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# NONLINEAR APPROXIMATION BY $N$-DIMENSIONAL SAMPLING TYPE DISCRETE OPERATORS WITH APPLICATIONS

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Abstract. In this paper, we explore $N$-dimensional nonlinear discrete operators, closely related to generalized sampling series. We investigate their approximation properties by using the supremum norm and employ a summability method to generalize the discrete operators. The order of convergence is studied by using suitable Lipschitz classes of uniformly continuous functions. We exemplify kernel functions that meet the necessary conditions. Additionally, in the final section of the paper, we propose an operator-based method for digital image zooming.

## 1. Introduction

In 1980s, the German mathematician Butzer introduces the theory of generalized sampling operators in [22] aiming to reconstruct signals that are not necessarily bandlimited (see [22, 23, 34]). As is well-known, these operators have numerous applications, particularly in signal theory [4, 5, 12–14, 18, 19, 23, 32, 34]. On the other hand, the discrete operators considered in the present paper are closely associated with generalized sampling series and have significant applications, including economic forecasting, geophysics, speech processing, and others [20–22].

In [4], Angeloni and Vinti investigate the convergence problem of generalized sampling series under a $\varphi$-variational functional using one-dimensional linear discrete operators. Inspired by [4], we construct a nonlinear setting of $N$-dimensional discrete operators and improve upon it by using Bell-type summability methods [16, 17] (which is also studied by Stieglitz in [36]) under the usual supremum

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norm (see also [6, 7, 38]). Our new operator is defined by

$$\mathcal{T}_{n,v}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right) l_{\mathbf{k},w} \quad \left(\mathbf{x}\in\mathbb{R}^N,\ n,v\in\mathbb{N}\right), \tag{1}$$

where $\mathcal{A} = \{A^v\}_{v\in\mathbb{N}} = \{[a_{nw}^v]\}_{v\in\mathbb{N}}$ is family of nonnegative regular matrices, $f : \mathbb{R}^N \to \mathbb{R}$ is bounded, $H_w : \mathbb{R} \to \mathbb{R}$, $H_w(0) = 0$ and $H_w$ is Lipschitz continuous, that is,

$$|H_w(u) - H_w(v)| \le C\,|u-v|$$

for some $C > 0$ and for every $w \in \mathbb{N}$, $u, v \in \mathbb{R}$. Here, $l_{\mathbf{k},w} := l_{(k_1,\dots,k_N),w} \in l^1\left(\mathbb{Z}^N\right)$ is a family of $N$-dimensional discrete kernels for each $w \in \mathbb{N}$.

We will prove that

$$\|\mathcal{T}_{n,v}(f) - f\| \to 0 \text{ (uniformly in } v\text{)}, \text{ as } n\to\infty$$

for all $f \in BUC\left(\mathbb{R}^N\right)$ (the space of bounded and uniformly continuous functions on $\mathbb{R}^N$), where $\|\cdot\|$ denotes the usual supremum norm on $\mathbb{R}^N$. Then, we examine the order of convergence by means of suitable Lipschitz class of continuous fuctions. Utilizing from the relation between operators (1) and nonlinear generalized sampling operators, introduced by

$$\mathcal{S}_{n,v}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\tfrac{\mathbf{k}}{w}\right)\right) \chi(w\mathbf{x}-\mathbf{k}) \quad \left(\mathbf{x}\in\mathbb{R}^N,\ n,v\in\mathbb{N}\right),$$

it is possible to show that, in some specific cases, $\mathcal{T}_{n,v}(f)$ coincides with $\mathcal{S}_{n,v}(f)$, and hence,

$$\|\mathcal{S}_{n,v}(f) - f\| \to 0 \text{ (uniformly in } v\text{), as } n\to\infty$$

holds. Some related recent articles on multidimensional sampling operators can be found in [1, 24].

For examples of $l_{\mathbf{k},w}$ that fulfill all the kernel assumptions, the reader can review the last section. Lastly, we prove that these types of discrete operators can be useful in digital zoom.

## 2. Preliminaries

In this section, some basic definitions, notations and kernel assumptions will be given.

Bell-type summability method is defined as follows:

Consider the following family of infinite matrices $\mathcal{A} = \{A^v\}_{v\in\mathbb{N}} = \{[a_{nw}^v]\}_{v\in\mathbb{N}}$ $(n, w \in \mathbb{N})$ with real or complex entries. For a given sequence $x = (x_w)$ and the double sequence $(\mathcal{A}x)_n^v$, $\mathcal{A}$-transform of $x$ is defined by

$$(\mathcal{A}x)_n^v := \left\{\sum_{w=1}^{\infty} a_{nw}^v x_w\right\} \quad (n, v \in \mathbb{N})$$

whenever the series is convergent for all $n, \upsilon \in \mathbb{N}$. Moreover, it is called that "$x$ is $\mathcal{A}$-summable to $L$" provided that

$$\lim_{n\to\infty} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w = L \text{ uniformly in } \upsilon,$$

and this convergence is denoted by

$$\mathcal{A} - \lim x = L \quad \text{(see [16])}.$$

Furthermore, $\mathcal{A}$ is called regular, if $\mathcal{A} - \lim x = L$ whenever $\lim_{k\to\infty} x_k = L$ ([16, 17]). A characterization of regularity is given by Bell in [17], such that $\mathcal{A}$ is regular if and only if

a) for every fixed $w \in \mathbb{N}$, $\lim_{n\to\infty} a_{nw}^{\upsilon} = 0$ (uniformly in $\upsilon$),
b) $\mathcal{A} - \lim e = 1$, where $e = (1, 1, \ldots)$
c) for every $n, \upsilon \in \mathbb{N}$, $\sum_{w=1}^{\infty} |a_{nw}^{\upsilon}| < \infty$ and there exist $N, M \in \mathbb{N}$ such that $\sup_{n\geq N, \upsilon\in\mathbb{N}} \sum_{w=1}^{\infty} |a_{nw}^{\upsilon}| \leq M$.

In the whole paper, it is supposed that $\mathcal{A}$ is regular and $a_{nw}^{\upsilon} \in \mathbb{R}_0^+$ for all $n, w, \upsilon \in \mathbb{N}$.

We should state that Bell's method has significant advantages in coping with the lack of convergence. In addition to classical convergence, by taking some definite matrices, $\mathcal{A}$-summability reduces to Cesàro summability [28], almost convergence [31], and more [29, 30]. For applications of the Bell-type summability method, we refer to [6, 8–11, 27, 33, 35, 37, 39].

Throughout the paper, the following notations and assumptions will be used.

Here are the notations:

- An $N$-dimensional vector $\mathbf{x} \in \mathbb{R}^N$ is denoted by $\mathbf{x} = (x_1, \ldots, x_N)$, where $|\mathbf{x}| = \sqrt{x_1^2 + \ldots + x_N^2}$.
- For a given $u_{\mathbf{k}} : \mathbb{Z}^N \to \mathbb{R}$ where $u_{\mathbf{k}} := u_{(k_1,\ldots,k_N)}$, we denote by $\|u_{\mathbf{k}}\|_{l^1}$, the $l^1$ norm of $u_{\mathbf{k}}$ on $\mathbb{Z}^N$, i.e., $\|u_{\mathbf{k}}\|_{l^1} := \sum \cdots \sum_{(k_1,\ldots k_N)\in\mathbb{Z}^N} |u_{\mathbf{k}}|$.
- We directly use $\sum_{\mathbf{k}\in\mathbb{Z}^N}$ instead of $\sum \cdots \sum_{(k_1,\ldots k_N)\in\mathbb{Z}^N}$.

Here are the assumptions:

$(l_1)$ $\sup_{n,\upsilon\in\mathbb{N}} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|l_{\mathbf{k},w}\|_{l^1} = A < \infty$

$(l_2)$ $\mathcal{A} - \lim \left( \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \right) = 1$

$(l_3)$ $\exists r > 0$ such that $\mathcal{A} - \lim \left( \sum_{|\mathbf{k}|\geq r} |l_{\mathbf{k},w}| \right) = 0.$

Here, conditions $(l_1) - (l_3)$ reduce to the approximate identities in [4] in the case of $\mathcal{A} = \{I\}$, where $I$ corresponds to the identity matrix.

Due to the nonlinearity of the kernel $H_w$, we also require condition (2):

$$\lim_{w\to\infty} \|G_w\|_J = 0 \tag{2}$$
(uniformly in every bounded interval $J \subset \mathbb{R}$)

where $G_w(u) := H_w(u) - u$ and $\|\cdot\|_J$ denotes the supremum norm on the bounded interval $J \subset \mathbb{R}$.

## 3. Approximation in Supremum Norm

First, we will investigate the well-definiteness of the operators of type (1).

**Lemma 1.** *Let $f$ be a bounded function on $\mathbb{R}^N$, $f \in L^1(\mathbb{R}^N)$ and $(l_1)$ hold. Then, $\|\mathcal{T}_{n,\upsilon}(f)\| < \infty$ for all $n, \upsilon \in \mathbb{N}$, namely, $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded functions into itself.*

*Proof.* Using the Lipschitz property of $H_w$ with $H_w(0) = 0$, we have

$$\begin{aligned} |\mathcal{T}_{n,\upsilon}(f;\mathbf{x})| &\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left|H_w(f(\mathbf{x} - \tfrac{\mathbf{k}}{w}))\right| |l_{\mathbf{k},w}| \\ &\le C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left|f(\mathbf{x} - \tfrac{\mathbf{k}}{w})\right| |l_{\mathbf{k},w}|, \end{aligned}$$

where $C$ is the Lipschitz constant of $H_w$. Since $f$ is bounded, from $(l_1)$

$$\begin{aligned} |\mathcal{T}_{n,\upsilon}(f;\mathbf{x})| &\le C\|f\| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \\ &\le C\|f\| A \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^N$. Therefore, taking supremum over all $\mathbf{x} \in \mathbb{R}^N$, we conclude that

$$\|\mathcal{T}_{n,\upsilon}(f;\mathbf{x})\| \le C\|f\| A$$

for all $n, \upsilon \in \mathbb{N}$. □

**Lemma 2.** *Let $f \in BUC(\mathbb{R}^N)$ and $(l_1)$ hold. Then, $\mathcal{T}_{n,\upsilon}(f) \in BUC(\mathbb{R}^N)$ for all $n, \upsilon \in \mathbb{N}$, namely, $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded and uniformly continuous functions into itself.*

*Proof.* By the uniform continuity of $f$ on $\mathbb{R}^N$, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f\left(\mathbf{x} - \frac{\mathbf{k}}{w}\right) - f\left(\mathbf{y} - \frac{\mathbf{k}}{w}\right)\right| < \varepsilon$ whenever $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \left(\mathbf{y} - \frac{\mathbf{k}}{w}\right)\right| = |\mathbf{x} - \mathbf{y}| < \delta$. Now, using the triangle inequality and Lipschitz property of $H_w$

$$\begin{aligned} |\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - \mathcal{T}_{n,\upsilon}(f;\mathbf{y})| &\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left|H_w\left(f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right) - H_w\left(f\left(\mathbf{y} - \tfrac{\mathbf{k}}{w}\right)\right)\right| \\ &\le C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f\left(\mathbf{y} - \tfrac{\mathbf{k}}{w}\right)\right| \end{aligned}$$

$$< \varepsilon C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left| l_{\mathbf{k},w} \right|,$$

hold, where $C$ is the Lipschitz constant of $H_w$. Then from $(l_1)$

$$\left| \mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - \mathcal{T}_{n,\upsilon}(f;\mathbf{y}) \right| < \varepsilon C A$$

which completes the proof. □

Our approximation theorem is as follows.

**Theorem 1.** *If $f \in BUC\left(\mathbb{R}^N\right)$ and $(l_1)-(l_3)$, (2) hold, then we have*

$$\lim_{n\to\infty} \left\| \mathcal{T}_{n,\upsilon}(f) - f \right\| = 0 \text{ (uniformly in } \upsilon).$$

*Proof.* Adding and subtracting some suitable terms, from the triangle inequality, we obtain

$$\begin{aligned}
\left| \mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - f(\mathbf{x}) \right| &= \Bigg| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \left( H_w\left(f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right) - f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right) \\
&\quad + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \left( f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f(\mathbf{x})\right) \\
&\quad + f(\mathbf{x}) \left( \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right) \Bigg| \\
&\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left| l_{\mathbf{k},w} \right| \left\| H_w\left(f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) \right\| \\
&\quad + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left| l_{\mathbf{k},w} \right| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\
&\quad + \|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right| =: A_1 + A_2 + A_3.
\end{aligned}$$

Since supremum is taken over $\mathbb{R}^N$, then we have

$$\left\| H_w\left(f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) \right\| = \left\| H_w(f) - f \right\|.$$

Moreover, since $f$ is bounded, then there exists an interval $J = [C_1, C_2]$ such that $C_1 \leq f(\mathbf{x}) \leq C_2$ and

$$\left| H_w(f(\mathbf{x})) - f(\mathbf{x}) \right| \leq \left\| G_w \right\|_J \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^N$, which implies

$$\left\| H_w(f) - f \right\| \leq \left\| G_w \right\|_J.$$

Then, from (2) for every $\varepsilon > 0$, one can find a positive number $w_0$ such that

$$\left\| H_w(f) - f \right\| < \varepsilon \tag{4}$$

for all $w > w_0$. One can write $A_1$ as follows

$$\begin{aligned} A_1 &= \sum_{w=1}^{w_0} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|H_w(f) - f\| \\ &+ \sum_{w=w_0+1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|H_w(f) - f\| \\ &:= A_1^1 + A_1^2, \end{aligned}$$

from $(l_1)$ and (4), we observe

$$A_1^2 < A\varepsilon$$

and

$$\begin{aligned} A_1^1 &\leq \sum_{w=1}^{w_0} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|G_w\|_J \\ &\leq D \sum_{w=1}^{w_0} a_{nw}^{\upsilon}, \end{aligned}$$

where $D := \max_{1\leq w\leq w_0} \left\{ \|G_w\|_J \, \|l_{\mathbf{k},w}\|_{l^1} \right\}$. In $A_1^1$, by the regularity of $\mathcal{A}$, for each $w \in \{1, 2, \cdots, w_0\}$, there exists a $n_0 = n_0(w, \varepsilon) \in \mathbb{N}$, such that $a_{nw}^{\upsilon} < \varepsilon$ for all $n > n_0$ and $\upsilon \in \mathbb{N}$. Since $w \in \{1, 2, \cdots, w_0\}$, one can find a common $\bar{n}_0 = \bar{n}_0(\varepsilon) := \max_{w\in\{1,2,\ldots,w_0\}} \{n_0(w, \varepsilon)\}$ such that

$$a_{nw}^{\upsilon} < \varepsilon$$

and hence

$$A_1^1 < Dw_0\varepsilon$$

for all $n > \bar{n}_0$, $\upsilon \in \mathbb{N}$ and $w \in \{1, 2, \cdots, w_0\}$.

In $A_2$, due to the uniform continuity of $f$, for every $\varepsilon > 0$ there can be found a number $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon \tag{5}$$

whenever $|\mathbf{x} - \mathbf{y}| < \delta$. Besides, for a given fixed $\bar{r}$ corresponding to assumption $(l_3)$, there exists a number $w_1 \in \mathbb{N}$ satisfying that

$$\left| \frac{\bar{r}}{w} \right| < \delta$$

for all $w > w_1$. Now, writing $A_2$ as follows

$$\begin{aligned} A_2 &= \sum_{w=1}^{w_1} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &+ \sum_{w=w_1+1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\geq\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &:= A_2^1 + A_2^2 + A_2^3 \end{aligned}$$

and considering $\left|\mathbf{x}-\frac{\mathbf{k}}{w}-\mathbf{x}\right|=\left|\frac{\mathbf{k}}{w}\right|<\frac{\bar{r}}{w}<\delta$ in $A_2^2$, one can obviously see from (5) and $(l_1)$ that

$$A_2^2 < A\varepsilon.$$

For $A_2^1$, using the regularity of $\mathcal{A}$, it is possible to find a number $n_1 = n_1(\varepsilon)$ such that

$$A_2^1 < D' w_1 \varepsilon$$

for all $n > n_1$, where $D' := \max_{1\le w\le w_1}\left\{\sum_{|\mathbf{k}|<\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right\}$.

For $A_2^3$, since $f$ is bounded, directly from $(l_3)$

$$A_2^3 < 2\|f\|\varepsilon$$

yields for sufficiently large $n \in \mathbb{N}$.

Finally, from $(l_2)$, we get

$$A_3 < \|f\|\varepsilon$$

for sufficiently large $n \in \mathbb{N}$, which completes the proof by the arbitrariness of $\varepsilon$. □

## 4. Order of Convergence

In this section, we study the order of convergence. For this reason, we introduce the following Lipschitz class.

Let $\alpha > 0$ be given. Then define $Lip_N(\alpha)$ such that:

$$Lip_N(\alpha) = \left\{ f \in BUC\left(\mathbb{R}^N\right) : \|f(\cdot - \mathbf{t}) - f(\cdot)\| = O\left(|\mathbf{t}|^{\alpha}\right) \text{ as } \mathbf{t} \to \mathbf{0}\right\}.$$

Here, with the notation $f(\mathbf{t}) = O(g(\mathbf{t}))$ as $\mathbf{t} \to \mathbf{0}$ we mean that one may find $\delta, R > 0$ such that $|f(\mathbf{t})| \le R|g(\mathbf{t})|$ whenever $|\mathbf{t}| < \delta$.

We require the following conditions on the kernel for the order of convergence.

Let $\alpha > 0$ and $\mathcal{A} = \{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ be fixed. Then, consider the followings:

$$\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1\right) = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{6}$$

there exists a constant $r_0 > 0$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<r_0} \frac{|l_{\mathbf{k},w}|}{w^{\alpha}} = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \tag{7}$$

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\ge r_0} |l_{\mathbf{k},w}| = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon) \tag{8}$$

and

$$\text{for each } w \in \mathbb{N},\ a_{nw}^{\upsilon} = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon). \tag{9}$$

**Theorem 2.** *Let* $\sup_{w\in\mathbb{N}} \|l_{\mathbf{k},w}\|_{l^1} = \breve{A} < \infty$. *Assume that for a fixed* $\mathcal{A} = \{[a^{\upsilon}_{nw}]\}_{\upsilon\in\mathbb{N}}$ *and* $\alpha > 0$, *(6)-(9) hold. Assume further that*

$$\sum_{w=1}^{\infty} a^{\upsilon}_{nw} \|G_w\|_J = O\left(1/n^{\alpha}\right) \text{ for every boınded interval } J \subset \mathbb{R} \quad \text{(uniformly in } \upsilon) \tag{10}$$

*hold. If* $f \in Lip_N(\alpha)$, *then*

$$\|\mathcal{T}_{n,\upsilon}(f) - f\| = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon).$$

*Proof.* We know from the proof of Theorem 1 that

$$\begin{aligned} \|\mathcal{T}_{n,\upsilon}(f) - f\| &\le \sum_{w=1}^{\infty} a^{\upsilon}_{nw} \|G_w\|_J \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \\ &+ \sum_{w=1}^{\infty} a^{\upsilon}_{nw} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \|f\| \left| \sum_{w=1}^{\infty} a^{\upsilon}_{nw} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right| \\ &=: B_1 + B_2 + B_3 \end{aligned}$$

for some bounded interval $J \subset \mathbb{R}$. From our assumption and (10), we immediately get

$$\begin{aligned} B_1 &\le \breve{A} \sum_{w=1}^{\infty} a^{\upsilon}_{nw} \|G_w\|_J \\ &= O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon). \end{aligned}$$

Let $\varepsilon > 0$ and $\delta > 0$ correspond to uniform continuity of $f$. Then, for the given fixed $r_0 > 0$, there exists a $w_2 > 0$ such that $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \mathbf{x}\right| = \left|\frac{\mathbf{k}}{w}\right| < \frac{r_0}{w} < \delta$ for all $w > w_2$. Dividing $B_2$ as follows,

$$\begin{aligned} B_2 &= \sum_{w=1}^{w_2} a^{\upsilon}_{nw} \sum_{|\mathbf{k}|<r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \sum_{w=w_2+1}^{\infty} a^{\upsilon}_{nw} \sum_{|\mathbf{k}|<r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \sum_{w=1}^{\infty} a^{\upsilon}_{nw} \sum_{|\mathbf{k}|\ge r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &:= B_2^1 + B_2^2 + B_2^3 \end{aligned}$$

then, from (9) we get

$$\begin{aligned} B_2^1 &\le D'' w_2 \max_{1\le w\le w_2} a^{\upsilon}_{nw} \\ &= O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon), \end{aligned}$$

where $D'' := \max_{1\leq w\leq w_2}\left\{\sum_{|\mathbf{k}|<r_0}|l_{\mathbf{k},w}|\left\|f\left(\cdot-\frac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\right\}$. Seeing that $f\in Lip_N(\alpha)$, then there can be found a number $R>0$ satisfying that

$$\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\leq R\left|\tfrac{\mathbf{k}}{w}\right|^{\alpha}.$$

Thus, from (7) there holds

$$\begin{aligned}B_2^2 &\leq Rr_0^{\alpha}\sum_{w=w_2+1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|<r_0}|l_{\mathbf{k},w}|\frac{1}{w^{\alpha}}\\ &=O(1/n^{\alpha})\text{ as } n\to\infty \text{ (uniformly in } \upsilon).\end{aligned}$$

For $B_2^3$, since $f$ is bounded, from (8) we obtain

$$\begin{aligned}B_2^3 &\leq 2\|f\|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|\geq r_0}|l_{\mathbf{k},w}|\\ &=O(1/n^{\alpha})\text{ as } n\to\infty \text{ (uniformly in } \upsilon).\end{aligned}$$

Finally, directly from (6), the following inequality yields

$$B_3=O(1/n^{\alpha})\text{ as } n\to\infty \text{ (uniformly in } \upsilon).$$

□

## 5. Main Results

Now, using the relation between operators (1) and the generalized sampling operators, following conclusions can be obtained.

For a given $f:\mathbb{R}^N\to\mathbb{R}$, let $l_{\mathbf{k},w}\equiv\chi(\mathbf{k})$ for all $w\in\mathbb{N}$, where $\chi:\mathbb{R}^N\to\mathbb{R}$. In this particular case, (1) turns into

$$\bar{\mathcal{T}}_{n,\upsilon}(f;\mathbf{x})=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right)\chi(\mathbf{k})\ \ (\mathbf{x}\in\mathbb{R}^N),$$

which is related to $\mathcal{A}-$transform of $N$-dimensional nonlinear generalized sampling series (for the linear and one dimensional case, see [4,6]), that is

$$\mathcal{S}_{n,\upsilon}(f;\mathbf{x})=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\tfrac{\mathbf{k}}{w}\right)\right)\chi(w\mathbf{x}-\mathbf{k}),\ \mathbf{x}\in\mathbb{R}^N.$$

Now, it is not hard to see that $(l_1)$ and $(l_2)$ turn out to be the following assumptions:

$(l_1')$ $\chi\in l^1\left(\mathbb{Z}^N\right)$,
$(l_2')$ $\sum_{\mathbf{k}\in\mathbb{Z}^N}\chi(\mathbf{k})=1$.

In this case, $(l_3)$ is not satisfied. On the other hand, instead of (2), now we may assume a more general condition (11), given by
for every bounded interval $J\subset\mathbb{R}$,

$$\mathcal{A}-\lim\|G_w\|_J=0 \tag{11}$$

in the following result.

**Theorem 3.** *Let $f \in BUC\left(\mathbb{R}^N\right)$. If $(l_1')$, $(l_2')$ and (11) hold, then*

$$\lim_{n\to\infty}\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\|=0 \text{ (uniformly in } \upsilon\in\mathbb{N}).$$

*Proof.* From the triangle inequality and $(l_2')$,

$$\begin{aligned}\left\|\bar{\mathcal{T}}_{n,\upsilon}(f)-f\right\| &\leq \sum_{w=1}^{\infty} a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}|\chi(\mathbf{k})|\left\|H_w\left(f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right)-f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right\| \\ &+\sum_{w=1}^{\infty} a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}|\chi(\mathbf{k})|\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\| \\ &+\|f\|\left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon}-1\right|\end{aligned}$$

holds. Since $\left\|H_w\left(f\left(\cdot-\frac{\mathbf{k}}{w}\right)\right)-f\left(\cdot-\frac{\mathbf{k}}{w}\right)\right\| = \|H_w(f)-f\|$, then from $(l_1')$ and (11) one can clearly see that

$$\begin{aligned}&\sum_{w=1}^{\infty} a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}|\chi(\mathbf{k})|\left\|H_w\left(f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right)-f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right\| \\ &=\left(\sum_{\mathbf{k}\in\mathbb{Z}^N}|\chi(\mathbf{k})|\right)\left(\sum_{w=1}^{\infty} a_{nw}\|H_w(f)-f\|\right) \\ &< \bar{A}\varepsilon\end{aligned}$$

for sufficiently large $n\in\mathbb{N}$, for which $\bar{A}=\|\chi\|_{l^1}$.

On the other hand, from $(l_1')$, for all $\varepsilon>0$ there can be found a number $\breve{r}>0$ such that

$$\sum_{|\mathbf{k}|\geq\breve{r}}|\chi(\mathbf{k})|<\varepsilon$$

and hence,

$$\begin{aligned}\sum_{w=1}^{\infty} a_{nw}^{\upsilon}\sum_{|\mathbf{k}|\geq\breve{r}}|\chi(\mathbf{k})|\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\| &< 2\|f\|\sum_{w=1}^{\infty} a_{nw}^{\upsilon}\varepsilon \\ &\leq 2\|f\|M\varepsilon\end{aligned}$$

holds for sufficiently large $n\in\mathbb{N}$. Here $M$ comes from c) in the regularity of $\mathcal{A}$. Using analogous lines of the proof of Theorem 1, one can find a number $\bar{w}_1\in\mathbb{N}$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon}\sum_{|\mathbf{k}|<\breve{r}}|\chi(\mathbf{k})|\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\|<\varepsilon\left(\bar{D}\bar{w}_1+\bar{A}M\right)$$

holds for sufficiently large $n\in\mathbb{N}$, where $\bar{D}$ is defined by

$$\bar{D}:=\max_{1\leq w\leq\bar{w}_1}\left\{\sum\nolimits_{|\mathbf{k}|<\breve{r}}|\chi(\mathbf{k})|\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\right\}.$$

Finally, using the regularity of $\mathcal{A}$, we obviously see that

$$\|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{v} - 1 \right| < \|f\| \varepsilon$$

for sufficiently large $n \in \mathbb{N}$. Consequently, since $f$ is bounded, the proof is done. □

We now take into account the following Paley-Wiener spaces to prove Corollary 1 below.

For $1 \leq p \leq \infty$,

$$B_{\pi w}^{p}(\mathbb{R}) = \{f \in L^{p}(\mathbb{R}) : f \text{ has an extension to whole } \mathbb{C} \text{ s.t. } |f(z)| \leq \exp(\pi w |z|) \|f\| \text{ for every } z \in \mathbb{C}\}$$

and

$$B_{\pi w,loc}^{p}(\mathbb{R}^{N}) = \left\{f \in L^{p}(\mathbb{R}^{N}) : \text{for every fixed } (x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N), \ f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N) \in B_{\pi w}^{p}(\mathbb{R}) \text{ for } 1 \leq j \leq N\right\}.$$

**Corollary 1.** *Let $f \in B_{\pi \hat{w},loc}^{1}(\mathbb{R}^{N}) \cap BUC(\mathbb{R}^{N})$ for some $\hat{w} > 0$ and $\chi \in B_{\pi,loc}^{\infty}(\mathbb{R}^{N})$. If $(l_1')$, $(l_2')$ and (11) are satisfied, then*

$$\lim_{n \to \infty} \|\mathcal{S}_{n,v}(f) - f\| = 0 \ \textit{(uniformly in } v \in \mathbb{N}\textit{)}.$$

*Proof.* Since $|H_w(f(\mathbf{x}))| \leq C|f(x)|$ for all $w \in \mathbb{N}$, then for every fixed $(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N)$, $H_w(f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N)) \in B_{\pi\hat{w}}^{1}(\mathbb{R})$ for all $f \in B_{\pi\hat{w},loc}^{1}(\mathbb{R}^{N})$ and $j = 1, \ldots, N$. On the other hand, assuming $g(\mathbf{x}) := \chi(w\mathbf{x})$ we observe that $g \in B_{\pi w,loc}^{1}(\mathbb{R}^{N})$ for all $w \geq \hat{w}$. Now, we can write the operators $\mathcal{S}_{n,v}(f;\mathbf{x})$ explicitly as follows

$$\begin{aligned}
&\mathcal{S}_{n,v}(f;\mathbf{x}) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k} \in \mathbb{Z}^N} (H_w \circ f)\left(\tfrac{\mathbf{k}}{w}\right) \chi(w\mathbf{x} - \mathbf{k}) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k} \in \mathbb{Z}^N} (H_w \circ f)\left(\tfrac{\mathbf{k}}{w}\right) g\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} (H_w \circ f)\left(\tfrac{k_1}{w}, \ldots, \tfrac{k_N}{w}\right) g\left(x_1 - \tfrac{k_1}{w}, \ldots, x_N - \tfrac{k_N}{w}\right).
\end{aligned}$$

Here, fixing the first $N-1$ terms of the previous expression and using Lemma 4.2 in [4], we get

$$\mathcal{S}_{n,v}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} (H_w \circ f)\left(\tfrac{k_1}{w}, \ldots, \tfrac{k_{N-1}}{w}, x_N - \tfrac{k_N}{w}\right) g\left(x_1 - \tfrac{k_1}{w}, \ldots, x_{N-1} - \tfrac{k_{N-1}}{w}, \tfrac{k_N}{w}\right).$$

Now, using Fubini-Tonelli theorem (discrete version) and applying the same process for every $k_j$ for $j = 1, \dots, N-1$, we conclude that

$$\begin{aligned}\mathcal{S}_{n,\upsilon}(f;\mathbf{x}) &= \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) g\left(\tfrac{\mathbf{k}}{w}\right) \\ &= \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) \chi(\mathbf{k}) \\ &= \bar{\mathcal{T}}_{n,\upsilon}(f;\mathbf{x}).\end{aligned}$$

Since $f \in BUC\left(\mathbb{R}^N\right)$, by Theorem 3, we complete the proof. □

Notice that, $B^1_{\pi w,loc}\left(\mathbb{R}^N\right) \subset UC_{loc}\left(\mathbb{R}^N\right)$, where $UC_{loc}\left(\mathbb{R}^N\right)$ is the space of all functions $f : \mathbb{R}^N \to \mathbb{R}$ such that for every fixed $(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)$, $f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)$ is uniformly continuous on $\mathbb{R}$ (see Proposition 4.3 in [4]).

**Remark 1.** *If $f \in B^p_{\pi w,loc}\left(\mathbb{R}^N\right) \cap BUC\left(\mathbb{R}^N\right)$ for $1 \le p \le 2$, then Corollary 1 is still applicable. In that case, we have to assume that $\chi \in B^q_{\pi,loc}\left(\mathbb{R}^N\right)$ to be able to apply Lemma 4.2 in [4], where $1/p + 1/q = 1$.*

**Remark 2.** *From Example 4.5 in [4], one can construct an $N$-dimensional kernel $\chi$ satisfying the conditions $(l_1')$ and $(l_2')$.*

Using the properties of $\mathcal{A}$-summability under suitable conditions $(l_1) - (l_3)$ and (2), the following results can easily be obtained for all $f \in BUC\left(\mathbb{R}^N\right)$:

Consider the operator $T_w(f;\mathbf{x})$, defined by

$$T_w(f;\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right) l_{\mathbf{k},w} \quad \left(\mathbf{x} \in \mathbb{R}^N,\ w \in \mathbb{N}\right).$$

Assume that $\mathcal{A} = \mathcal{F}, \{C_1\}$ and $\{I\}$, where

- $\mathcal{F}$ is the sequences of infinite matrices given by $\{[a^{\upsilon}_{nw}]\}_{\upsilon\in\mathbb{N}}$ such that $a^{\upsilon}_{nw} = 1/n$, if $\upsilon \le w \le n + \upsilon - 1$; $a^{\upsilon}_{nw} = 0$, if otherwise (see [31]),
- $C_1$ is the Cesàro matrix [28] such that $c_{nw} = 1/n$, if $1 \le w \le n$; $c_{nw} = 0$, if otherwise

and

- $I$ is the identity matrix.

Then we obtain

$$\lim_{n\to\infty} \left\| \frac{T_{\upsilon}(f) + T_{\upsilon+1}(f) + \cdots + T_{n+\upsilon-1}(f)}{n} - f \right\| = 0 \quad \text{(uniformly in } \upsilon)$$

$$(T_w(f) \text{ is almost convergent to } f),$$

$$\lim_{n\to\infty}\left\|\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right\|=0$$
$$(T_w(f) \text{ is arithmetic mean convergent to } f)$$

and

$$\lim_{n\to\infty}\|T_n(f)-f\|=0$$

respectively.

Moreover, under suitable conditions, all the previous convergence methods are valid for nonlinear generalized sampling series, given by

$$S_w(f;\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\tfrac{\mathbf{k}}{w}\right)\right)\chi(w\mathbf{x}-\mathbf{k}).$$

## 6. Applications

In this section, we begin by providing a detailed example of the discrete kernel $l_{\mathbf{k},w}$. Following this, we explore an application of our operator in the field of digital image processing.

First, consider the 2-dimensional case and substitute the matrix $\mathcal{A}$ with matrix $\mathcal{F}$ as defined above.

Let $l_{\mathbf{k},w}$ be defined by

$$l_{\mathbf{k},w}:=\begin{cases} 2\left(\dfrac{1}{2a_w-1/2}\right)^2\frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}}; & \text{if } w=m^2\ (m\in\mathbb{N})\\ \left(\dfrac{1}{2a_w-1/2}\right)^2\frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}}; & \text{if } w\neq m^2\ (m\in\mathbb{N})\end{cases},$$

where

$$a_w=\sum_{k=0}^{\infty}\frac{1}{2^{(w+1)^k}}.$$

Note that, by the ratio test one can observe that $a_w$ is finite for all $w\in\mathbb{N}$. Then, considering the following equality

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\frac{1}{2^{(w+1)^{|k|}}}&=2\sum_{k=0}^{\infty}\frac{1}{2^{(w+1)^k}}-\frac{1}{2}\\&=2a_w-1/2\end{aligned}$$

we may obtain

$$\sup_{n,v\in\mathbb{N}}\sum_{w=1}^{\infty}a_{nw}^{v}\|\,l_{\mathbf{k},w}\|_{l^1}=\sup_{n,v\in\mathbb{N}}\sum_{w=v}^{n+v-1}\frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2}|l_{\mathbf{k},w}|$$

$$\leq \sup_{n,v\in\mathbb{N}} \sum_{w=v}^{n+v-1} \frac{2}{n}\left(\frac{1}{2a_w - 1/2}\right)^2 \sum_{k_1\in\mathbb{Z}} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{k_2\in\mathbb{Z}} \frac{1}{2^{(w+2)^{|k2|}}}$$
$$= 2$$

which shows that condition $(l_1)$ is satisfied. Furthermore, for $(l_2)$ consider the following

$$\begin{aligned}
\left|\sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| &= \left|\sum_{w=v}^{n+v-1} \frac{1}{n} \sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| \\
&= \left|\sum_{\substack{v\leq w\leq n+v-1\\ w\neq m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} + \sum_{\substack{v\leq w\leq n+v-1\\ w\neq =m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| \\
&\leq \sum_{\substack{v\leq w\leq n+v-1\\ w\neq =m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} \\
&= \sum_{\substack{v\leq w\leq n+v-1\\ w\neq =m^2}} \frac{1}{n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} 2\left(\frac{1}{2a_w - 1/2}\right)^2 \frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}} \\
&= \sum_{\substack{v\leq w\leq n+v-1\\ w\neq =m^2}} \frac{2}{n} \\
&= \frac{2}{n}\left(\sqrt{n+v-1} - \sqrt{v} + 1\right) \\
&\leq \frac{4}{\sqrt{n}}
\end{aligned}$$

we obtain $(l_2)$.

For $(l_3)$, taking $r = 1$, since

$$\{\mathbf{k} = (k_1, k_2) : |\mathbf{k}| \geq 1\} \subset \left\{\mathbf{k} = (k_1, k_2) : |k_1| \geq \frac{1}{\sqrt{2}} \text{ and } |k_2| \geq \frac{1}{\sqrt{2}}\right\},$$

we write

$$\begin{aligned}
\sum_{w=1}^{\infty} a_{nw}^{v} \sum_{|\mathbf{k}|\geq 1} |l_{\mathbf{k},w}| &= \sum_{w=v}^{n+v-1} \frac{1}{n}\sum_{|\mathbf{k}|\geq 1} |l_{\mathbf{k},w}| \\
&\leq \sum_{w=v}^{n+v-1} \frac{2}{n}\left(\frac{1}{2a_w - 1/2}\right)^2 \sum_{|k_1|\geq \frac{1}{\sqrt{2}}} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{|k_2|\geq \frac{1}{\sqrt{2}}} \frac{1}{2^{(w+1)^{|k2|}}} \\
&= \sum_{w=v}^{n+v-1} \frac{2}{n}\left(\frac{1}{2a_w - 1/2}\right)^2 \sum_{|k_1|\geq 1} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{|k_2|\geq 1} \frac{1}{2^{(w+1)^{|k2|}}}
\end{aligned}$$

$$= \sum_{w=v}^{n+v-1} \frac{2}{n} \left(1 - \frac{1}{4a_w - 1}\right)^2 .$$

On the other hand, since

$$\lim_{w\to\infty} \frac{1}{2^{(w+1)^{|k|}}} = \begin{cases} \frac{1}{2}; & k = 0 \\ 0; & \text{otherwise,} \end{cases}$$

by the discrete version of dominated convergence theorem, we obtain

$$\lim_{w\to\infty} 2 \left(1 - \frac{1}{4a_w - 1}\right)^2 = 0$$

and by the regularity of $\mathcal{F}$, we get

$$\lim_{n\to\infty} \sum_{w=v}^{n+v-1} \frac{2}{n} \left(1 - \frac{1}{4a_w - 1}\right)^2 = 0$$

uniformly in $v \in \mathbb{N}$. Therefore, $(l_3)$ is satisfied. However, our kernel $l_{\mathbf{k},w}$ does not adhere the classical approximate identities, since

$$\lim_{w\to\infty} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} = \begin{cases} 2; & \text{if } w = m^2 \ (m \in \mathbb{N}) \\ 1; & \text{if } w \neq m^2 \ (m \in \mathbb{N}) \end{cases}$$
$$\neq 1.$$

This non-fulfillment suggests that our approximation is non-trivial.

6.1. **Application on Images.** With the development of modern technology, zooming in on digital images has become common in many areas such as digital cameras, medical imaging, and mobile phones. In the literature, there are different types of zooming methods such as pixel replication, interpolation, zero-order hold method, and more. In this part of the application, we propose an operator method for zooming in on images. We should note that approximating operators can be very useful in image processing [15, 25, 26].

In classical zoom techniques, the neighborhood of a pixel is often processed. In contrast, our proposed method requires all pixel values of the zoomed image for each pixel value. Although this may reduce computational efficiency, it helps prevent issues such as loss of sharpness in the corners. Now, we apply our approximation method to zoom in on digital images.

It is known that, a grayscale $m \times m$ pixel valued digital image can be represented by a step function as follows (see [25]):

$$I(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{m} u_{ij} 1_{ij}(x, y)$$

where $u_{ij}$ is the $(i, j)$'th pixel value of the given image and $1_{ij}$ is defined by

$$1_{ij}(x, y) = \begin{cases} 1; & \text{if } (x, y) \in (i-1, i] \times (j-1, j] \\ 0; & \text{otherwise} \end{cases} \quad (i, j = 1, 2, \ldots, m).$$

It is clear that $I$ is a step function with compact support and therefore $I \in L^1\left(\mathbb{R}^2\right)$. Using the density of the continuous functions in $L^1\left(\mathbb{R}^2\right)$, we may approximate this image by the operator (1).

Let $\mathcal{A} = \{I\}$ and $l_{k,w}$, $H_w$ be defined by

$$l_{\mathbf{k},w} = \left(\frac{2^w - 1}{2^w + 1}\right)^2 \frac{1}{2^{w(|k_1|+|k_2|)}}$$

and

$$H_w\left(x\right) = \begin{cases} x + \log_{10}\left(1 + \frac{x}{w}\right); & \text{if } 0 \leq x < 1 \\ x + \log_{10}\left(1 + \frac{1}{wx}\right); & \text{if } x \geq 1. \end{cases} \quad \text{(see [2,3])}$$

respectively. We assume that $H_w$ is extended symmetrically in the odd way. One can clearly observe that $l_{k,w}$ fulfills the assumptions $(l_1) - (l_3)$ and $H_w$ satisfies (2).

Now, consider the following image, named by "baboon" in Figure 1. We will

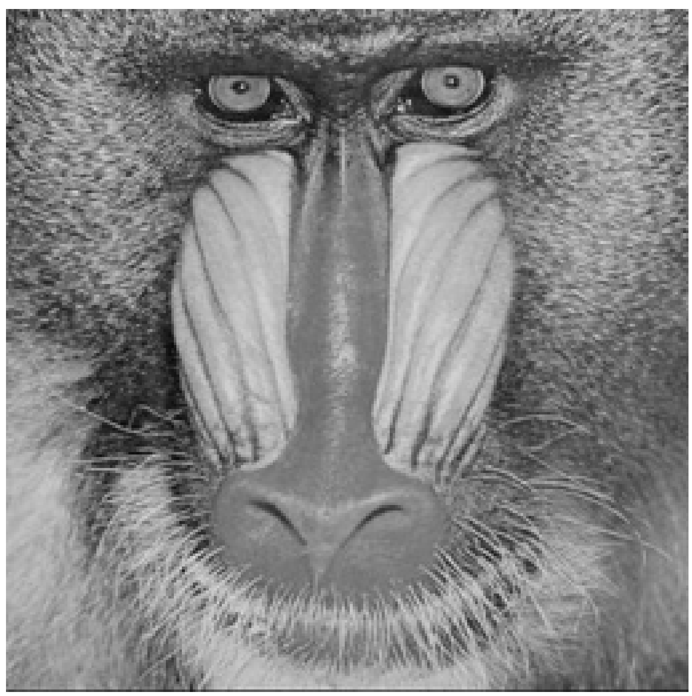

FIGURE 1. Original 256×256 pixel resolution Baboon

focus on the left eye of the baboon shown in the Figure 2. By using our nonlinear operator, we approximate the Baboon's eye for $w = 4$. By increasing the sampling rate, the following new zoomed images can be obtained (see Figure 3 and Figure 4).

FIGURE 2. Original 50×50 pixel resolution eye of the Baboon

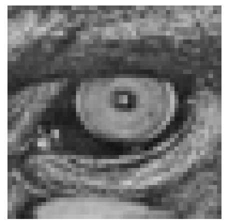

FIGURE 3. The eye of the Baboon zoomed in with a resolution of 100×100 pixels, obtained by nonlinear operator for $w = 4$

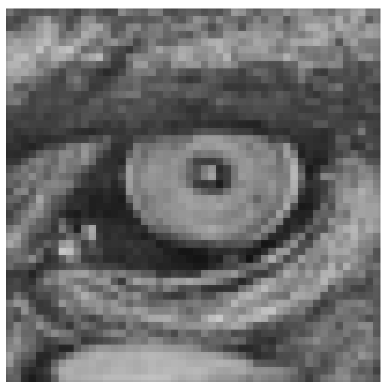

FIGURE 4. The eye of the Baboon zoomed in with a resolution of 200×200 pixels, obtained by nonlinear operator for $w = 4$

These new images demonstrate that our proposed method could be useful for digital image zooming. We should note that changing or scaling the kernels for different values of $w$ may result in higher quality zoomed images.

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