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Multiple Positive Symmetric Solutions for the Fourth-Order Iterative Differential Equations Involving p-Laplacian with Integral Boundary Conditions

Rajendra Prasad Kapula[1*], Kosuri Bhushanam[1] and Sreedhar Namburi[2]

[1]*Department of Applied Mathematics, College of Science and Technology, Andhra University, 530003, Andhra Pradesh, India*
[2]*Department of Mathematics, School of Science, GITAM (Deemed to be University), 530045, Andhra Pradesh, India*
[*]*Corresponding author*

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Abstract

The purpose of this paper is to investigate the existence of multiple positive symmetric solutions for fourth order p-Laplacian iterative system with integral boundary conditions. Initially, we establish the existence of at least one and two positive symmetric solutions for the fourth order problem using Krasnosel'skii fixed point theorem. Subsequently, we establish the existence of at least three positive symmetric solutions by applying five-functionals fixed point theorem.

1. Introduction

Boundary value problems (BVPs) associated with ordinary differential equations play a significant role in various fields, including physics, chemistry, engineering, biotechnology, and social sciences. The higher order differential equations with specific types of iterative differential equations are important for analyzing the characteristics like monotonicity, convexity, equivariance, smoothness, and numerical solutions (see [1–5]). It is also worth noting that differential equations with integral boundary conditions are crucial in modeling phenomena such as plasma physics, underground water flow, chemical engineering, heat conduction, and thermo-elasticity.

In the theory of differential equations, one of the most significant operators is one dimensional p-Laplacian operator and is defined as $\phi_{\mathrm{p}}(z)=|z|^{\mathrm{p}-2}z$, where $\mathrm{p}>1$, $\phi_{\mathrm{p}}^{-1}=\phi_{\mathrm{q}}$ and $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1$. Such problems can be found in the mathematical modeling of image processing, heat radiation, glaciology, biophysics, plasma physics, rheology, plastic molding, etc (see [6,7]). In particular, fourth-order BVPs with the p-Laplacian operator, have diverse applications in brain warping, fluids in lungs, ice formation, beam theory, and designing special curves on surfaces. The applications highlight the wide range of uses and significance of the p-Laplacian operator in several fields (see [8–14]). Various approaches, like fixed point theorems, iterative techniques, and shooting methods, are employed to establish the existence of solutions for such problems (see [15–17]). In 2000, Avery and Henderson [18] considered the problem

$$\begin{aligned}&y''(z)+\mathtt{f}(y)=0,\ \ 0\leqslant z\leqslant 1,\\&y(0)=0=y(1),\end{aligned}$$

and established the existence of at least three symmetric positive solutions by using the generalization of Leggett-Williams fixed point theorem. In 2015, Akcan and Hamal [19] established the existence of concave symmetric positive solutions for the BVP

$$\begin{aligned}&y''(z)+\mathtt{f}(z,y(z),y'(z))=0,\ 0<z<1,\\&y(0)=y(1)=\psi\int_0^{\eta}y(\mathrm{x})\mathrm{dx},\end{aligned}$$

Email addresses and ORCID numbers: **rajendra92@rediffmail.com, 0000-0001-8162-1391 (P. R. Kapula), kosuribhushanam@gmail.com, 0009-0003-5617-0092 (K. Bhushanam), sreedharnamburi@rediffmail.com, 0000-0002-3916-3689 (S. Namburi)**
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where $\psi,\eta\in(0,1)$ by applying monotone iterative technique. In 2016, [20] Ding established the existence of symmetric positive solutions for the p-Laplacian BVP

$$\begin{aligned}&(\phi_p(y'(z)))'+\mathtt{f}(z,y(z),y'(z))=0,\ \ 0\le z\le 1,\\ &y(0)=y(1)=\int_0^1 y(x)\mathtt{g}(x)dx,\end{aligned}$$

by using the fixed point theorem due to Avery and Peterson. In 2020, [21] Asaduzzamana and Ali established the existence of symmetric positive solutions for the BVP

$$\begin{aligned}&-y^{(4)}(z)=\mathtt{f}(y,v),\ z\in[0,1],\\ &-v^{(4)}(z)=\mathtt{f}(y,v),\ z\in[0,1],\\ &y(z)=y(1-z),\ y'''(0)-y'''(1)=y''(z_1)+y''(z_2),\\ &v(z)=v(1-z),\ y'''(0)-v'''(1)=v''(z_1)+v''(z_2),\ \ 0<z_1<z_2<1,\end{aligned}$$

by applying Krasnoselskii's fixed point theorem. Following that, the researchers have explored the study of symmetric positive solutions, see [22–30]. Inspired by the works mentioned above, we investigate the existence of multiple positive symmetric solutions for the fourth order p-Laplacian iterative system with integral boundary conditions

$$\left.\begin{aligned}&(\phi_p(v(z)y''_{\mathtt{n}}(z)))''=w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\le\mathtt{n}\le i,\ \ z_1\le z\le z_2,\\ &y_{i+1}(z)=y_1(z),\ \ z_1\le z\le z_2,\end{aligned}\right\}\tag{1.1}$$

satisfying boundary conditions

$$\left.\begin{aligned}&y_{\mathtt{n}}(z_1)=\int_{z_1}^{z_2}\mathtt{g}(s)y_{\mathtt{n}}(s)ds,\ \ y_{\mathtt{n}}(z_2)=\int_{z_1}^{z_2}\mathtt{g}(s)y_{\mathtt{n}}(s)ds,\ \ 1\le\mathtt{n}\le i,\\ &\phi_p(v(z_1)y''_{\mathtt{n}}(z_1))=\int_{z_1}^{z_2}\mathtt{h}(s)\phi_p(v(s)y''_{\mathtt{n}}(s))ds,\ \ \phi_p(v(z_2)y''_{\mathtt{n}}(z_2))=\int_{z_1}^{z_2}\mathtt{h}(s)\phi_p(v(s)y''_{\mathtt{n}}(s))ds,\ \ 1\le\mathtt{n}\le i,\end{aligned}\right\}\tag{1.2}$$

where $i\in\mathbb{N}$ with $2z_1<z_2,\ \phi_p(z)=|z|^{p-2}z,\ p>1,\ \phi_p^{-1}=\phi_q,\ \ \frac{1}{p}+\frac{1}{q}=1$. The following conditions are presumed to be valid in the entire paper:

(I1) $\mathtt{f}_{\mathtt{n}}:[z_1,z_2]\times[z_1,\infty)\to[z_1,\infty)$ is continuous, $\mathtt{f}_{\mathtt{n}}(z_2+z_1-z,y)=\mathtt{f}_{\mathtt{n}}(z,y),\ \ 1\leqslant\mathtt{n}\leqslant i$ for all $(z,y)\in[z_1,z_2]\times[z_1,\infty)$. (For existence of solution)
(I2) $v(z),w(z)\in L^1[z_1,z_2]$ are positive, symmetric on $[z_1,z_2]$ (i.e., $v(z_2+z_1-z)=v(z)$ for $z\in[z_1,z_2]$). (For positive symmetric solution)
(I3) $\mathtt{g}(z),\mathtt{h}(z)\in L^1[z_1,z_2]$ are non-negative, symmetric on $[z_1,z_2]$, and $\mu_1,\ \mu_2\in(z_1,z_2),\ \mu_1=\int_{z_1}^{z_2}\mathtt{g}(s)ds,\ \ \mu_2=\int_{z_1}^{z_2}\mathtt{h}(s)ds$. (For positive symmetric solution)

The organization of the remaining part of the paper is as follows. In Section 2, we construct Green's function and estimate the bounds for Green's function for the problem (1.1)-(1.2). In section 3, we establish the existence of at least one and two positive symmetric solutions by using Krasnoselskii's fixed point theorem. Using the five-functional fixed point theorem, we establish the existence of at least three positive symmetric solutions. In Section 4, we provide examples to check the validity of the results.

2. Green's Function and Its Bounds

Here, we determine the solution of (1.1)-(1.2) as a solution of the integral equation that includes Green's function. After that, we establish a few characteristics of the Green's function which are useful in establishing our main results.

Lemma 2.1. *Assume that* (I2) − (I3) *hold. Then for any* $u_1(z)\in C([z_1,z_2],\mathbb{R})$*, the BVP*

$$\phi_p(v(z)y''_1(z))=u_1(z),\ \ z_1\le z\le z_2,\tag{2.1}$$

$$y_1(z_1)=\int_{z_1}^{z_2}\mathtt{g}(s)y_1(s)ds,\ \ y_1(z_2)=\int_{z_1}^{z_2}\mathtt{g}(s)y_1(s)ds,\tag{2.2}$$

has one and only one solution

$$y_1(z)=-\int_{z_1}^{z_2}\mathtt{H}_1(z,t)v^{-1}(t)\phi_q(u_1(t))dt,$$

where $\mathtt{H}_1(z,t)$ *is the Green's function and is given by*

$$\mathtt{H}_1(z,t)=G(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}G(s,t)\mathtt{g}(s)ds,\tag{2.3}$$

in which

$$G(z,t)=\frac{1}{z_2-z_1}\begin{cases}(z-z_1)(z_2-t),\ \ z\leqslant t,\\ (t-z_1)(z_2-z),\ \ t\leqslant z.\end{cases}\tag{2.4}$$

Proof. Integrating (2.1) twice from z_1 to z, we get

$$y_1(z)=\int_{z_1}^{z}(z-t)v^{-1}(t)\phi_q(u_1(t))dt+c_1(z-z_1)+c_2.$$

By using boundary conditions (2.2), we get

$$c_1=\frac{-1}{z_2-z_1}\int_{z_1}^{z_2}(z_2-t)v^{-1}(t)\phi_q(u_1(t))dt,\ \text{and}\ c_2=\int_{z_1}^{z_2}g(s)y_1(s)ds.$$

So, we have

$$\begin{aligned}y_1(z)&=\int_{z_1}^{z}(z-t)v^{-1}(t)\phi_q(u_1(t))dt+\frac{-1}{z_2-z_1}\int_{z_1}^{z_2}(z-z_1)(z_2-t)v^{-1}(t)\phi_q(u_1(t))dt+\int_{z_1}^{z_2}g(s)y_1(s)ds\\&=-\int_{z_1}^{z_2}G(z,t)v^{-1}(t)\phi_q(u_1(t))dt+\int_{z_1}^{z_2}g(s)y_1(s)ds.\end{aligned}$$

After certain computations, we obtain

$$\int_{z_1}^{z_2}g(s)y_1(s)ds=\frac{-1}{1-\mu_1}\int_{z_1}^{z_2}\int_{z_1}^{z_2}G(s,t)v^{-1}(t)\phi_q(u_1(t))dtds.$$

Therefore,

$$\begin{aligned}y_1(z)&=-\int_{z_1}^{z_2}G(z,t)v^{-1}(t)\phi_q(u_1(t))dt+\frac{-1}{1-\mu_1}\int_{z_1}^{z_2}\int_{z_1}^{z_2}G(s,t)v^{-1}(t)\phi_q(u_1(t))dtds\\&=-\int_{z_1}^{z_2}\left[G(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}G(s,t)g(s)ds\right]v^{-1}(t)\phi_q(u_1(t))dt\\&=-\int_{z_1}^{z_2}H_1(z,t)v^{-1}(t)\phi_q(u_1(t))dt.\end{aligned}$$

□

Lemma 2.2. *Suppose* (I3) *holds. For* $\lambda\in(z_1,\frac{z_2}{2})$, *let* $\sigma(\lambda)=\frac{\lambda-z_1}{z_2-z_1}$, $\alpha_1=\frac{1}{1-\mu_1}$. *Then* $G(z,t)$, $H_1(z,t)$ *have the following properties:*

(A1) $0\leqslant G(z,t)\leqslant G(t,t),\ \forall\, z,t\in[z_1,z_2]$,
(A2) $0\leqslant H_1(z,t)\leqslant \alpha_1 G(t,t),\ \forall\, z,t\in[z_1,z_2]$,
(A3) $G(z,t)\geqslant\sigma(\lambda)G(t,t),\ \forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
(A4) $H_1(z,t)\geqslant\sigma(\lambda)\alpha_1G(t,t),\ \forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
(A5) $G(z_2+z_1-z,z_2+z_1-t)=G(z,t),\ H_1(z_2+z_1-z,z_2+z_1-t)=H_1(z,t),\ \forall\, z,t\in[z_1,z_2]$.

Proof. From (2.3) and (2.4), it is clear that the properties (A1) and (A2) hold.

For inequality (A3), let $z\in[\lambda,z_2-\lambda]$ and $z\leqslant t$, then

$$\frac{G(z,t)}{G(t,t)}=\frac{(z-z_1)(z_2-t)}{(t-z_1)(z_2-t)}\geqslant\sigma(\lambda),$$

and for $t\leqslant z$,

$$\frac{G(z,t)}{G(t,t)}=\frac{(t-z_1)(z_2-z)}{(t-z_1)(z_2-t)}\geqslant\sigma(\lambda).$$

Hence, the inequality (A3). For the inequality (A4), consider

$$\begin{aligned}H_1(z,t)&=G(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}G(s,t)g(s)ds\\&\geqslant\sigma(\lambda)G(t,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\sigma(\lambda)G(t,t)g(s)ds.\end{aligned}$$

Hence, $H_1(z,t)\geqslant\sigma(\lambda)\alpha_1G(t,t)$. For inequality (A5), consider

$$\begin{aligned}G(z_2+z_1-z,z_2+z_1-t)&=\frac{1}{z_2-z_1}\begin{cases}(z_2+z_1-z-z_1)(z_2-(z_2+z_1-t)), & z_2+z_1-z\leqslant z_2+z_1-t,\\(z_2+z_1-t-z_1)(z_2-(z_2+z_1-z)), & z_2+z_1-t\leqslant z_2+z_1-z,\end{cases}\\&=\frac{1}{z_2-z_1}\begin{cases}(z-z_1)(z_2-t), & z\leqslant t,\\(t-z_1)(z_2-z), & t\leqslant z,\end{cases}\\&=G(z,t).\end{aligned}$$

Consider

$$\begin{aligned}\mathtt{H}_1(z_2+z_1-z,z_2+z_1-t)=&\mathrm{G}(z_2+z_1-z,z_2+z_1-t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},z_2+z_1-t)\mathtt{g}(\mathtt{s})\mathtt{ds}\\ =&\mathrm{G}(z,t)+\frac{1}{1-\mu_1}\int_{z_2}^{z_1}\mathrm{G}(z_2+z_1-\mathtt{s},z_2+z_1-t)\mathtt{g}(z_2+z_1-\mathtt{s})\mathtt{d}(z_2+z_1-\mathtt{s})\\ =&\mathrm{G}(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds}\\ =&\mathtt{H}_1(z,t).\end{aligned}$$

□

Lemma 2.3. *Assume that* $(\mathtt{I2})-(\mathtt{I3})$ *hold. Then for any* $u_2(z)\in C([z_1,z_2],\mathbb{R})$*, the BVP*

$$(\phi_{\mathrm{p}}(v(z)y_1''(z)))''=u_2(z),\ \ z_1\le z\le z_2,$$

satisfying boundary conditions

$$y_1(z_1)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},\ \ y_1(z_2)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},$$
$$\phi_{\mathrm{p}}(v(z_1)y_1''(z_1))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(z_2)y_1''(z_2))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},$$

has a unique solution

$$y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t)v^{-1}(t)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t,\mathrm{s})u_2(\mathrm{s})\mathtt{ds}\right]\mathtt{d}t,$$

where $\mathtt{H}_1(z,t)$ *is given in (2.3) and*

$$\mathtt{H}_2(z,t)=\mathrm{G}(z,t)+\frac{1}{1-\ \mu_2}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds}.$$

Proof. Let, $u_1(z)=\phi_{\mathrm{p}}(v(z)y_1''(z))$ for $z_1\le z\le z_2$. Then the BVP

$$(\phi_{\mathrm{p}}(v(z)y_1''(z)))''=u_2(z),\ \ z_1\le z\le z_2,$$

$$\phi_{\mathrm{p}}(v(z_1)y_1''(z_1))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(z_2)y_1''(z_2))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds}$$

is equivalent to the problem

$$u_1''(z)=u_2(z),\ \ z_1\le z\le z_2, \tag{2.5}$$

$$u_1(z_1)=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})u_1(\mathtt{s})\mathtt{ds},\ \ u_1(z_2)=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})u_1(\mathtt{s})\mathtt{ds}. \tag{2.6}$$

By Lemma 2.1, the BVP (2.5)-(2.6) has unique solution $u_1(z)=-\int_{z_1}^{z_2}\mathtt{H}_2(z,t)u_2(t)\mathtt{d}t$. That is

$$\phi_{\mathrm{p}}(v(z)y_1''(z))=-\int_{z_1}^{z_2}\mathtt{H}_2(z,t)u_2(t)\mathtt{d}t \tag{2.7}$$

Again by Lemma 2.1, the differential equation (2.7) with boundary conditions

$$y_1(z_1)=y_1(z_2)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},$$

has a unique solution

$$y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t)v^{-1}(t)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t,\mathrm{s})u_2(\mathrm{s})\mathtt{ds}\right]\mathtt{d}t.$$

This completes the proof. □

Lemma 2.4. *Suppose* $(\mathtt{I3})$ *holds. For* $\lambda\in(z_1,\frac{z_2}{2})$*, let* $\sigma(\lambda)=\frac{\lambda-z_1}{z_2-z_1}$*,* $\alpha_2=\frac{1}{1-\mu_2}$*. Then,* $\mathtt{H}_2(z,t)$ *has the following properties:*

$(\mathtt{A6})$ $0\leqslant\mathtt{H}_2(z,t)\leqslant\alpha_2\mathrm{G}(t,t),\ \forall\, z,t\in[z_1,z_2]$,
$(\mathtt{A7})$ $\mathtt{H}_2(z,t)\geqslant\sigma(\lambda)\alpha_2\mathrm{G}(t,t),\ \forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
$(\mathtt{A8})$ $\mathtt{H}_2(z_2+z_1-z,z_2+z_1-t)=\mathtt{H}_2(z,t),\ \forall\, z,t\in[z_1,z_2]$.

Note that an i-tuple $\big(y_1(z),y_2(z),\cdots,y_i(z)\big)$ is a solution of (1.1)-(1.2) if and only if

$$y_{\mathtt{n}}(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_{\mathtt{n}}(t_2,y_{\mathtt{n}+1}(t_2))\mathtt{d}t_2\right]\mathtt{d}t_1,\ \mathtt{n}=1,2,\cdots,i,$$
$$y_{i+1}(z)=y_1(z),\ \ z\in[z_1,z_2],\ \ 1\le\mathtt{n}\le i,$$

i.e.,

$$y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\right.\right.\right.$$
$$\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.$$
$$\left.\left.\left.\left.\left.\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1.$$

3. Existence of Positive Symmetric Solutions

Let $\mathrm{B}=\{y:y\in\mathrm{C}([z_1,z_2],\mathbb{R})\}$ be a Banach space with norm $\|y\|=\max\limits_{z\in[z_1,z_2]}|y(z)|$. For $\lambda\in(z_1,\frac{z_2}{2})$, we define the cone $\mathrm{K}\subset\mathrm{B}$ as

$$\mathrm{K}=\Big\{y\in\mathrm{B}:y(z)\geqslant0, y(z)\text{ is concave, symmetric on }[z_1,z_2]\text{ and }\min_{z\in[\lambda,z_2-\lambda]}y(z)\geqslant\sigma(\lambda)\|y\|\Big\}.$$

Define operator $\mathrm{T}:\mathrm{K}\to\mathrm{B}$ by

$$\mathrm{T}y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1.$$

Let,

$$m\leqslant\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_j,t_j)v^{-1}(t_j)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{f}\mathrm{G}(t_{j+1},t_{j+1})w(t_{j+1})\mathtt{d}t_{j+1}\right]\mathtt{d}t_j,\ j=1,2,\cdots2i-1,$$
$$M\geqslant\sigma(\lambda)\alpha_1\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_j,t_j)v^{-1}(t_j)\phi_{\mathtt{q}}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{j+1},t_{j+1})w(t_{j+1})\mathtt{d}t_{j+1}\right]\mathtt{d}t_j,\ j=1,2,\cdots2i-1.$$

Lemma 3.1. *For each* $\lambda\in(z_1,\frac{z_2}{2})$, $\mathrm{T}(\mathrm{K})\subset\mathrm{K}$ *and* $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ *is completely continuous.*

Proof. Since $\mathtt{H}_1(z,t)\geqslant0,\ \mathtt{H}_2(z,t)\geqslant0,\ \forall\,z,t\in[z_1,z_2],\ (\mathrm{T}y_1)(z)\geqslant0$. Let $y_1\in\mathrm{K}$, then consider

$$(\mathrm{T}y_1)(z_2+z_1-z)=\int_{z_1}^{z_2}\mathtt{H}_1(z_2+z_1-z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\right.\right.\right.$$
$$w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.$$
$$\left.\left.\left.\left.\left.\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1$$
$$=\int_{z_2}^{z_1}\mathtt{H}_1(z_2+z_1-z,z_2+z_1-t_1)v^{-1}(z_2+z_1-t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(z_2+z_1-t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)\right.\right.$$
$$\left.\left.\cdots\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}(z_2+z_1-t_1)$$
$$=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_2}^{z_1}\mathtt{H}_2(z_2+z_1-t_1,z_2+z_1-t_2)w(z_2+z_1-t_2)\mathtt{f}_1\left(z_2+z_1-t_2,\right.\right.$$
$$\left.\int_{z_1}^{z_2}\mathtt{H}_1(z_2+z_1-t_2,t_3)\cdots\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)$$
$$\left.\mathtt{d}(z_2+z_1-t_2)\right]\mathtt{d}t_1$$
$$\vdots$$
$$=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.$$
$$\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)$$
$$\left.\left.\left.\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1$$
$$=(\mathrm{T}y_1)(z).$$

Hence $\mathrm{T}y_1$ is symmetric on $[z_1,z_2]$. From Lemma 2.2, we get

$$\begin{aligned}(\mathrm{T}y_1)(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\\leqslant\ &\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1.\end{aligned}$$

So,

$$\begin{aligned}\|\mathrm{T}y_1\|\leqslant&\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1.\end{aligned}$$

Again from Lemma 2.2, we get

$$\begin{aligned}\min_{z\in[\lambda,z_2-\lambda]}\{(\mathrm{T}y_1)(z)\}=&\min_{z\in[\lambda,z_2-\lambda]}\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\right.\right.\right.\\&w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.\\&\left.\left.\left.\left.\left.\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\\geqslant&\alpha_1\ \sigma(\lambda)\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\right.\right.\right.\\&w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.\\&\left.\left.\left.\left.\left.\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1.\end{aligned}$$

By using above two inequalities one can write

$$\min_{z\in[\lambda,z_2-\lambda]}\{(\mathrm{T}y_1)(z)\}\geqslant\sigma(\lambda)\|\mathrm{T}y_1\|.$$

So, $\mathrm{T}y_1\in\mathrm{K}$ and thus $\mathrm{T(K)}\subset\mathrm{K}$. By using Arzela-Ascoli theorem and standard methods it can be prove T is completely continuous. □

Theorem 3.2. *Let* (I1) − (I3) *hold. Also assume that the following hold,*

(I4) $\displaystyle\lim_{y\to0^+}\frac{\mathtt{f}_n(z,y)}{\phi_p(y)}=0,\ \lim_{y\to+\infty}\frac{\mathtt{f}_n(z,y)}{\phi_p(y)}=+\infty,\quad 1\leq n\leq i$ *for* $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least one positive symmetric solution.

Proof. Since $\displaystyle\lim_{y\to0^+}\frac{\mathtt{f}_n(z,y)}{\phi_p(y)}=0$, there exists $l_1>0$ such that

$$\mathtt{f}_n(z,y)\leqslant\eta\phi_p(y),\ 0\leqslant y\leqslant l_1,\ z\in[z_1,z_2],\ \text{where}\ \eta\leqslant\phi_p\left(\frac{1}{m}\right).$$

Let $\Theta_1=\{y\in\mathrm{B}:\|y\|<l_1\}$, if $y_1\in\mathrm{K}\cap\partial\Theta_1$, and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\phi_p(y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\leqslant\phi_q(\eta)\alpha_1l_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\leqslant l_1m\phi_q(\eta)\leqslant l_1.\end{aligned}$$

Similarly for $t_{2i-4}\in[z_1,z_2]$

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\right.\right.\\
&\left.\left.\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\leqslant\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},l_1\right)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i-3},t_{2i-2})w(t_{2i-2})\eta\phi_p(l_1)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\leqslant\phi_q(\eta)\alpha_1 l_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-3},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i-2},t_{2i-2})w(t_{2i-2})\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\leqslant l_1 m\phi_q(\eta)\leqslant l_1.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\
&\leqslant l_1=\|y_1\|.
\end{aligned}$$

So $\|\mathrm{T}y_1\|\leqslant\|y_1\|$ for all $y_1\in\mathrm{K}\cap\partial\Theta_1$.

Since $\lim_{y\to+\infty}\dfrac{\mathtt{f}_n(z,y)}{\phi_p(y)}=+\infty$, there exists $\overline{l_2}>0$ such that

$$\mathtt{f}_n(z,y)\geqslant\zeta\phi_p(y),\ y\geqslant\overline{l_2},\ z\in[z_1,z_2],\ \text{where }\zeta\geqslant\phi_p\left(\frac{1}{M}\right).$$

Let $l_2=\max\{2l_1,\frac{\overline{l_2}}{\sigma(\lambda)}\}$ and $\Theta_2=\{y\in\mathrm{B}:\|y\|<l_2\}$. For $y_1\in\mathrm{K}\cap\partial\Theta_2$, we have

$$\min_{z\in[\lambda,z_2-\lambda]}y_1(z)\geqslant\sigma(\lambda)\|y_1\|\geqslant\sigma(\lambda)l_2\geqslant\overline{l_2}.$$

For $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\zeta\phi_p(y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\phi_q(\zeta)\sigma(\lambda)\alpha_1 l_2\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant l_2 M\phi_q(\zeta)\geqslant l_2.
\end{aligned}$$

Similarly for $t_{2i-4}\in[z_1,z_2]$

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\right.\right.\\
&\left.\left.\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\geqslant\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},l_2\right)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-3},t_{2i-2})w(t_{2i-2})\zeta\phi_p(l_1)\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\geqslant\phi_q(\zeta)\sigma(\lambda)\alpha_1 l_2\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-3},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-2},t_{2i-2})w(t_{2i-2})\mathtt{d}t_{2i-1}\right]\mathtt{d}t_{2i-3}\\
&\geqslant l_2 M\phi_q(\zeta)\geqslant l_2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\
&\geqslant l_2=\|y_1\|.
\end{aligned}$$

So $\|\mathrm{T}y_1\|\geqslant\|y_1\|$ for all $y_1\in\mathrm{K}\cap\partial\Theta_2$. Consequently, Krasnoselskii's fixed point theorem [31, 32] guarantees that T has a fixed point $\mathrm{K}\cap(\overline{\Theta}_2\setminus\Theta_1)$. □

Theorem 3.3. *Let* $(\mathtt{I1})-(\mathtt{I3})$ *hold. Also assume that the following conditions hold,*

$(\mathtt{I5})$ $\lim_{y\to 0^+}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=+\infty,\ \lim_{y\to+\infty}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=0,\ 1\le \mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least one positive symmetric solution.

Proof. We can establish the result by using the previous argument is in Theorem 3.2. □

Theorem 3.4. *Let* $(\mathtt{I1})-(\mathtt{I3})$ *hold. Also assume that the following conditions hold,*

$(\mathtt{I6})$ $\lim_{y\to 0^+}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=+\infty,\ \lim_{y\to+\infty}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=+\infty,\ 1\le \mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

$(\mathtt{I7})$ *There exists a constant* r_1 *such that* $\mathtt{f_n}(z,y)\leqslant\phi_p(\frac{r_1}{m})$ *for* $y\in[0,r_1]$, $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least two positive symmetric solutions y_1^* *and* y_1^{**} *such that* $0<\|y_1^*\|<r_1<\|y_1^{**}\|$.

Proof. Since $\lim_{y\to 0^+}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=+\infty$, there exists $r_*\in(0,r_1)$ such that $\mathtt{f_n}(z,y)\geqslant\zeta_1\phi_p(y)$, for $0\leqslant y\leqslant r_*$, $z\in[z_1,z_2]$, where $\zeta_1\geqslant\zeta$; here ζ is given in the proof of Theorem 3.2. Set $\Theta_3=\{y\in\mathrm{B}:\|y\|<r_*\}$. For $y_1\in\mathrm{K}\cap\partial\Theta_3$, and $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\zeta_1\phi_p(y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\geqslant\phi_q(\zeta_1)\sigma(\lambda)\alpha_1 r_*\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\&\geqslant r_*M\phi_q(\zeta_1)\geqslant r_*.\end{aligned}$$

Similarly for $t_{2i-4}\in[z_1,z_2]$

$$\begin{aligned}&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})\right.\right.\right.\\&\left.\left.\left.w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\&\geqslant\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},r_*\right)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-3},t_{2i-2})w(t_{2i-2})\zeta_1\phi_p(r_*)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\&\geqslant\phi_q(\zeta_1)\sigma(\lambda)\alpha_1 r_*\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-3},t_{2i-3})v^{-1}(t_{2i-3})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-2},t_{2i-2})w(t_{2i-2})\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\&\geqslant r_*M\phi_q(\zeta_1)\geqslant r_*.\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\&\geqslant r_*=\|y_1\|.\end{aligned}$$

So,

$$\|\mathrm{T}y_1\|\geqslant\|y_1\|\text{ for all }y_1\in\mathrm{K}\cap\partial\Theta_3.\tag{3.1}$$

Since $\lim_{y\to+\infty}\frac{\mathtt{f_n}(z,y)}{\phi_p(y)}=+\infty$, there exists $r^*>r_1$ such that $\mathtt{f_n}(z,y)\geqslant\zeta_2\phi_p(y)$, for $y\geqslant r^*$, $z\in[z_1,z_2]$, where $\zeta_2\geqslant\zeta$; here ζ is given in the proof of Theorem 3.2. Choose $\overline{r^*}>\max\{\frac{r^*}{\sigma(\lambda)},r_1\}$ and set $\Theta_4=\{y\in\mathrm{B}:\|y\|<\overline{r^*}\}$. For any $y_1\in\mathrm{K}\cap\partial\Theta_4$, we get

$$y_1\geqslant\min_{z\in[\lambda,z_2-\lambda]}y_1(z)\geqslant\sigma(\lambda)\|y_1\|\geqslant\sigma(\lambda)\overline{r^*}\geqslant r^*.$$

For $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant \int_{\lambda}^{z_2-\lambda} \sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2-\lambda} \sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\zeta_2\phi_{\mathrm{p}}(y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant \phi_{\mathrm{q}}(\zeta_2)\sigma(\lambda)\alpha_1\overline{r^*}\int_{\lambda}^{z_2-\lambda} \mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2-\lambda} \sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant \overline{r^*}M\phi_{\mathrm{q}}(\zeta_1)\\
&\geqslant \overline{r^*}.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z) = &\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\\
&\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\right.\\
&\left.\left.\left.\left.\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\
\geqslant &\overline{r^*} = \|y_1\|.
\end{aligned}$$

So,

$$\|\mathrm{T}y_1\| \geqslant \|y_1\| \ \text{ for all } y_1 \in \mathrm{K}\cap\partial\Theta_4. \tag{3.2}$$

Let $\Theta_5 = \{y\in\mathrm{B} : \|y\| < r_1\}$, if $y_1 \in \mathrm{K}\cap\partial\Theta_5$, and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\leqslant \int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\phi_{\mathrm{p}}(\frac{r_1}{m})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\leqslant \frac{r_1}{m}\alpha_1\int_{z_1}^{z_2} \mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\leqslant r_1.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z) = &\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\
\leqslant &r_1 = \|y_1\|.
\end{aligned}$$

So,

$$\|\mathrm{T}y_1\| \leqslant \|y_1\| \text{ for all } y_1 \in \mathrm{K}\cap\partial\Theta_5. \tag{3.3}$$

Since $r_* \leqslant r_1 < r^*$ and from (3.1), (3.2), and (3.3) it follows from Krasnoselskii's fixed point theorem [31, 32] T has a fixed point y_1^* in $\mathrm{K}\cap(\overline{\Theta}_5\setminus\Theta_3)$ and a fixed point y_1^{**} in $\mathrm{K}\cap(\overline{\Theta}_4\setminus\Theta_5)$ such that $0 < \|y_1^*\| < r_1 < \|y_1^{**}\|$. □

Theorem 3.5. *Let* (I1) – (I3) *hold. Also assume that the following conditions hold,*

(I8) $\lim_{y\to 0^+}\frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = 0,\ \lim_{y\to+\infty}\frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = 0,\ 1\le \mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

(I9) *There exists a constant* r_2 *such that* $\mathtt{f}_{\mathtt{n}}(z,y) \geqslant \phi_{\mathrm{p}}(\frac{r_2}{M})$ *for* $y\in[\sigma(\lambda)r_2, r_2]$, $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least two positive symmetric solutions y_1^* *and* y_1^{**} *such that* $0 < \|y_1^*\| < r_2 < \|y_1^{**}\|$.

Proof. We can establish the result by using the previous argument is in Theorem 3.4. □

Next, we establish sufficient conditions for the existence of at least three positive symmetric solutions for the BVP (1.1)-(1.2) by using the five functionals fixed point theorem. For that we define the nonnegative continuous concave functionals ψ_1,ψ_2 and the nonnegative continuous convex functionals $\gamma_1,\gamma_2,\gamma_3$ on K by

$$\psi_1(y) = \min_{z\in I}|y|,\quad \psi_2(y) = \min_{z\in I_1}|y|,\quad \gamma_1(y) = \max_{z\in[z_1,z_2]}|y|,\quad \gamma_2(y) = \max_{z\in I_1}|y|,\quad \gamma_3(y) = \max_{z\in I},|y|,$$

$$\psi_1(y) = \min_{z\in I}|y| \leqslant \max_{z\in I_1}|y| = \gamma_2(y), \tag{3.4}$$

$$\|y\| = \frac{1}{\sigma(\lambda)}\min_{z\in I}|y| \leqslant \frac{1}{\sigma(\lambda)}\max_{z\in[z_1,z_2]}|y| = \frac{1}{\sigma(\lambda)}\gamma_1(y), \tag{3.5}$$

where $I=[\lambda, z_2-\lambda]$, $I_1=[\lambda_1,\lambda_2]$, $\lambda<\lambda_1<\lambda_2<z_2-\lambda$. Then for nonnegative numbers $\mathtt{d}_1,\mathtt{d}_2,\mathtt{d}_3,\mathtt{d}_4,$ and $\mathtt{d}_5$, convex sets are defined as follows

$$\begin{aligned}
&\mathrm{K}(\gamma_1,\mathtt{d}_3)=\{y\in \mathrm{K}:\gamma_1(y)<\mathtt{d}_3\},\\
&\mathrm{K}(\gamma_1,\psi_1,\mathtt{d}_1,\mathtt{d}_3)=\{y\in \mathrm{K}:\mathtt{d}_1\leqslant\psi_1(y);\gamma_1(y)\leqslant\mathtt{d}_3\},\\
&\overline{\mathrm{K}}(\gamma_1,\gamma_2,\mathtt{d}_4,\mathtt{d}_3)=\{y\in \mathrm{K}:\gamma_2(y)\leqslant\mathtt{d}_4;\gamma_1(y)\leqslant\mathtt{d}_3\},\\
&\mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_1,\mathtt{d}_2,\mathtt{d}_3)=\{y\in \mathrm{K}:\mathtt{d}_1\leqslant\psi_1(y);\gamma_3(y)\leqslant\mathtt{d}_2;\gamma_1(y)\leqslant\mathtt{d}_3\},\ \text{and}\\
&\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\mathtt{d}_5,\mathtt{d}_4,\mathtt{d}_3)=\{y\in \mathrm{K}:\mathtt{d}_5\leqslant\psi_2(y);\gamma_2(y)\leqslant\mathtt{d}_4;\gamma_1(y)\leqslant\mathtt{d}_3\}.
\end{aligned}$$

Theorem 3.6. *Suppose that* $0<\mathtt{d}_1<\mathtt{d}_2<\frac{\mathtt{d}_2}{\sigma(\lambda)}<\mathtt{d}_3$ *such that* $\mathtt{f}_\mathtt{n}$ *satisfies the following conditions:*

(I10) $\mathtt{f}_\mathtt{n}(z,y)\leqslant\phi_\mathtt{p}(\frac{\mathtt{d}_1}{m})$ *for* $y\in[\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1]$, $z\in[z_1,z_2]$,
(I11) $\mathtt{f}_\mathtt{n}(z,y)\geqslant\phi_\mathtt{p}(\frac{\mathtt{d}_2}{M})$ *for* $y\in[\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)}]$, $z\in I$,
(I12) $\mathtt{f}_\mathtt{n}(z,y)\leqslant\phi_\mathtt{p}(\frac{\mathtt{d}_3}{m})$ *for* $y\in[0,\mathtt{d}_3]$, $z\in[z_1,z_2]$,

Then the BVP (1.1)-(1.2) has at least three positive symmetric solutions y_1^*, y_1^{**}, *and* y_1^{***} *such that* $\gamma_2(y_1^*)<\mathtt{d}_1$, $\mathtt{d}_2<\psi_1(y_1^{**})$ *and* $\mathtt{d}_1<\gamma_2(y_1^{***})$ *with* $\psi_1(y_1^{***})<\mathtt{d}_2$.

Proof. From Lemma 3.1 the operator T is completely continuous. From (3.4) and (3.5), for each $y\in \mathrm{K}$, $\psi_1(y)\leqslant\gamma_2(y)$ and $\|y\|\leqslant\frac{1}{\sigma(\lambda)}\gamma_1(y)$. Now to show that $\mathrm{T}:\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}\to\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$. Let $y\in\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$, then $0\leqslant|y|\leqslant\mathtt{d}_3$.
By (I12), and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\phi_\mathtt{p}(\frac{\mathtt{d}_3}{m})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\frac{\mathtt{d}_3}{m}\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\leqslant\mathtt{d}_3.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\gamma_1(\mathrm{T}y_1(z))=\max_{z\in[z_1,z_2]}&\left[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\
&\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\
&\left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\
&\leqslant\mathtt{d}_3.
\end{aligned}$$

Therefore $\mathrm{T}:\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}\to\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$. It obvious that

$$\frac{\mathtt{d}_2(\sigma(\lambda)+1)}{\sigma(\lambda)}\in\{y\in \mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)},\mathtt{d}_3):\psi_1(y)>\mathtt{d}_2\}\neq\emptyset\quad\text{and}$$

$$\mathtt{d}_1(\sigma(\lambda)+1)\in\{y\in\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1,\mathtt{d}_3):\gamma_2(y)<\mathtt{d}_1\}\neq\emptyset.$$

Next, let $y\in \mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)},\mathtt{d}_3)$ or $y\in\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1,\mathtt{d}_3)$. Then, $\mathtt{d}_2\leqslant|y|\leqslant\frac{\mathtt{d}_2}{\sigma(\lambda)}$ and $\mathtt{d}_1\sigma(\lambda)\leqslant|y|\leqslant\mathtt{d}_1$.
By (I11) and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\quad\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\phi_\mathtt{p}(\frac{\mathtt{d}_2}{M})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\quad\geqslant\frac{\mathtt{d}_2}{M}\sigma(\lambda)\alpha_1\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\quad\geqslant\mathtt{d}_2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\psi_1(\mathrm{T}y_1(z)) = \min_{z\in I} & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ & \geqslant \mathsf{d}_2.\end{aligned}$$

By (I10), and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\,\phi_{\mathrm{p}}\left(\frac{\mathsf{d}_1}{m}\right)\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \frac{\mathsf{d}_1}{m}\alpha_1\int_{z_1}^{z_2} \mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \mathsf{d}_1.\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\gamma_2(\mathrm{T}y_1(z)) = \max_{z\in I_1} & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ & \leqslant \mathsf{d}_1.\end{aligned}$$

Next, let $y\in \mathrm{K}(\gamma_1,\psi_1,\mathsf{d}_2,\mathsf{d}_3)$ with $\gamma_3(\mathrm{T}y_1(z)) > \frac{\mathsf{d}_2}{\sigma(\lambda)}$. Then

$$\begin{aligned}\psi_1(\mathrm{T}y_1(z)) = \min_{z\in I} & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1}v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant \sigma(\lambda) & \left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant \sigma(\lambda)\max_{z\in[z_1,z_2]} & \left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2}\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant \sigma(\lambda)\max_{z\in I} & \left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ = \sigma(\lambda) & \gamma_3(\mathrm{T}y_1(z)) > \mathsf{d}_2.\end{aligned}$$

Let $y\in \overline{\mathrm{K}}(\gamma_1,\gamma_2,\mathsf{d}_1,\mathsf{d}_3)$ with $\psi_2(\mathrm{T}y) < \sigma(\lambda)\mathsf{d}_1$. Then we have

$$\begin{aligned}
\gamma_2(\mathrm{T}y_1(z)) &= \max_{z\in I_1}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\\
&\quad \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots\\
&\quad \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg]\\
&\leqslant \max_{z\in[z_1,z_2]}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\\
&\quad \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots\\
&\quad \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg]\\
&\leqslant \frac{1}{\sigma(\lambda)}\min_{z\in I}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\\
&\quad \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots\\
&\quad \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg]\\
&\leqslant \frac{1}{\sigma(\lambda)}\min_{z\in I_1}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\\
&\quad \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\ \mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots\\
&\quad \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg]\\
&= \frac{1}{\sigma(\lambda)}\psi_2(\mathrm{T}y_1(z)) < \mathtt{d}_1.
\end{aligned}$$

So, proved all the conditions of the five functionals fixed point theorem [33]. Therefore, the BVP (1.1)-(1.2) has at least three positive symmetric solutions y_1^*, y_1^{**}, and y_1^{***} such that $\gamma_2(y_1^*) < \mathtt{d}_1$, $\mathtt{d}_2 < \psi_1(y_1^{**})$ and $\mathtt{d}_1 < \gamma_2(y_1^{***})$ with $\psi_1(y_1^{***}) < \mathtt{d}_2$. □

4. Examples

In this section, as an application, the results are demonstrated with examples.

Example 4.1.

Consider the following problem

$$\left.\begin{aligned}
&(\phi_{\mathrm{p}}(v(z)y_{\mathtt{n}}''(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\le \mathtt{n}\le 2,\ \ 0\le z\le 1,\\
&y_3(z) = y_1(z),
\end{aligned}\right\} \tag{4.1}$$

satisfying boundary conditions

$$\left.\begin{aligned}
&y_{\mathtt{n}}(0) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(1) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\\
&\phi_{\mathrm{p}}(v(0)y_{\mathtt{n}}''(0)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(1)y_{\mathtt{n}}''(1)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},
\end{aligned}\right\} \tag{4.2}$$

where $v(z) = 2+z-z^2$, $w(z) = 10$, $\mathtt{g}(z) = \frac{1}{4}$, $\mathtt{h}(z) = \frac{5}{9}$,

$$\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} z^2(1-z)^2y^3, & (z,y)\in[0,1]\times(0,6];\\ 6z^2(1-z)^2y^2, & (z,y)\in[0,1]\times[6,\infty).\end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{5}{9}$, $\alpha_1 = \frac{4}{3}$, $\mathtt{f} = \frac{9}{4}$,

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},$$

$$\mathtt{H}_2(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\ \mu_2}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},$$

in which

$$\mathrm{G}(z,t) = \begin{cases} z(1-t), & z\leqslant t,\\ t(1-z), & t\leqslant z.\end{cases}$$

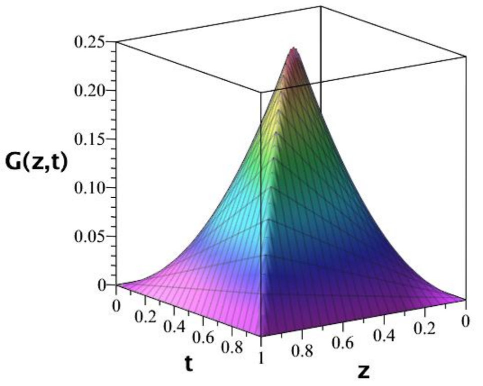

Figure 4.1: Pictorial representation of $\mathrm{G}(z,t)$

Let $\lambda = \frac{103}{356}$ then $\sigma(\lambda) = \frac{103}{356}$ and $M = 0.3790187963$, $M = 0.01103127360$

$$\lim_{y\to 0^+} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \lim_{y\to 0^+} \frac{z^2(1-z)^2y^3}{y} = \lim_{y\to 0^+} \left(\frac{1}{2}\right)^4 y^2 = 0,$$

$$\lim_{y\to \infty} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \lim_{y\to \infty} \frac{6z^2(1-z)^2y^2}{y} = \lim_{y\to \infty} 6\times(0.0423)y = \infty,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \eta\phi_{\mathrm{p}}(y) = 2y, \forall z\in[0,1],\ 0\leqslant y\leqslant 5,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \geqslant \zeta\phi_{\mathrm{p}}(y) = 101y, \forall z\in[0,1],\ y\geqslant 53.$$

Hence by Theorem 3.2, the BVP (4.1)-(4.2) has at least one positive symmetric solution.

Example 4.2.

Consider the following problem

$$\left.\begin{array}{l}(\phi_{\mathrm{p}}(v(z)y_{\mathtt{n}}''(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\leq \mathtt{n}\leq 2,\ \ 1\leq z\leq 3,\\ y_3(z) = y_1(z),\end{array}\right\} \tag{4.3}$$

satisfying boundary conditions

$$\left.\begin{array}{l}y_{\mathtt{n}}(1) = \int_1^3 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(3) = \int_1^3 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\\ \phi_{\mathrm{p}}(v(1)y_{\mathtt{n}}''(1)) = \int_1^3 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(3)y_{\mathtt{n}}''(3)) = \int_1^3 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\end{array}\right\} \tag{4.4}$$

where $v(z)=2,\ w(z)=z^2(4-z)^2,\ \mathtt{g}(z)=\frac{2}{7},\ \mathtt{h}(z)=\frac{3}{5},$

$$\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} \frac{1}{5}z(4-z)y, & (z,y)\in[1,3]\times(0,20];\\ 4z(4-z)+(y-20)e^y, & (z,y)\in[1,3]\times[20,\infty).\end{cases}$$

After algebraic computations, we get $\mu_1=\frac{2}{7},\ \mu_2=\frac{3}{5},\ \alpha_1=\frac{7}{5},\ \mathtt{f}=\frac{5}{2},$

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_1^3 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},$$

$$\mathtt{H}_2(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_2}\int_1^3 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},$$

in which

$$\mathrm{G}(z,t) = \frac{1}{2}\begin{cases}(z-1)(3-t), & z\leqslant t,\\ (t-1)(3-z), & t\leqslant z.\end{cases}$$

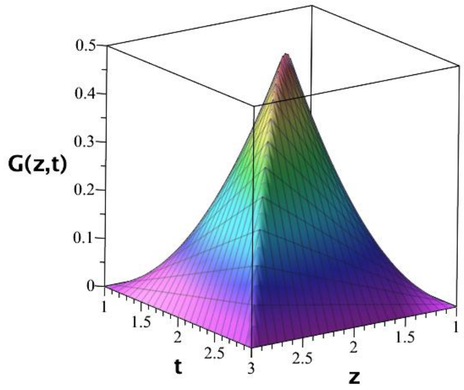

Figure 4.2: Pictorial representation of $G(z,t)$

Let $\lambda = 1.5$ then $\sigma(\lambda) = 0.25$ and $m = 11.276666$,

$$\lim_{y\to 0^+} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \frac{\frac{1}{5}z(4-z)y}{y^2} = +\infty, \ \lim_{y\to +\infty} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \frac{4z(4-z)+(y-20)e^y}{y^2} = +\infty \text{ for } z\in[1,3].$$

Choose a constant $r_1 = 20$ such that

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{r_1}{M}\right) = 76.07776843 \text{ for } y\in[0,20],\ z\in[1,3].$$

Hence by Theorem 3.4, the BVP (4.3)-(4.4) has at least two positive symmetric solutions y_1^* and y_1^{**}. such that

$$0 < \|y_1^*\| < 20 < \|y_1^{**}\|.$$

Example 4.3.

Consider the following problem

$$\left.\begin{array}{l}(\phi_{\mathrm{p}}(v(z)y_{\mathtt{n}}''(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\le \mathtt{n}\le 2,\ \ 0\le z\le 1,\\ y_3(z) = y_1(z),\end{array}\right\} \tag{4.5}$$

satisfying boundary conditions

$$\left.\begin{array}{l} y_{\mathtt{n}}(0) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(1) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\\ \phi_{\mathrm{p}}(v(0)y_{\mathtt{n}}''(0)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(1)y_{\mathtt{n}}''(1)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\end{array}\right\} \tag{4.6}$$

where $v(z) = \frac{2}{5}$, $w(z) = \frac{4}{11}$, $\mathtt{g}(z) = \frac{1+z-z^2}{2}$, $\mathtt{h}(z) = \frac{10}{17}$,

$$\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} e^{z(1-z)} + \dfrac{\sin(y)}{4} + \dfrac{6y^4}{7}, & (z,y)\in[0,1]\times(0,5];\\ e^{z(1-z)} + \dfrac{\sin(y)}{4} + \dfrac{3750}{7}, & (z,y)\in[0,1]\times[5,\infty).\end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{7}{12}$, $\mu_2 = \frac{10}{17}$, $\alpha_1 = \frac{12}{5}$, $\mathtt{f} = \frac{17}{7}$,

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},$$

$$\mathtt{H}_2(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\ \mu_2}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},$$

in which

$$\mathrm{G}(z,t) = \begin{cases} z(1-t), & z\leqslant t,\\ t(1-z), & t\leqslant z.\end{cases}$$

Let $\lambda = \frac{1}{3}$ then $\sigma(\lambda) = \frac{1}{3}$ and $m = .1471861472$, $M = 0.003791260308$. Choose $\mathtt{d}_1 = 1.5$, $\mathtt{d}_2 = 5$, $\mathtt{d}_3 = 100$ then

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_1}{m}\right) = 10.19117647 \ \text{ for } \ y\in[0.5,1.5],\ z\in[0,1],$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \geqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_2}{M}\right) = 211.6628959 \ \text{ for } \ y\in[5,15],\ z\in I,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_3}{m}\right) = 679.4521 \ \text{ for } \ y\in[0,100],\ z\in[0,1],$$

Hence by Theorem 3.6, the BVP has (4.5)-(4.6) has at least three positive symmetric solutions y_1^*, y_1^{**}, and y_1^{***}. such that $\gamma_2(y_1^*) < 1.5$, $5 < \psi_1(y_1^{**})$ and $1.5 < \gamma_2(y_1^{***})$ with $\psi_1(y_1^{***}) < 5$.

5. Conclusion

The current research work is devoted to establish the presence and characteristics of positive symmetric solutions for iterative system of p-Laplacian problem with integral boundary conditions based on the Krasnoselskii's and five functionals fixed point theorems. We anticipate that our findings will inspire and serve as a reference for future developments in this field.

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