorem 2.1].

Let P_i be the indecomposable projective right R-module corresponding to the vertex $i \in \{1, 2, \cdots, n+1\}$. Write $M_s = \overline{\delta_s}R$ and $G_{n+1} = \overline{\alpha_{n+1}}R$. We have naturally the following exact sequences of right R-modules

$$0 \longrightarrow G_{n+1} \longrightarrow P_{n+1} \longrightarrow L \longrightarrow 0,$$

$$0 \to \mathrm{rad}\ P_1 \to P_1 \to P_2 \to \cdots \to P_n \to G_{n+1} \to 0,$$

where $L = P_{n+1}/G_{n+1}$. Since rad $P_1 = \bigoplus_{\delta \in S} M_s$ is not finitely generated, we see from the two exact sequences above that L is n-presented but not $(n+1)$-presented. Therefore, R is not a right n-coherent ring. □

Let $\mathscr{C}$ be a class of right R-modules. We say that a $\mathscr{C}$-cover $f : C \to A$ of a module A *completes the diagrams in a unique way* if for any homomorphism $g : C' \to A$ with $C' \in \mathscr{C}$, there is a unique homomorphism $h : C' \to C$ such that $fh = g$.

Remark 5.13. (1) The implication of (2) $\Rightarrow$ (1) in Theorem 5.6 has been proven by Zhou (see [50, Proposition 4.3]). But it seems that there is a gap in the proof there because an $\mathcal{I}_{n,d}$-precover can not complete the diagrams in a unique way in general. In fact, for a right n-coherent ring R, R is a right $(n, d+2)$-ring if and only if R is a right weak $(n, d+2)$-ring (see [50, Proposition 2.6(3)]) if and only if every right R-module has an $\mathcal{I}_{n,d}$-cover which completes the diagrams in a unique way (see [35, Proposition 4.11]); however, there are right n-coherent rings which are not right $(n, d+2)$-rings for any $n \geq 2$ and d (see [33, Theorem 2.1(3, 4)]).

(2) Let $n \geq 1$. It is asked in [34, Remark 4.4] that whether R must necessarily be right n-coherent in order that $\mathcal{I}_{n,d}$ is covering for any non-negative integer d. Theorem 5.6 gives an affirmative answer to this question.

Theorem 5.6 and Proposition 3.2 tell us that, if R is a right n-coherent ring $(n \geq 1)$, then $\mathcal{SI}_{n,n-1}$ is (pre)covering. We will see that the converse is also true for right perfect rings and right (n, d)-rings.

Theorem 5.14. *The following are equivalent for a right perfect ring R and a non-negative integer n.*

(1) *R is a right n-coherent ring.*
(2) *$\mathcal{SI}_{n,t}$ is (pre)covering for some non-negative integer $t \leq n$.*
(3) *$\mathcal{SI}_{n,t}$ is closed under direct sums for some non-negative integer $t \leq n$.*

Proof. (1) $\Rightarrow$ (2) By [34, Theorem 3.6] and Proposition 3.2.
(2) $\Rightarrow$ (3) See [34, Proposition 2.6].
(3) $\Rightarrow$ (1) Let P be an n-presented right R-module. Then there is an exact sequence

$$F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow P \longrightarrow 0, \qquad (\clubsuit)$$

where each F_i is finitely generated and free, from which we obtain the exact sequence

$$0 \longrightarrow K = \ker(f_n) \xrightarrow{\eta} F_n \xrightarrow{f_n} L = \operatorname{im}(f_n) \longrightarrow 0. \qquad (\sharp)$$

Since R is right perfect, K has a minimal generating set $\mathcal{X}$ (this means that any proper subset of $\mathcal{X}$ no longer generates K) by [41, Theorem 3]. If Card $\mathcal{X}$ is finite, then we are done. Now assume that Card $\mathcal{X}$ is infinite. Pick a countable subset $\mathcal{Y} = \{y_1, y_2, \cdots, y_i, \cdots\}$ of $\mathcal{X}$. Write $\mathcal{U}_i = (\mathcal{X}\backslash\mathcal{Y})\cup\{y_1, y_2, \cdots, y_i\}$, $i \geq 1$. Let $K_i = \mathrm{Span}(\mathcal{U}_i)$ and let E_i be the injective envelope of K/K_i. The natural homomorphisms $\pi_i : K \to K/K_i$ and the inclusions $\tau_i : K/K_i \to E_i$ induce a homomorphism $g\colon K \to \bigoplus_{i=1}^{\infty} E_i$ via $g(x) = (\tau_i\pi_i(x))$. Then g is well defined because, for any $x \in K$, $\pi_i(x) = 0$ for $i \gg 0$.

Note that $\bigoplus_{i=1}^{\infty} E_i \in \mathcal{SI}_{n,t}$ for some non-negative integer $t \leq n$ by (3). It follows from the exact sequence (♣) that

$$\mathrm{Ext}^1(L, \bigoplus_{i=1}^{\infty} E_i) \cong \mathrm{Ext}^{n+1}(P, \bigoplus_{i=1}^{\infty} E_i) = 0.$$

Hence, the exactness of the sequence (♯) yields a homomorphism $h : F_n \to \bigoplus_{i=1}^{\infty} E_i$ making the following diagram commutative

$$\begin{array}{ccccccc} E_i & \xleftarrow{\tau_i} & K/K_i & \xleftarrow{\pi_i} & K & \xrightarrow{\eta} & F_n \\ & & & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ & & & & \bigoplus_{i=1}^{\infty} E_i. & & \end{array}$$

As F_n is finitely generated and free, there exists a sufficiently large l such that $\mathrm{im}(h) \bigcap E_j = 0$ whenever $j > l$. But $\mathrm{im}(g) \subseteq \mathrm{im}(h)$. Thus $\mathrm{im}(g) \bigcap E_j = 0$ whenever $j > l$.

On the other hand, the generating set $\mathcal{X}$ of K is minimal. Hence, for any i, there is $x_i \in K$ such that $x_i \notin K_i$, i.e., $\pi_i(x_i) \neq 0$. This forces that $g(x_i) = (\tau_i\pi_i(x_i)) \neq 0$. So $\mathrm{im}(g) \bigcap E_i \neq 0$ for any i, a contradiction.

Therefore, Card $\mathcal{X}$ is finite, as desired. □

Remark 5.15. There are right perfect rings which are not right n-coherent; the ring constructed in Example 5.12 is such a ring.

Though a right (n, d)-ring is always right $\max\{n,\ d\}$-coherent, it need not be right n-coherent (see Example 5.12). Next we explore equivalent conditions on a right (n, d)-ring R which imply that R is right n-coherent. Before doing that, we state the following result which appears in [5, Theorem 2.5].

Lemma 5.16. *Let $(\mathscr{C},\ \mathscr{D})$ be a hereditary complete cotorsion theory of right R-modules. Then the following are equivalent for a non-negative integer m.*

(1) $\mathscr{C} \subseteq \mathcal{P}_m$.

(2) *For any right R-module M, there is an exact sequence* $0 \to M \to D^0 \to D^1 \to \cdots \to D^{m-1} \to D^m \to 0$ *with each* $D^i \in \mathscr{D}$.

Corollary 5.17. *The following statements hold for any ring R.*

(1) *R is a right $(n, d+m)$-ring if and only if* ${}^{\perp}\mathcal{SI}_{n,d} \subseteq \mathcal{P}_m$.

(2) *R is a right weak $(n, d+m)$-ring if and only if* $\mathcal{SF}_{n,d}^{\perp} \subseteq \mathcal{SI}_{0,m}$.

Proof. (1) holds by Remark 3.10, Theorem 3.8, Corollary 3.9 and Lemma 5.16. (2) is a dual version of (1). □

For a module M, we denote by Add M (resp. Prod M) the class of all direct summands of arbitrary direct sums (resp. products) of copies of M.

Let m be a non-negative integer. A right R-module T is called *m-tilting* [3] if it satisfies the following three conditions:

(T1) $T \in \mathcal{P}_m$;

(T2) $\mathrm{Ext}^i(T, T^{(\mathcal{S})}) = 0$ for any positive integer i and all sets $\mathcal{S}$;

(T3) there exist $r \geq 0$ and a long exact sequence $0 \to R \to T^0 \to \cdots \to T^r \to 0$ such that $T^i \in$ Add T for all $0 \leq i \leq r$.

A class of modules $\mathscr{T}$ is *m-tilting* provided there is an m-tilting module T such that $\mathscr{T} = T^{\perp\infty}$. In this case, $({}^{\perp}(T^{\perp\infty}), T^{\perp\infty})$ is a hereditary complete cotorsion theory (cf. [18]), called the *m-tilting cotorsion theory* induced by T. Moreover, if there exists $\mathcal{S} \subseteq \mathcal{P}_m^{<\infty}$ such that $\mathscr{T} = T^{\perp\infty} = \mathcal{S}^{\perp\infty}$, then T and $T^{\perp\infty}$ are called *m-tilting of finite type*.

Dually, a left R-module C is called *m-cotilting* [3] if it satisfies the following three conditions:

(C1) $C \in \mathcal{SI}_{0,m}$;

(C2) $\mathrm{Ext}^i(C^{\mathcal{S}}, C) = 0$ for any positive integer i and all sets $\mathcal{S}$;

(C3) there exist $r \geq 0$ and a long exact sequence $0 \to C^r \to \cdots \to C^0 \to Q \to 0$ such that $C^i \in$ Prod C for all $0 \leq i \leq r$ and Q is an injective cogenerator.

A class of modules $\mathscr{C}$ is *m-cotilting* provided there is an m-cotilting module C such that $\mathscr{C} = {}^{\perp\infty}C$. In this case, $({}^{\perp\infty}C, ({}^{\perp\infty}C)^{\perp})$ is a hereditary complete cotorsion theory (cf. [3]), called the *m-cotilting cotorsion theory* induced by C. Moreover, if there exists $\mathcal{S} \subseteq \mathcal{P}_m^{<\infty}$ such that $\mathscr{C} = {}^{\perp\infty}C = \mathcal{S}^{\top\infty}$, then C and $\mathscr{C}$ are called *m-cotilting of cofinite type*.

It is known that every tilting class is of finite type (see [28, Theorem 5.2.20]); however there are cotilting classes that are not of cofinite type (see [28, Example 8.2.13]).

Proposition 5.18. *Every tilting class contains $\mathcal{WI}_0$, and every cotilting classs of cofinite type contains $\mathcal{WF}_0$.*

Proof. This follows directly by definitions. □

We know that tilting (resp. cotilting) classes are special preenveloping (resp. special precovering). Here we have:

Proposition 5.19. *Every tilting class $\mathscr{T}$ is covering, and every cotilting class $\mathscr{C}$ is preenveloping.*

Proof. Note that every tilting class $\mathscr{T}$ is closed under pure submodules and direct sums (see [28, Corollary 5.2.17]). Thus $\mathscr{T}$ is closed under pure quotients by [4, Theorem 2.1(1)(b)]. Hence $\mathscr{T}$ is covering by [29, Theorem 2.5].

Since every cotilting class $\mathscr{C}$ is closed under pure submodules and direct products by [28, Theorem 8.1.7], it follows from [29, Remark 2.6] that $\mathscr{C}$ is preenveloping. □

Now we determine when a right (n, d)-ring is right n-coherent.

Theorem 5.20. *Let R be a ring and m a non-negative integer. Consider the following statements:*

(1) *R is a right $(n, d+m)$-ring and R is right n-coherent;*
(2) *R is a right $(n, d+m)$-ring and $\mathcal{SI}_{n,d}$ is closed under direct sums;*
(3) *R is a right $(n, d+m)$-ring and $\mathcal{SI}_{n,d}$ is (pre)covering;*
(4) *$\mathcal{SI}_{n,d}$ is an m-tilting class;*
(5) *$\mathcal{SF}_{n,d}$ is an m-cotilting class of cofinite type;*
(6) *$\mathcal{SF}_{n,d}$ is an m-cotilting class;*
(7) *R is a right weak $(n, d+m)$-ring and $\mathcal{SF}_{n,d}$ is closed under direct products.*

Then (1) ⇒ (2) ⇔ (3) ⇔ (4) ⇒ (5) ⇒ (6) ⇔ (7). *Moreover, if $n \geq d$, then* (4) ⇒ (1)*; if $n \geq d+1$, then* (6) ⇒ (1).

Proof. (1) ⇒ (3) By Proposition 3.2, we obtain that $\mathcal{I}_{n,d} = \mathcal{SI}_{n,d}$. On the other hand, notice that in this case the n-presented right R-modules coincide with the right R-modules of type FP_∞. Hence, we deduce the equalities $\mathcal{WI}_d = \mathcal{I}_{n,d} = \mathcal{SI}_{n,d}$. It follows from Proposition 4.1(5) that $\mathcal{SI}_{n,d}$ is covering.
(3) ⇒ (2) is a consequence of [34, Proposition 2.6].
(2) ⇔ (4) Note that R is a right $(n, d+m)$-ring if and only if ${}^{\perp}\mathcal{SI}_{n,d} \subseteq \mathcal{P}_m$ by Corollary 5.17(1). Also note that $({}^{\perp}\mathcal{SI}_{n,d}, \mathcal{SI}_{n,d})$ is a hereditary complete cotorsion theory (see Theorem 3.8). The equivalence then follows from [4, Theorem 2.1(1)].
(4) ⇒ (3) By (2) and Proposition 5.19.

(4) $\Rightarrow$ (5) Let F be a left R-module and P any n-presented right R-module. Then $F \in \mathcal{SF}_{n,d}$ if and only if $\mathrm{Tor}_1(K_i, F) = 0$ for all $i > d$, where K_i denotes the ith syzygy of P. Let $\mathcal{X}$ be a set of representatives of ith (for any $i > d$) syzygy modules of all n-presented right R-modules. Then $\mathcal{SF}_{n,d} = \mathcal{X}^{\top}$ and $\mathcal{SI}_{n,d} = \mathcal{X}^{\perp}$.

Let $\mathcal{U} = \mathcal{X}^{<\infty}$. By [28, Theorem 5.2.20], $\mathcal{SI}_{n,d}$ is of finite type, so $\mathcal{SI}_{n,d} = \mathcal{U}^{\perp}$. Thus $\mathcal{U}^{\top}$ is an m-cotilting class of cofinite type by [28, Theorem 8.1.2]. Next we show that $\mathcal{SF}_{n,d} = \mathcal{U}^{\top}$.

Note that ${}^{\perp}(\mathcal{U}^{\perp}) = {}^{\perp}\mathcal{SI}_{n,d} = {}^{\perp}(\mathcal{X}^{\perp})$. We may assume that both $\mathcal{U}$ and $\mathcal{X}$ contain R. So, by [28, Corollary 3.2.4], every module in $\mathcal{X}$ is a direct summand of a $\mathcal{U}$-filtered module, and every module in $\mathcal{U}$ is a direct summand of an $\mathcal{X}$-filtered module; for the definitions of $\mathscr{C}$-filtered modules we refer to [28, Definition 3.1.1]. Therefore, we infer from [28, Corollary 3.1.3] that $\mathcal{U}^{\top} = \mathcal{X}^{\top} = \mathcal{SF}_{n,d}$, as desired.
(5) $\Rightarrow$ (6) is trivial.
(6) $\Leftrightarrow$ (7) Similar to that of (2) $\Leftrightarrow$ (4).
(4) $\Rightarrow$ (1) Assume that $n \geq d$ and $\mathcal{SI}_{n,d}$ is an m-tilting class. Then R is a right $(n, d+m)$-ring since (4) and (2) are equivalent. It remains to show that R is right n-coherent.

If $n = 0$, then $d = 0$. So $\mathcal{SI}_{0,0}$ is closed under direct limits. It follows from [23, Theorem 3.1.17] that R is right noetherian.

If $n > 0$, we then conclude from the equivalence of (1) $\Leftrightarrow$ (4) in Theorem 5.6 that R is right n-coherent.
(6) $\Rightarrow$ (1) Suppose that $n \geq d+1$ and $\mathcal{SF}_{n,d}$ is an m-cotilting class. Then every direct product of copies of the left module ${}_RR$ belongs to $\mathcal{SF}_{n,d}$ by [4, Theorem 2.1(2)]. Hence R is right n-coherent by [50, Proposition 3.1]. On the other hand, we can mimic the proof of (4) $\Rightarrow$ (1) to obtain that R is a right weak $(n, d+m)$-ring. But then R is a right $(n, d+m)$-ring by [50, Proposition 2.6(3)]. □

Corollary 5.21. *Suppose R is a right $(n, d+1)$-ring and R is right n-coherent. If R is commutative, then $\mathcal{SF}_{n,d}$ is closed under taking injective envelopes.*

Proof. Combine Theorem 5.20 with [31, Proposition 3.11]. □

Remark 5.22. (1) Theorem 5.20 tells us that, a right (n, d)-ring R is right n-coherent if and only if $\mathcal{SI}_{n,t}$ is closed under direct sums for some non-negative integer $t \leq n$ if and only if $\mathcal{SI}_{n,t}$ is (pre)covering for some non-negative integer $t \leq n$. This generalizes and improves [32, Theorem 4.5], and answers the problem in [32, Remark 4.7] when R is a right $(1, d)$-ring.

(2) It seems reasonable to conjecture that a ring R is right n-coherent if and only if $\mathcal{SI}_{n,t}$ is closed under direct sums for some non-negative integer $t \leq n$ if and only if $\mathcal{SI}_{n,t}$ is (pre)covering for some non-negative integer $t \leq n$.

6. G-(n,d)-rings: a Gorenstein analogue of Costa's first conjecture

In this section, we deal with a Gorenstein analogue of Costa's first conjecture. First, we recall the definitions of Gorenstein projective and flat modules introduced by Enochs and Jenda in [22] and [24]:

A *complete projective resolution* is an exact sequence of projective R-modules,

$$\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots ,$$

such that $\mathrm{Hom}_R(-,\, Q)$ leaves the sequence exact whenever Q is a projective R-module; the module $M = \mathrm{im}(P_0 \to P_{-1})$ is then said to be *Gorenstein projective*.

A right R-module M is said to be *Gorenstein $flat$* [24] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat right R-modules with $M = \mathrm{im}(F_0 \to F^0)$ such that -$\otimes E$ leaves the sequence exact whenever E is an injective left R-module.

Let M be an R-module. We say that M has *Gorenstein projective dimension* at most n, and we write $\mathrm{Gpd}_R(M) \leq n$, if there exists an exact sequence of R-modules $0 \to G_n \to \cdots \to G_0 \to M \to 0$ where each G_i is Gorenstein projective. If there is no such n, set $\mathrm{Gpd}_R(M) = \infty$. The *Gorenstein flat dimension*, $\mathrm{Gfd}_R(M)$, is defined similarly.

The *right Gorenstein global dimension* of rings is introduced in [8] as follows:

$$r.\mathrm{Ggldim}(R) = \sup\{\mathrm{Gpd}_R(M) \mid M \text{ is a right } R\text{-module}\}.$$

Recently, Christensen, Estrada and Thompson (see [15, Corollary 1.5 and Remark 1.6]) showed that

$$\sup\{\mathrm{Gfd}_R(M) \mid M \text{ is a right } R\text{-module}\} = \sup\{\mathrm{Gfd}_R(N) \mid N \text{ is a left } R\text{-module}\}$$

for any ring R. The common value of the quantities above is called the *Gorenstein weak global dimension* of R and we denote it by $\mathrm{Gwgldim}(R)$.

Mahdou and Ouarghi [37] called a commutative ring R a *G-(n,d)-ring* if every n-presented right R-module has Gorenstein projective dimension at most d. For a general ring R, we give the following definition.

Definition 6.1. Let n and d be non-negative integers. R is called a *right G-(n,d)-ring* if every n-presented right R-module has Gorenstein projective dimension at most d; R is called a *right weak G-(n,d)-ring* if every n-presented right R-module has Gorenstein flat dimension at most d.

Proposition 6.2. *Let R be a ring.*

(1) *R is a right G-$(0,d)$-ring if and only if $r.\mathrm{Ggldim}(R) \leq d$.*
(2) *R is a right weak G-$(1,d)$-ring if and only if $\mathrm{Gwgldim}(R) \leq d$.*

Proof. The assertions (1) and (2) follow respectively from [20, Proposition 3.5] and [36, Theorem 2.10]. □

Costa's paper [16] concludes with a number of open problems for commutative rings, including his first conjecture: given non-negative integers n and d, there is an (n,d)-ring which is neither an $(n,d-1)$-ring nor an $(n-1,d)$-ring. This has been answered positively for non-commutative settings in [33, Theorem 2.1]. In addition, a right (n,d)-ring is always a right G-(n,d)-ring. So one might be interested to ask the following question:

Question 1. *For all non-negative integers n and d, give examples of rings R satisfying the following conditions:*

(1) *R is a right G-(n,d)-ring;*
(2) *R is neither a right G-$(n,d-1)$-ring nor a right G-$(n-1,d)$-ring;*
(3) *R is not a right (n,d)-ring.*

Such examples of rings for $n = 0$, 1 can be easily constructed by using Theorem 4.2(1) and Corollary 6.6 (see [7, Examples 3.4 and 3.8]). For $n = 2$, 3, examples of rings R satisfying the conditions (1) and (2) in Question 1 are provided in [37, Theorems 3.1 and 3.3].

Before answering this question in the positive for all non-negative integers n and d, we need to study the transfer of the G-(n,d)-property to the finite direct sum of rings; this requires two lemmas.

Lemma 6.3. *Let R_1 and R_2 be two rings and let $R = R_1 \oplus R_2$. Then every right R-module M has a decomposition that $M = A \oplus B$, where $A = M(R_1, 0)$ is a right R_1-module and $B = M(0, R_2)$ is a right R_2-module via $ar_1 = a(r_1, 0)$ for $a \in A$, $r_1 \in R_1$, and $br_2 = b(0, r_2)$ for $b \in B$, $r_2 \in R_2$. Consequently, if $M' = A' \oplus B'$ with $A' \in \mathcal{M}_{R_1}$ and $B' \in \mathcal{M}_{R_2}$, then*

$$\mathrm{Hom}_R(M,\, M') \cong \mathrm{Hom}_{R_1}(A,\, A') \oplus \mathrm{Hom}_{R_2}(B,\, B').$$

Proof. The assertion that $M = A \oplus B$ is obvious; see also [38, Lemma 3.14]. Now let $f \in \mathrm{Hom}_R(M,\, M')$. Then for arbitrary $a \in A$ and $b \in B$, one has

$$f(a+b) = f(a) + f(b) = f(a(1_{R_1})) + f(b(1_{R_2})) = f(a)1_{R_1} + f(b)1_{R_2}.$$

But $f(a)1_{R_1} \in A'$ and $f(b)1_{R_2} \in B'$. It follows from this observation that

$$\mathrm{Hom}_R(M,\, M') \cong \mathrm{Hom}_{R_1}(A,\, A') \oplus \mathrm{Hom}_{R_2}(B,\, B').$$

□

For an R-module M, as in [45], we set $\lambda_R(M) = \sup\{n\colon M \text{ is } n\text{-presented}\}$ (if M is not finitely generated, set $\lambda_R(M) = -1$; if M is n-presented for each $n \geq 0$, set $\lambda_R(M) = \infty$).

Lemma 6.4. *Let R_1 and R_2 be two rings and let $R = R_1 \oplus R_2$. If $M = A \oplus B$ with $A \in \mathcal{M}_{R_1}$ and $B \in \mathcal{M}_{R_2}$, then the following statements hold for any non-negative integer n.*

(1) $\lambda_R(M) \geq n$ *if and only if* $\lambda_{R_1}(A) \geq n$ *and* $\lambda_{R_2}(B) \geq n$.
(2) $\mathrm{pd}_R(M) \leq n$ *if and only if* $\mathrm{pd}_{R_1}(A) \leq n$ *and* $\mathrm{pd}_{R_2}(B) \leq n$.
(3) $\mathrm{Gpd}_R(M) \leq n$ *if and only if* $\mathrm{Gpd}_{R_1}(A) \leq n$ *and* $\mathrm{Gpd}_{R_2}(B) \leq n$.

Proof. (1) See [37, Lemma 2.8] or [40, Lemma 3.2].
(2) This is well-known; we include an elementary proof for the sake of completeness.

By induction on n, it suffices to prove the assertion for $n = 0$. If $\mathrm{pd}_R(M) = 0$, then it is obvious that $\mathrm{pd}_{R_1}(A) = \mathrm{pd}_{R_2}(B) = 0$.

Let $\varepsilon_R : X \to Y \to 0$ be an arbitrary exact sequence in $\mathcal{M}_R$. Then, by Lemma 6.3, there exist an exact sequence $\varepsilon_{R_1} : X_1 \to Y_1 \to 0$ in $\mathcal{M}_{R_1}$ and an exact sequence $\varepsilon_{R_2} : X_2 \to Y_2 \to 0$ in $\mathcal{M}_{R_2}$ such that $\varepsilon_R = \varepsilon_{R_1} \oplus \varepsilon_{R_2}$. Note that

$$\mathrm{Hom}_R(M, \varepsilon_R) \cong \mathrm{Hom}_{R_1}(A, \varepsilon_{R_1}) \oplus \mathrm{Hom}_{R_2}(B, \varepsilon_{R_2})$$

again by Lemma 6.3. Hence, $\mathrm{pd}_{R_1}(A) = \mathrm{pd}_{R_2}(B) = 0$ implies that $\mathrm{pd}_R(M) = 0$.
(3) By induction on n, it suffices to prove the assertion for $n = 0$. If $\mathrm{Gpd}_R(M) = 0$, then $\mathrm{Gpd}_{R_1}(A) = \mathrm{Gpd}_{R_2}(B) = 0$ by [7, Lemma 3.2]. Now assume that there exist a complete projective resolution in $\mathcal{M}_{R_1}$

$$\mathbf{F}: \quad \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with $A = \mathrm{im}(F_0 \to F^0)$, and a complete projective resolution in $\mathcal{M}_{R_2}$

$$\mathbf{P}: \quad \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $B = \mathrm{im}(P_0 \to P^0)$. By Lemma 6.3, any projective right R-module Q is a direct sum of a projective right R_1-module Q_1 and a projective right R_2-module Q_2. Then

$$\mathrm{Hom}_R(\mathbf{F} \oplus \mathbf{P}, Q) \cong \mathrm{Hom}_{R_1}(\mathbf{F}, Q_1) \oplus \mathrm{Hom}_{R_2}(\mathbf{P}, Q_2)$$

again by Lemma 6.3. Hence $\mathbf{F} \oplus \mathbf{P}$ is a complete projective resolution in $\mathcal{M}_R$. Thus $\mathrm{Gpd}_R(M) = 0$. □

Remark 6.5. Lemma 6.4(3) has been established in [7, Lemma 3.3] under the additional assumption that the rings R_1 and R_2 are commutative and all projective modules have finite injective dimensions.

Corollary 6.6. *Let R_1 and R_2 be two rings and let $R = R_1 \oplus R_2$. Then the following statements hold for any non-negative integers n and d.*

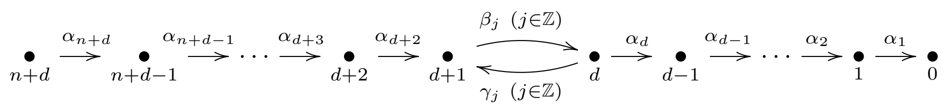
(1) *R is a right (n,d)-ring if and only if both R_1 and R_2 are right (n,d)-rings.*
(2) *R is a right G-(n,d)-ring if and only if both R_1 and R_2 are right G-(n,d)-rings.*

Proof. This is a direct consequence of Lemma 6.4. □

Remark 6.7. Corollary 6.6(2) has been proved in [37, Theorem 2.7] under the additional assumption that the rings R_1 and R_2 are commutative and have finite Gorenstein global dimensions.

Now we answer Question 1 for all $n \geq 2$.

Example 6.8. Let $n \geq 2$ and $d \geq 0$ be fixed integers. Let Q be the quiver with $n+d+1$ vertices, one arrow α_{i+1} from vertex $i+1$ to vertex i for each $i \in \{0, 1, \cdots, n+d-1\} \backslash \{d\}$, infinitely many arrows $\{\beta_j \mid j \in \mathbb{Z}\}$ from vertex $d+1$ to vertex d, and infinitely many arrows $\{\gamma_j \mid j \in \mathbb{Z}\}$ from vertex d to vertex $d+1$.



Set $R = S \oplus T$. Here S is the quotient of the path algebra of Q over an algebraically closed field F by the ideal generated by the set of all paths of length $\ell \geq 2$, and T is a quasi-Frobenius ring with $\mathrm{rD}(T) = \infty$. Then the following are true for R:

(1) R is a right G-(n,d)-ring;
(2) R is not a right G-$(n-1,t)$-ring for each non-negative integer t;
(3) R is not a right G-$(m,d-1)$-ring for each non-negative integer m;
(4) R is not a right (n,d)-ring.

Proof. It has been shown in [33, Theorem 2.1] that S is a right (n,d)-ring. So R is a right G-(n,d)-ring by Corollary 6.6(1). Note that finitely generated right T-modules are n-presented, and T is not a right $(0,d)$-ring. Hence T is not a right (n,d)-ring. Thus R is not a right (n,d)-ring by Corollary 6.6(2). This gives (1) and (4).

Now we consider the following exact sequences of right R-modules (see the proof of [33, Theorem 2.1])

$$0 \to \bigoplus_{j\in\mathbb{Z}} \overline{\beta_j}R \to P_{d+1} \to \cdots \to P_{n+d-1} \to P_{n+d} \to S_{n+d} \to 0, \qquad (\zeta_1)$$

$$0 \to \bigoplus_{j\in\mathbb{Z}} \overline{\beta_j}R \xrightarrow{\eta} P_{d+1} \to \overline{\gamma_k}R \to 0, \quad k \in \mathbb{Z}, \qquad (\zeta_2)$$

$$0 \to \bigoplus_{j\in\mathbb{Z}} \overline{\gamma_j}R \oplus \overline{\alpha_d}R \to P_d \to \overline{\beta_k}R \to 0, \qquad (\zeta_3)$$

$$0 \to P_0 \cong \overline{\alpha_1}R \to P_1 \to \cdots \to P_{d-1} \to \overline{\alpha_d}R \to 0, \qquad (\zeta_4)$$

where P_i is the indecomposable projective right S-module corresponding to the vertex $i \in \{0, 1, 2, \cdots, n+d\}$, and S_{n+d} is the simple right S-module corresponding to the vertex $n + d$.

Since projective S-modules are also projective R-modules, we see from (ζ_1) that $\lambda_R(S_{n+d}) = n - 1$; hence, to prove (2), it suffices to show that $\mathrm{Gpd}_R(S_{n+d}) = \infty$. First, we argue that $\mathrm{Gpd}_R(\overline{\gamma_k}R) \neq 0$ for any $k \in \mathbb{Z}$; otherwise, we see from (ζ_2) that, the composition of the natural projection $\pi : \bigoplus_{j\in\mathbb{Z}} \overline{\beta_j}R \twoheadrightarrow \overline{\beta_k}R$ and the injection $\iota : \overline{\beta_k}R \rightarrowtail P_{d+1}$ can be extended to P_{d+1}, i.e., there exists a non-zero endomorphism f of P_{d+1} such that $\iota\pi = f\eta$. By the construction of S, one can easily verify that $f(e_{d+1}) = ue_{d+1}$ (here e_{d+1} denotes the stationary path at the vertex $d + 1$) for some non-zero element $u \in F$, i.e., f is an isomorphism of P_{d+1}. This forces that π is monic, a contradiction. Thus $\mathrm{Gpd}_R(\overline{\gamma_k}R) \neq 0$, and we conclude from (ζ_1), (ζ_2), (ζ_3) and [30, Proposition 2.7] that $\mathrm{Gpd}_R(S_{n+d}) = \infty$. So (2) is true.

Finally we prove (3). From (ζ_4) and the short exact sequence $0 \to \overline{\alpha_d}R \to P_d \to L \longrightarrow 0$ we get that $\lambda_R(L) = \infty$ and $\mathrm{pd}_R(L) = d$. So $\mathrm{Gpd}_R(L) = d > d - 1$, and (3) follows. □

We see from Corollary 5.11 that, if R is a right (n, d)-ring, then R is a right $\max\{n,\, d\}$-coherent ring. This raises the following:

Problem 1. Is every right G-(n, d)-ring right $\max\{n,\, d\}$-coherent?

Costa [16, Sec. 7] asked whether $R[x]$ is a right $(n, d + 1)$-ring whenever R is a right (n, d)-ring. We end this article with the following:

Problem 2. Let R be a right G-(n, d)-ring. Is $R[x]$ a right G-$(n, d + 1)$-ring?

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Declarations

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