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Comparative Analysis of Optimization Methods for Grey Fuzzy Transportation Problems in Logistics

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Abstract

This study examines the Grey Fuzzy Transportation Problem, which represents decision-making processes under uncertainty in the transportation problem, a significant issue in the logistics sector and academic studies. The study provides comprehensive analysis and recommendations that contribute to the effective solution of the Grey Fuzzy Transportation Problem and better management of uncertain transportation problems. The research compares four different optimization methods, Closed Path Method, Interval Optimization, Robust Optimization, and Interval Optimization with Penalty Function, for the Grey Fuzzy Transportation Problem (GFTP). The analyses were conducted on a total of 40 test problems across four different problem sizes: small, medium, large, and extra-large. The results showed that the Interval Optimization and Robust Optimization methods demonstrated the best performance in terms of solution quality and computation time. Specifically, detailed analyses of the Interval Optimization with Penalty Function method confirmed that this method provides an effective and consistent solution approach for the GFTP.

Keywords Julia, JuMP, SCIP, Grey Fuzzy Linear Programming, Transportation Problem

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1. Introduction

Logistics is an integrated system that manages the movement of products from raw materials to the final consumer. Emerging as physical distribution management in the 1960s, this field rapidly developed in the 1980s and 1990s due to globalization, technological advancements, and increasing competitive pressure (Christopher, 1992). Scientific Management, proposed by Taylor in the early 20th century, aims to systematically analyze and improve work processes to increase efficiency (Taylor, 1911). Operations Research, developed during World War II for the optimization of military operations, began to rise academic and industrial applications in business management and decision-making processes in the post-war period (Gass & Assad, 2005).

Fuzzy Logic is a decision-making approach that mimics the human thought process using linguistic variables and membership functions (Zadeh, 1965). Grey System Theory is a decision analysis approach developed to analyze systems that contain partially known, partially unknown, incomplete, or uncertain information (Deng, 1982). Both approaches have found widespread use in decision-making processes under uncertainty in both academic studies and industrial applications and have been extensively studied in the fields of Operations Research, Scientific Management, and Decision Making.

Transportation Problems are one of the oldest and most fundamental application areas of Operations Research. First formulated by Hitchcock in 1941, the Transportation Problem is a mathematical approach developed to determine how to transport goods from supply points to demand points at the lowest cost (Hitchcock, 1941). Decision Making under Uncertainty is a core focus of Operations Research and Scientific Management (Simon, 1960). Grey System Theory and Fuzzy Logic are widely studied in modelling real industrial decision problems involving uncertain Transportation Problems (Bai & Sarkis, 2010).

Nasseri & Khabiri (2019) considered the cost coefficients of the Transportation Problem as grey numbers and the supply and demand quantities as fuzzy numbers. The problem is called the Grey Fuzzy Transportation Problem.

In this research, 40 test problems were generated randomly in four different dimensions depending on the parameters determined for the Grey Fuzzy Transportation Problem. The test problems were coded in Julia language to ensure that the test problems were generated according to the specified parameters. In this study, the initial solution algorithm proposed by Nasseri & Khabiri (2019) for the Grey Fuzzy Transportation Problem and the Closed Path approach expressed as an improvement algorithm are coded in Julia language and analyzed on test problems. In addition, Interval Optimization and Robust Optimization algorithms used in the literature for different problems are adapted to solve the Grey Fuzzy Transportation Problem and the algorithms are coded in Julia language. In addition, a penalty function is added to the Interval Optimization approach and an approach that will enable intelligent positioning of optimization parameters is proposed for this problem type. In terms of the literature, the development of a test set for Grey Fuzzy Transportation problems proposed by Nasseri & Khabiri (2019), the adaptation of the Interval and Robust Optimization approach to solve these problems, and the design of an intelligent and adaptive solution approach with the penalty function of the Interval Optimization approach can be stated as innovations.

The second section of the study includes a literature review. The third section provides basic information on Grey System Theory and Fuzzy Logic and explains the Grey Cost Fuzzy Transportation Problem. The fourth section describes the solution approaches used. The fifth section presents a set of test problems and analyses, and the final section shares the discussion and conclusions.

2. Literature Review

Fuzzy and Grey analysis and solution approaches are widely used in optimization studies in fields such as artificial intelligence, production management, operations research, economics, and decision theory. Therefore, the development of general and applicable fuzzy and optimization methods is important both theoretically and practically.

Various approaches have been proposed in the literature for solving problems involving uncertainty. Yu et al. (2024) developed an interval-constrained multi-objective optimization algorithm using a new penalty function to directly address problems with uncertain objectives and constraints. Jayswal et al. (2022) established robust sufficient optimality conditions for multi-time first-order partial differential equation-constrained control optimization problems in the face of data uncertainty. Fu & Cao (2019) proposed adaptive sub-interval decomposition analysis and interval differential evolution approaches to solve these uncertain optimization problems when the parameters of nonlinear optimization problems take interval values.

Karmakar & Bhunia (2014) presented an interval-focused solution approach aimed at obtaining solutions with low cost and high efficiency for optimization problems where uncertainty is expressed in intervals. Steuer (1981) proposed three different algorithms for linear programming problems where the objective function coefficients are expressed in intervals. Guerra et al. (2017) developed linear programming solutions based on the Hukuhara difference for optimization problems where uncertainty is expressed as intervals.

Fuzzy logic and fuzzy set theory are important methods used to solve problems involving uncertainty. Klir & Yuan (1995) highlighted theoretical advancements and application opportunities in the field of fuzzy set theory and fuzzy logic. Zimmermann (1996) provided an instructive and guiding study on fuzzy set models, addressing linear programming, logistics, transportation problems, and their relationships with fuzzy logic.

Fu et al. (2006), presented a framework where different heuristic algorithms can produce effective solutions for the rapid solution of transportation problems. Aydemir (2020) proposed a new approach for determining and analyzing the nth degree of greyness for the characterization and dimension measurement of uncertain information. Aydemir et al. (2020) examined the analysis of production planning under uncertainty conditions using fuzzy linear programming and four different grey linear programming models.

Grey system theory and fuzzy logic are also used in logistics and transportation. Şahin & Karagül (2023) examined the motivation for purchasing tractors in a company engaged in road transportation in the logistics sector using grey relational analysis. Tokat et al. (2022) designed key performance indicators for warehouse loading operations using a fuzzy logic clustering approach. Aydemir et al. (2023) analyzed the relationships between customer expectations, requirements, and prices using grey system theory.

Li & Jin (2008) proposed a fuzzy optimization approach based on a comparison of indices and developed a new solution approach by integrating it with genetic algorithms. Teodorović (1999) modelled traffic and transportation processes using fuzzy logic approaches and argued that fuzzy logic is a universal structure for solving engineering problems in this field.

Voskoglou (2018), solved different linear programming examples structured with grey numbers using the simplex algorithm. Moore et al. (2009) provided an important and fundamental resource for interval numbers. Pourofoghi et al. (2019) defined the transportation problem as a grey transportation problem when transportation costs, supply, and demand data are interval grey numbers and proposed a new solution approach using the concepts of center and width of grey numbers. Ben-Tal & Nemirovski (1998; 2002) proposed the Robust Optimization approach to formulate optimization problems where data is uncertain and belongs to a certain uncertainty set. Nasseri & Khabiri (2019), considered the cost coefficients of the Transportation Problem as grey numbers and the supply and demand quantities as fuzzy numbers and proposed improving the solution using the closed path approach with classical Transportation Problem initial solutions.

When the recent literature was searched with the keywords Grey, fuzzy, and transportation problem, no studies that directly correspond to the subject of this study were found. However, indirectly related studies are summarized below.

Table 1. Recent literature with the *grey*, *fuzzy* and *transportation problem* keywords

Paper	Summary of Research
Moslem et al. (2023a)	IMF SWARA and Fuzzy Bonferroni methods were used to determine the preferences of decision-makers to improve the supply quality of urban bus transportation in Mersin. While traceability was identified as the most important criterion, service quality was the least important.
Çelikbilek et al. (2022)	A study was conducted to evaluate Budapest's public transportation system using a grey decision model. BWM, AHP and MOORA methods were combined.
Kumar et al. (2023)	The uncertain multiobjective transportation problem with a neutrosophic hyperbolic programming approach is considered and optimal solutions with different confidence levels are analyzed.
Mardanya & Roy (2023)	The Multi-objective Multi-product Solid Transportation Problem (MMSTP) in a fuzzy environment is investigated. Fuzzy parameters are treated with trapezoidal fuzzy numbers and different models are developed.
Ghosh et al. (2022)	A model has been developed with time window and preservation technology for the transportation of perishable products. The spoilage rate was minimized by finding optimal solutions.
Moslem et al. (2023b)	A study was conducted to systematically examine the applications of AHP in solving transportation problems. The research shows that there is widespread use in solving public transportation and logistics problems with AHP.
Bilişik et al. (2024)	An IVIFS-based CRITIC-TOPSIS methodology is proposed for transportation mode selection for a glass manufacturing company. The railway is identified as the most suitable transportation mode.
Kacher & Singh (2023)	A general parameter approach for fuzzy parameter-based multi-objective transportation problems is proposed. This approach covers the decomposition of fuzzy data into different levels and the solution of classical transportation problems.

Paper	Summary of Research
Zhang et al. (2024)	The multi-objective low carbon multimodal transportation planning problem under fuzzy demand and time conditions is considered and a new model is developed with the sparrow search algorithm.
Baidya (2024)	A multi-stage solid transportation problem is addressed using grey number theory. This approach focuses on solving logistics problems with uncertainty.

As can be seen from the literature review, the Grey Fuzzy Transportation Problem emerges as a relatively new logistics problem. The solution approaches proposed and developed for this problem also indicate a new break in the literature. Although the solution approaches proposed in this research are used in different problem areas, they can be considered as new solution approaches for Grey Fuzzy Transportation. From this point on, we will continue to explain the approaches that introduce the concept of uncertainty to reveal the details of the research.

3. Approaches Under Uncertainty

In classical optimization problems, parameters take on specific values, and in such cases, exact solutions can be obtained using classical optimization methods. However, real industrial applications do not always meet this certainty condition, and therefore, uncertainties arise due to incomplete, incorrect, or insufficient information. In this case, scientific management requires new approaches for the process of compiling and modeling problem data under these uncertainties. Approaches that can be proposed for decision problems where uncertainty arises can be grouped under three headings:

- Probabilistic approach: Uncertain parameters are designed as random variables with probability distributions.
- Fuzzy approach: Uncertain parameters are designed as fuzzy sets and/or fuzzy numbers.
- The grey system approach: Uncertain parameters are designed as grey numbers using the Grey System Theory.

For decision problems under uncertainty conditions that arise in the industry, these approaches can be used individually or in various combinations. In this study, the Grey System Theory approach and Fuzzy Logic approaches will be considered together to define the optimization problem and seek solution approaches. In this context, basic information about Fuzzy Logic and Grey System Theory will be provided within the scope of the article.

3.1. Grey Numbers and Operations

A grey number is a number whose exact value is unknown but is known to lie within a certain interval. It is usually denoted by the symbol (⊗). It is expressed as follows (Aydemir et al., 2020; Liu & Lin, 2006; Nasseri & Khabiri, 2019):

$$\otimes x = [\underline{x}, \overline{x}] = t \in x : \underline{x} \leqslant x \leqslant \overline{x} \tag{1}$$

Here, ($\underline{x}$) is the lower bound, and ($\overline{x}$) is the upper bound.

An interval grey number is denoted as $\otimes x = [a, b]$ and is a grey number with a lower and upper bound. If $\otimes x = [a, a]$, it is called a white number, and if $\otimes x = [-\infty, \infty]$, it is called a black number.

Every real number (a) can be expressed as a grey number ($\otimes a = [a, a]$).

The grey zero is ($\otimes 0 = [0, 0]$).

The set of all grey numbers is denoted by ($\mathfrak{R}(\otimes)$). Let ($\otimes a = [\underline{a}, \overline{a}]$) and ($\otimes b = \left[\underline{b}, \overline{b}\right]$) be two grey numbers.

The addition operation is as follows: $\otimes a + \otimes b = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$

The multiplication of two grey numbers ($\otimes a = [\underline{a}, \overline{a}]$) and ($\otimes b = \left[\underline{b}, \overline{b}\right]$) is:

$\otimes a * \otimes b = [\min P, \max P]$ Here, ($P = \underline{a} * \underline{b}, \underline{a} * \overline{b}, \overline{a} * \underline{b}, \overline{a} * \overline{b}$) contains the products. The multiplication of the grey number ($\otimes a = [\underline{a}, \overline{a}]$) by a positive real number (k) is: $k \otimes a = [k\underline{a}, k\overline{a}]$ For the grey number ($\otimes x = [\underline{x}, \overline{x}]$), the whitenization is:

$x = \alpha \underline{x} + (1 - \alpha)\overline{x}, \alpha \in [0, 1]$

is converted to a white number. Typically, ($\alpha = 0.5$) is used. For the grey number ($\otimes x = [\underline{x}, \overline{x}]$), the kernel (center) is: $\hat{x} = \frac{1}{2}(\underline{x} + \overline{x})$

The whitening function is used to rank grey numbers. ($G : \mathfrak{R}(\otimes) \to \mathfrak{R}$) assigns a real number value to each grey number. Accordingly:

- $G(\otimes x) < G(\otimes y) \Rightarrow \otimes x < \otimes y$
- $G(\otimes x) = G(\otimes y) \Rightarrow \otimes x = \otimes y$
- $G(\otimes x) > G(\otimes y) \Rightarrow \otimes x > \otimes y$

3.2. Fuzzy Sets and Fuzzy Numbers

The definitions related to fuzzy sets and fuzzy numbers are provided at a very basic level below (Nasseri & Khabiri, 2019; Zimmermann, 1996).

Let X be a universal set, and A be a fuzzy set on X defined by the membership function $\mu_A(x), x \in X$. The value $\mu_A(x)$ represents the degree of membership of x in A within the interval $[0, 1]$. For the fuzzy set A, the λ-cut is defined as $\lambda \in [0, 1]$ and $A_\lambda = x \in X : \mu_A(x) \geq \lambda$. The support set of the fuzzy set A is defined as $S(A) = x \in X : \mu_A(x) > 0$. A fuzzy set is called a fuzzy number if it is convex, normal, and bounded. A triangular fuzzy number is usually expressed as $\mathcal{a} = (a^L, a, a^U)$. The membership function is:

$$\mu_{\mathcal{A}}(X) = \begin{cases} \frac{x - a^L}{a - a^L} \text{ , if } a^L \leqslant x \leqslant a \\ \frac{a^U - x}{a^U - a} \text{ , if } a \leqslant x \leqslant a^U \\ 0 \text{ , otherwise} \end{cases} \quad (2)$$

is as follows a^L: lower bound, a: peak value, a^U: upper bound. The set of fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$. For ranking, the Yager index is:

$R(\mathcal{A}) = \frac{1}{4}(a^L + 2a + a^U)$

Defuzzification: The process of converting a fuzzy number into a single number. Usually, the center of gravity method is used. For $\mathcal{a} = (a^L, a, a^U) \Rightarrow a^R = \frac{(a^L + 2a + a^U)}{4}$

3.3. Grey Fuzzy Transportation Problem

Let there be m supply points $(A_1, ..., A_m)$ and n demand points $(B_1, ..., B_n)$. The supplies and demands are given by the triangular fuzzy numbers $(\tilde{a}_i, i = 1, ..., m)$ and $(\tilde{b}_j, j = 1, ..., n)$, respectively, and the transportation costs are given by the grey numbers $\otimes c_{ij}, i = 1, ..., m, j = 1, ..., n$. By defuzzifying the fuzzy values and whitening the grey values, we will transform the problem into a classical transportation problem (Nasseri & Khabiri, 2019).

Grey Fuzzy Transportation Problem (GFTP):

$$\begin{aligned} \min Z &= \sum_{i=1}^{m} \sum_{j=1}^{n} \otimes c_{ij} x_{ij} \\ \text{s.t.} \sum_{j=1}^{n} x_{ij} &\leq \tilde{a}_i, i = 1, 2, 3, ..., m \\ \sum_{i=1}^{m} x_{ij} &\geq \tilde{b}_j, j = 1, 2, 3, ..., n \\ x_{ij} &\geq 0, i = 1, 2, 3, ..., m; j = 1, 2, 3, ..., n \end{aligned} \tag{3}$$

Here, x_{ij} is the quantity to be transported from point A_i to point B_j

4. Solution Approaches for the Grey Fuzzy Transportation Problem

This section of the research includes the Closed Path Approach proposed by Nasseri & Khabiri (2019), the Interval Optimization Approach, the Robust Optimization, and the Interval Optimization with Penalty Function methods.

4.1. Closed Path Method

Algorithm 1 step-by-step explanations summarize the main steps of the “Grey Fuzzy Transportation Problem Solver” approach. The code is created by implementing the approach proposed by Nasseri & Khabiri (2019). The authors find the initial solution by transforming the structure into a classical Transportation problem and using the North-West Corner Method or the Least Cost Cell method. Then, the Closed Path Approach is applied to the initial solution, and in each iteration, the improvement index for the Transportation Problem with uncertainty conditions is calculated to check if the solution has improved. The least-cost cell method is used as the initial solution method.

Algorithm 1. Closed Path Method Step-by-Step Explanation

1. Load the test data from the “testset.json” file.
2. Create an empty dictionary to store the results.
3. For each problem group:
   - Get the problem list.
   - Create an empty list to store the solutions.
   - For each problem:
     - Create the transportation data.
     - Solve the fuzzy situation and calculate the crisp supply/demand values.
     - Solve the grey situation and calculate the whitenized costs.
     - Find the initial solution.
     - Calculate the initial cost.
     - Find the optimal solution.
     - Calculate the optimal cost.
     - Calculate the improvement ratio.
     - Create the solution object.
     - Add the solution to the solutions list.
     - Generate a report.
   - Add the solutions to the results dictionary.
4. Return the results.

4.2. Interval Optimization

The mathematical model defining the Interval Optimization approach for the Grey Fuzzy Transportation Problem is provided. This model is designed to be solved using the Julia JuMP and SCIP open-source solvers. The detailed explanation of the Interval Optimization model for the Grey Fuzzy Transportation Problem is as follows:

Interval Optimization Mathematical Model:

- Decision Variables:
  - x_{ij}: The quantity transported from supply point i to demand point j
  - α_{ij}: The weighting factor between the lower and upper bounds of the transportation cost from supply point i to demand point j ($0 \leq \alpha_{ij} \leq 1$)
  - β_i: The weighting factor between the lower and upper bounds of supply point i ($0 \leq \beta_i \leq 1$)
  - γ_j: The weighting factor between the lower and upper bounds of demand point j ($0 \leq \gamma_j \leq 1$)
- Constraints:
  - Supply Constraint: For each supply point, the total quantity transported cannot exceed the weighted sum of the lower and upper bounds of the supply quantity $\sum_j x_{ij} \leq \beta_i * \mathrm{supply}_i[3] + (1 - \beta_i) * \mathrm{supply}_i[1], \forall i$
  - Demand Constraint: For each demand point, the total quantity transported must be greater than or equal to the weighted sum of the lower and upper bounds of the demand quantity $\sum_i x_{ij} \geq \gamma_j * \mathrm{demand}_j[1] + \left(1 - \gamma_j\right) * \mathrm{demand}_j[3], \forall j$
- Objective Function:

- To minimize the total transportation cost:
  $\min \sum_i \sum_j (\alpha_{ij} * \mathrm{costs}_{ij}[1] + (1-\alpha_{ij}) * \mathrm{costs}_{ij}[2]) * x_{ij}$

In the interval optimization model, lower and upper bounds are used for supply and demand quantities, as well as transportation costs. Values between these bounds are selected using weighting factors (α, β, γ). The objective is to minimize the total transportation cost.

Below is the formal structure of the mathematical model for the Interval Optimization approach:

Nomenclature:

- I: Supply points set ($i \in I$)
- J: Demand points set ($j \in J$)
- $[\underline{S}_i, \overline{S}_i]$: i-th supply point's lower and upper bounds
- $[\underline{D}_j, \overline{D}_j]$: j-th demand point's lower and upper bounds
- $[\underline{C}_{ij}, \overline{C}_{ij}]$: The lower and upper bounds of the unit transportation cost from the i-th supply point to the j-th demand point
- x_{ij}: The quantity transported from the i-th supply point to the j-th demand point
- α_{ij}: The weighting factor between the lower and upper bounds of the transportation cost from the i-th supply point to the j-th demand point ($0 \leq \alpha_{ij} \leq 1$)
- β_i: The weighting factor between the lower and upper bounds of the i-th supply point ($0 \leq \beta_i \leq 1$)
- γ_j: The weighting factor between the lower and upper bounds of the j-th demand point ($0 \leq \gamma_j \leq 1$)

Interval Optimization Formal Model:

- Objective Function:
  - $\min \sum_i \sum_j \alpha_{ij} * \underline{C}_{ij} + (1-\alpha_{ij}) * \overline{C}_{ij}) * x_{ij}$
- Constraints:
  - Supply Constraint:
    $\sum_j x_{ij} \leqslant \beta_i * \overline{S}_i + (1-\beta_i) * \underline{S}_i, \forall i \in I$
  - Demand Constraint:
    $\sum_i x_{ij} \geq \gamma_j * \underline{D}_j + (1-\gamma_j) * \overline{D}_j, \forall j \in J$
  - Non-negativity Constraints:
    $x_{ij} \geq 0, \forall i \in I, \forall j \in J$
  - Weighting Factors Constraints:
    $0 \leq \alpha_{ij} \leq 1, \forall i \in I, \forall j \in J$
    $0 \leq \beta_i \leq 1, \forall i \in I$
    $0 \leq \gamma_j \leq 1, \forall j \in J$

In the interval optimization model, lower and upper bounds are used for supply and demand quantities as well as transportation costs. By using weighting factors ($\alpha_{ij}, \beta_i, \gamma_j$), a value is selected within these bounds. The objective function aims to minimize the total transportation cost.

4.3. The Robust Optimization Model

- Decision Variables:
  x_{ij}: The quantity transported from supply point i to demand point j

α_{ij}: The weighting factor between the lower and upper bounds of the transportation cost from supply point i to demand point j ($0 \le \alpha_{ij} \le 1$)
- Constraints:
  - Supply Constraint: For each supply point, the total quantity transported cannot exceed the weighted sum of the lower and upper bounds of the supply quantity $\sum_j x_{ij} \leqslant \mathrm{supply}_i[3], \forall i$
  - Demand Constraint: For each demand point, the total quantity transported must be greater than or equal to the weighted sum of the lower and upper bounds of the demand quantity $\sum_i x_{ij} \ge \mathrm{demand}_j[1], \forall j$
- Objective Function: To minimize the total transportation cost: $\min \sum_i \sum_j \left(\alpha_{ij} * \mathrm{costs}_{ij}[1] + \left(1 - \alpha_{ij}\right) * \mathrm{costs}_{ij}[2]\right) * x_{ij}$

In the robust optimization model, upper bounds are used for supply quantities and lower bounds are used for demand quantities. A weighting factor (α) is again used for transportation costs. The objective is to minimize the total transportation cost even in the worst-case scenario.

Robust Optimization Formal Model:
- Objective Function: $\min \sum_i \sum_j (\alpha_{ij} * \underline{C}_{ij} + \left(1 - \alpha_{ij} * \overline{C}_{ij}\right) * x_{ij}$
- Constraints:
  - Supply Constraint: $\sum_j x_{ij} \leqslant \overline{S}_i, \forall i \in I$
  - Demand Constraint: $\sum_i x_{ij} \ge \underline{D}_j, \forall j \in J$
  - Non-negativity Constraints: $x_{ij} \ge 0, \forall i \in I, \forall j \in J$
  - Weighting Factors Constraints: $0 \le \alpha_{ij} \le 1, \forall i \in I, \forall j \in J$

In the robust optimization model, upper bounds are used for supply quantities and lower bounds are used for demand quantities. A weighting factor α_ij is also used for transportation costs. The objective function aims to minimize the total transportation cost even in the worst-case scenario.

These models can be used to address transportation problems involving uncertainty. Interval optimization expresses uncertainty as intervals, while robust optimization offers a more protective approach against uncertainty.

4.4. Interval Optimization Approach with Penalty Function

This model defines the interval optimization approach with a penalty function. Here, the decision variables ((x,α,β,γ)), constraints, and objective function are provided. A term that penalizes deviations from the mid-interval values is added to the objective function.

Nomenclature:

- J: $\mathrm{demandpointsset}(j \in J)$
- I: $\mathrm{supplypointsset}(i \in I)$
- $\left[\underline{S}_i, \overline{S}_i\right]$: Lower and upper bounds of the i-th supply point
- $\left[\underline{D}_j, \overline{D}_j\right]$: Lower and upper bounds of the j-th demand point

- $\left[\underline{C}_{ij}, \overline{C}_{ij}\right]$: Lower and upper bounds of the unit transportation cost from the i-th supply point to the j-th demand point
- x_{ij}: Quantity transported from the i-th supply point to the j-th demand point
- α_{ij}: Weighting factor between the lower and upper bounds of the transportation cost from the i -th supply point to the j-th demand point ($0 \leq \alpha_{ij} \leq 1$)
- β_i: Weighting factor between the lower and upper bounds of the i-th supply point ($0 \leq \beta_i \leq 1$)
- γ_j: Weighting factor between the lower and upper bounds of the j-th demand point ($0 \leq \gamma_j \leq 1$)
- λ: Penalty factor for deviations from the mid-interval values

Interval Optimization Approach with Penalty Function Mathematical Model:

- Objective Function: $\min \sum_i \sum_j \alpha_{ij} * \underline{C}_{ij} + (1 - \alpha_{ij} * \overline{C}_{ij} * x_{ij} + \lambda * \left(\sum_i \sum_j \left(\alpha_{ij} - 0.5\right)^2 + \sum_i \left(\beta_i - 0.5\right)^2 + \sum_j \left(\gamma_j - 0.5\right)^2\right)$
- Constraints:
  - Supply Constraint:
    $\sum_j x_{ij} \leqslant \beta_i * \overline{S}_i + (1 - \beta_i) * \underline{S}_i, \forall i \in I$
  - Demand Constraint:
    $\sum_i x_{ij} \geq \gamma_j * \underline{D}_j + \left(1 - \gamma_j\right) * \overline{D}_j, \forall j \in J$
  - Non-negativity Constraint: $x_{ij} \geq 0, \forall i \in I, \forall j \in J$
  - Weighting Factors Constraints: $0 \leq \alpha_{ij} \leq 1, \forall i \in I, \forall j \in J$ $0 \leq \beta_i \leq 1, \forall i \in I$ $0 \leq \gamma_j \leq 1, \forall j \in J$

The objective function consists of two components:

1. Total transportation cost:
   $\sum_i \sum_j (\alpha_{ij} * \underline{C}_{ij} + (1 - \alpha_{ij} * \overline{C}_{ij} * x_{ij}$
   Here, each transportation cost is calculated by interpolating between the lower and upper bounds using the corresponding weighting factor (α_{ij}).
2. Penalty for deviations from the mid-interval values:
   $\lambda * \left(\sum_i \sum_j \left(\alpha_{ij} - 0.5\right)^2 + \sum_i \left(\beta_i - 0.5\right)^2 + \sum_j \left(\gamma_j - 0.5\right)^2\right)$
   Here, the sum of the squares of the deviations of the weighting factors ($\alpha_{ij}, \beta_i, \gamma_j$) from 0.5 (the mid-value) is taken and multiplied by the penalty factor (λ). This term encourages the weighting factors to be close to the mid-interval values and prevents the selection of extreme values. The constraints ensure that the supply and demand remain within their bounds and guarantee that the weighting factors are between 0 and 1.

This model addresses transportation problems involving uncertainty by expressing transportation costs and supply/demand quantities as intervals. The objective is to minimize the total transportation cost while ensuring that the weighting factors are close to the mid-interval values. This balances the effect of uncertainty, resulting in more reliable solutions.

5. Computational Analysis

In this section, the analysis of the four proposed approaches for the Grey Fuzzy Transportation Problem will be conducted. All proposed algorithms were implemented using the Julia language, JuMP mathematical programming language, and the open-source solver SCIP. To analyze the algorithms, 40 synthetic Grey Fuzzy Transportation Problems were randomly generated. These problems were categorized into four groups: small, medium, large, and extra-large, with 10 problems in each group. The computer used for analyzing the test set is described as having 8 GB RAM, an Intel(R) Core™ i7-3520M CPU @ 2.90GHz, and the operating system Linux Mint 21.3/Virginia.

Table 2 provides the design parameters for the test problems. When looking at the design parameters of the test problems, the problem group 'small' is planned to have 10 problems, with problem sizes of 2x3, a supply range of (10.0, 50.0), a demand range of (5.0, 30.0), and unit variable cost ranges of (1.0, 20.0).

Table 2. Designed test instances parameters

Group	Number of Problems	Problem Size	Supply Range (Min, Max)	Demand Range (Min, Max)	Cost Range (Min, Max)
Small	10	2x3	(10.0,50.0)	(5.0,30.0)	(1.0,20.0)
Medium	10	4x5	(50.0, 200.0)	(30.0, 100.0)	(10.0, 50.0)
Large	10	6x7	(200.0, 500.0)	(100.0, 300.0)	(20.0, 100.0)
Extra-large	10	8x10	(500.0, 1000.0)	(300.0, 800.0)	(50.0, 200.0)

Table 3 provides the actual parameters for the generated test problems. When examining the generated test problems, the problem group 'small' has 10 problems, with problem sizes of 2x3, a supply range of (10.89, 49.97), a demand range of (6.26, 29.99), and a cost range of (1.43, 19.99).

Table 3. Produced test instances parameters

Group	Number of Problems	Problem Size	Supply Range (Min, Max)	Demand Range (Min, Max)	Cost Range (Min, Max)
Small	10	2x3	(10.89, 49.97)	(6.16, 29.99)	(1.43, 19.99)
Medium	10	4x5	(54.87, 199.94)	(30.04, 100.0)	(10.0, 50.0)
Large	10	6x7	(200.25, 499.97)	(102.19, 299.69)	(20.44, 99.99)
Extra-large	10	8x10	(500.87, 999.99)	(302.46, 799.43)	(50.02, 199.99)

Table 4 provides the solutions for 10 problems in the 'small' problem group. When examining the solutions for this group, the initial solutions and the closed path approaches are listed with the same results. This indicates that the closed path approach did not improve any solutions in this problem group. When looking at the mathematical programming approaches, it can be asserted that the Interval Optimization (IO), Robust Optimization (RO), and Interval Optimization with Penalty Function (IOWPF) approaches yielded the same results.

Table 4. Small size problems solutions with algorithms

No of Problem	Problem Size	Initial Cost	CP Cost	Imp (%)	IO Cost	RO Cost	IOWPF Cost
1	2x3	422.6	422.6	0	171.54	171.54	171.69
2	2x3	740.75	740.75	0	362.84	362.84	362.99
3	2x3	798.65	798.65	0	468.32	468.32	468.47
4	2x3	836.7	836.7	0	424.71	424.71	424.88
5	2x3	610.7	610.7	0	283.34	283.34	283.49
6	2x3	569.35	569.35	0	237.8	237.8	237.95
7	2x3	582.7	582.7	0	311.06	311.06	311.21
8	2x3	647.25	647.25	0	292.64	292.64	292.8
9	2x3	799.4	799.4	0	552.73	552.73	552.88
10	2x3	935.65	935.65	0	346.49	346.49	346.64

Table 5 provides the solutions for 10 problems in the 'medium' problem group. When examining the solutions for this group, it is observed that the closed path approach improved the solutions for all, but four problems compared to the initial solutions. When looking at the mathematical programming approaches, it can be asserted that the Interval Optimization (IO), Robust Optimization (RO), and Interval Optimization with Penalty Function (IOWPF) approaches yielded the same results. Additionally, these methods have produced significantly superior solutions compared to the closed path approach.

Table 5. Medium size problems solutions with algorithms

No of Problem	Problem Size	Initial Cost	CP Cost	Imp (%)	IO Cost	RO Cost	IOWPF Cost
1	4x5	10043.9	10011.2	0.33	4447.03	4447.03	4447.28
2	4x5	12796.85	12639.8	1.23	5854.59	5854.59	5854.89
3	4x5	12318.45	12318.45	0	5844.04	5844.04	5844.29
4	4x5	11253.65	10749.95	4.48	6241.67	6241.67	6241.97
5	4x5	13164.85	13066.4	0.75	6728.95	6728.95	6729.25
6	4x5	12736.5	12736.5	0	6800.01	6800.01	6800.31
7	4x5	14480.85	14466.9	0.1	8307.64	8307.64	8307.99
8	4x5	11465.55	10796.7	5.83	6369	6369	6369.25
9	4x5	10947.55	10865.05	0.75	5845.73	5845.73	5845.98
10	4x5	11493.55	11493.55	0	6217.61	6217.61	6217.86

Table 6 provides the solutions for 10 problems in the 'large' problem group. When examining the solutions for this group, it is observed that the closed path approach improved all solutions compared to the initial solutions. When looking at the mathematical programming approaches, it can be asserted that the Interval Optimization (IO), Robust Optimization (RO), and Interval Optimization with Penalty Function (IOWPF) approaches yielded the same results. Additionally, these methods have produced significantly superior solutions compared to the closed-path approach.

Table 6. Large size problems solutions with algorithms

No of Problem	Problem Size	Initial Cost	CP Cost	Imp (%)	IO Cost	RO Cost	IOWPF Cost
1	6x7	104024.45	102541.15	1.43	49822.87	49822.87	49823.32
2	6x7	105871.25	95250.25	10.03	54568.56	54568.56	54568.96
3	6x7	78391.9	72318.7	7.75	44861.31	44861.31	44861.66
4	6x7	82868.7	82847.8	0.03	44197.48	44197.48	44197.83
5	6x7	80411.95	79699.15	0.89	42602.21	42602.21	42602.61
6	6x7	98358.65	97120.2	1.26	49163.55	49163.55	49164.05
7	6x7	86516.95	85997.7	0.6	40124.66	40124.66	40125.16
8	6x7	89493.45	86491.25	3.35	45460.44	45460.44	45460.84
9	6x7	111170.3	101898.8	8.34	65531.9	65531.9	65532.35
10	6x7	102810.25	102483.7	0.32	44406.46	44406.46	44406.85

Table 7 provides the solutions for 10 problems in the 'extra large' problem group. When examining the solutions for this group, it is observed that the closed path approach improved all solutions except for one problem compared to the initial solutions. When looking at the mathematical programming approaches, it can be asserted that the Interval Optimization (IO), Robust Optimization (RO), and Interval Optimization with Penalty Function (IOWPF) approaches yielded the same results. Additionally, these methods have produced significantly superior solutions compared to the closed-path approach.

Table 7. Extra large size problems solutions with algorithms

No of Problem	Problem Size	Initial Cost	CP Cost	Imp (%)	IO Cost	RO Cost	IOWPF Cost
1	8x10	686598.85	686150.85	0.07	305855.81	305855.81	305856.38
2	8x10	672057.2	663530.3	1.27	403755.99	403755.99	403756.73
3	8x10	618454.3	610538.35	1.28	353941.8	353941.8	353942.44
4	8x10	679429.8	664164.45	2.25	322422.59	322422.59	322423.28
5	8x10	612326.95	610274.95	0.34	346980.83	346980.83	346981.52
6	8x10	678037.85	678037.85	0	380024.59	380024.59	380025.28
7	8x10	741827.65	717347.3	3.3	401703.37	401703.37	401704.12
8	8x10	621685.5	616008.3	0.91	334730.08	334730.08	334730.73
9	8x10	751334.95	723157.25	3.75	322584.71	322584.71	322585.45
10	8x10	631604.15	610758.7	3.3	355147.62	355147.62	355148.31

The variables used in Table 8 are defined as follows:

- T_0: Initial Solution Time (seconds)
- T_1: Closed Path Solution Time (seconds)
- T_2: Interval Optimization Solution Time (seconds)
- T_3: Robust Optimization Solution Time (seconds)
- T_4: Interval Optimization with Penalty Function Solution Time (seconds)

The maximum value for Initial Solution Time, T_0, has been determined as 0.0001 seconds in 40 problems. The remaining times have been recorded as 0.0000 seconds. Therefore, it has been excluded from Table 8. Table 8 is presented in four groups as (a), (b), (c), and (d). Each group shows the solution times in seconds for small (a), medium (b), large (c), and extra-large (d), respectively.

Table 8. Solution times of algorithms by problem groups (seconds)

	Problem	Problem Size	T1	T2	T3	T4
(a)	1	2x3	0.0072	0.0091	0.0072	0.049
	2	2x3	0.0058	0.0103	0.0068	0.1027
	3	2x3	0.0028	0.0158	0.0061	0.0697
	4	2x3	0.008	0.0075	0.0063	0.0487
	5	2x3	0.0082	0.0105	0.0079	0.0778
	6	2x3	0.0049	0.0083	0.0096	0.0848
	7	2x3	0.003	0.0093	0.0076	0.0635
	8	2x3	0.0037	0.0141	0.0074	0.0662
	9	2x3	0.0049	0.0128	0.0191	0.0741
	10	2x3	0.0076	0.0066	0.0278	0.0773
(b)	1	4x5	0.0147	0.012	0.0065	0.2199
	2	4x5	0.0188	0.0162	0.0133	0.1479
	3	4x5	0.0078	0.0175	0.0167	0.1161
	4	4x5	0.0209	0.0185	0.0105	0.1354
	5	4x5	0.0137	0.0167	0.0075	0.1018
	6	4x5	0.0037	0.0099	0.0119	0.144
	7	4x5	0.0206	0.0074	0.0101	0.2113
	8	4x5	0.0359	0.009	0.0083	0.1382
	9	4x5	0.0166	0.0075	0.0229	0.1258
	10	4x5	0.0067	0.0078	0.0145	0.1216
(c)	1	6x7	0.0357	0.0127	0.0231	0.3654
	2	6x7	0.187	0.0171	0.0093	0.2758
	3	6x7	0.0992	0.0109	0.0209	0.2788
	4	6x7	0.0481	0.0181	0.0114	0.2593
	5	6x7	0.0489	0.025	0.007	0.2382
	6	6x7	0.0783	0.0065	0.0145	0.3185
	7	6x7	0.0495	0.0105	0.0227	0.2721
	8	6x7	0.0589	0.0178	0.0203	0.2874
	9	6x7	0.1604	0.0161	0.0112	0.7377
	10	6x7	0.0582	0.0224	0.0064	0.1988
(d)	1	8x10	1.475	2.2918	0.0078	3.645
	2	8x10	0.2461	0.0066	0.0074	0.8444
	3	8x10	0.4781	0.0107	0.0129	1.0813
	4	8x10	0.1551	0.0378	0.0366	0.9456
	5	8x10	0.0689	0.0246	0.0271	0.6429
	6	8x10	0.0451	0.0181	0.0118	0.6398
	7	8x10	0.1862	0.0173	0.014	0.7765
	8	8x10	0.1479	0.0083	0.0238	0.7431

Problem	Problem Size	T1	T2	T3	T4
9	8x10	0.2543	0.0276	0.0076	0.826
10	8x10	0.4399	0.0172	0.0207	0.9082

Table 9 provides solutions for Problem 5 in the large problem group regarding the behavior of parameters in mathematical models. Additionally, Appendix 2 shares the detailed solution records for all solutions obtained with the Interval Optimization with Penalized Function algorithm concerning the α, β, and γ parameters on a problem-by-problem basis.

Table 9. Solution details and parameters for the algorithms

```
------------------------------------------------------------
Interval Optimization Solution Report
------------------------------------------------------------
Problem Group: large
Problem Size: 6x7
Problem No: 5
Best Objective Value: 42602.20999999999
Best Alpha: [1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0]
Best Beta: [1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
Best Gamma: [1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
Solution Time: 0.024971961975097656 seconds
----------------------------------------------------------
Robust Optimization Solution Report
----------------------------------------------------------
Problem Group: large
Problem Size: 6x7
Problem No: 5
Best Objective Value: 42602.20999999999
Best Alpha: [1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0;
             1.0 1.0 1.0 1.0 1.0 1.0 1.0]
Solution Time: 0.007044076919555664 seconds
--------------------------------------------------------------------------------------------------
Interval Optimization with Penalized Function Solution Report
--------------------------------------------------------------------------------------------------
Problem Group: large
Problem Size: 6x7
Problem No: 5
Best Objective Value: 42602.6
Best Alpha: [0.5 0.5 0.5 0.5 0.5 0.5 0.5;
             1.0 1.0 0.5 0.5 0.5 0.5 0.5;
             0.5 1.0 0.5 0.5 0.5 1.0 1.0;
             0.5 0.5 0.5 1.0 0.5 0.5 0.5;
             0.5 0.5 0.5 0.5 0.5 0.5 0.5;
```

```
             0.5 0.5 1.0 0.5 1.0 0.5 0.5]
Best Beta: [0.5, 1.0, 0.5, 0.5, 0.5, 0.5]
Best Gamma: [1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
Solution Time: 0.2382 seconds
```

Comparative comparison table according to T_3 solution time in Table 10. In this table, T_3 is considered as 1 unit; values less than 1 indicate better solution times, and values greater than 1 indicate higher solution times. A detailed comparative comparison table on a problem-by-problem basis is provided in Appendix 1.

Table 10. Comparative comparison table according to T_3 solution time

Group	Size	T1	T2	T3	T4	Units
Small	2x3	0.0056	0.0104	0.0106	0.0714	Average-seconds
	T_j/T_3	0.5	1	1	6.7	rate
Medium	4x5	0.0159	0.0123	0.0122	0.1462	Average-seconds
	T_j/T_3	1.3	1	1	12	rate
Large	6x7	0.0824	0.0157	0.0147	0.3232	Average-seconds
	T_j/T_3	5.6	1.1	1	22	rate
extra-large	8x10	0.3497	0.246	0.017	1.1053	Average-seconds
	T_j/T_3	20.6	14.5	1	65.1	rate

Experimental studies conducted on a test set of 40 problems have observed that all methods provided better results compared to the initial solution. Interval Optimization and Robust Optimization methods achieved the same optimal cost in most of the tested problems, while the Interval Optimization with Penalty Function method produced slightly higher-cost solutions compared to the other methods. The Closed Path Method showed improvements ranging from 0.0% to 3.75% compared to the initial solution in the extra-large problem group, with improvements of 0.0% to 4.48% in the medium group, and 0.03% to 10.03% in the large group.

In terms of computation time, the comparative time coefficients of the Robust Optimization algorithm varied depending on the problem size. For small and medium-sized problems, the Interval Optimization and Robust Optimization methods showed similar computation times, while for large and extra-large problems, the Robust Optimization method provided faster results. The Closed Path Algorithm and Interval Optimization with Penalty Function methods became computationally disadvantaged as the problem size increased. When compared to the Robust Optimization algorithm, the Closed Path Algorithm, Interval Optimization, Robust Optimization, and Interval Optimization with Penalty Function methods had comparative time coefficients of 0.5, 1.0, 1.0, and 6.7 for small problems, 1.3, 1.0, 1.0, and 12.0 for medium problems, 5.6, 1.1, 1.0, and 22.0 for large problems, and 20.6, 14.5, 1.0, and 65.1 for extra-large problems, respectively.

In conclusion, among the four methods examined, Interval Optimization and Robust Optimization methods were found to be more effective in solving the Grey Fuzzy Transportation Problem (GFTP). However, it is thought that the performance of the Interval Optimization with Penalty Function method could be improved by dynamically adjusting the weight factors.

The Interval Optimization with Penalty Function method uses lower and upper bounds for supply and demand quantities and transportation costs to solve the GFTP. The effectiveness of the method depends on the appropriate assignment of weight factors (α, β, γ). The method manages uncertainty by assigning weight factors (α, β, γ) and adds a term to the objective function that penalizes deviations from the middle interval values. In all problem sizes, the demand weight factor (γ) values were consistently set to 1.0, indicating that the method focuses on the upper bound of demand quantities. The transportation cost weight factor (α) values were balanced between 0.5 and 1.0 in all problem sizes, indicating that the method balances the lower and upper bounds of transportation costs. The supply weight factor (β) values were mostly set to 0.5 or 1.0, but some intermediate values such as 0.59, 0.65, 0.81, 0.92, and 0.99 were also observed, showing that the method adapts to the characteristics of the problems. In terms of solution time, it was observed that the computation time of the Interval Optimization with the Penalty Function method increased as the problem size increased. However, even for the largest problems, the average solution time was approximately 1 second, indicating that the method can solve large-scale problems in a reasonable time.

6. Conclusion and Recommendations

The Grey Fuzzy Transportation Problem (GFTP) is a model developed to address transportation problems involving uncertainty. In this study, four different optimization methods (Closed Path Method, Interval Optimization, Robust Optimization, and Interval Optimization with Penalty Function) were used to solve the GFTP, and their performances were comprehensively analyzed. The analyses were conducted on a total of 40 test problems across four different problem sizes: small, medium, large, and extra-large. The results showed that the Interval Optimization and Robust Optimization methods demonstrated the best performance in terms of solution quality and computation time.

Detailed analyses focusing on the Interval Optimization with the Penalty Function method revealed that this method provides an effective and consistent solution approach for the GFTP. The method focuses on the upper bound of demand quantities while adopting a more balanced approach for supply quantities and transportation costs. In assigning weight factors (α, β, γ), the method adapts to the characteristics of the problems. Additionally, it consistently performs well across different problem sizes, successfully managing uncertainty. Although the solution times increase with problem size, even the largest problems can be solved in a reasonable time.

This study highlights the potential of the Interval Optimization with Penalty Function method for solving the GFTP, while also showing that other methods can be effective. Future research can focus on determining the optimal value of the penalty factor, exploring different weight factor assignment strategies, and applying the methods to real-world problems. Comparative analyses with other fuzzy optimization methods will also help determine the most suitable solution approaches for the GFTP. The comprehensive analysis and suggestions provided by this study will contribute to the effective solution of the GFTP and better management of uncertain transportation problems.

It can be argued that the solution approaches used in this study can be used for the Fixed-Charge Transportation Problem (FCTP) in addition to the Transportation Problem. Analyses can be improved by adapting the proposed solution approaches for FCTP.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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Appendix

Appendix 1. Detailed Solution Times of all test problems and Time Comparisons

Problem	Problem Size	T1	T2	T3	T4	T1/T3	T2/T3	T3/T3	T4/T3
1	2x3	0.0072	0.0091	0.0072	0.049	1	1.3	1	6.8
2	2x3	0.0058	0.0103	0.0068	0.1027	0.9	1.5	1	15.1
3	2x3	0.0028	0.0158	0.0061	0.0697	0.5	2.6	1	11.4
4	2x3	0.008	0.0075	0.0063	0.0487	1.3	1.2	1	7.7
5	2x3	0.0082	0.0105	0.0079	0.0778	1	1.3	1	9.8
6	2x3	0.0049	0.0083	0.0096	0.0848	0.5	0.9	1	8.8
7	2x3	0.003	0.0093	0.0076	0.0635	0.4	1.2	1	8.4
8	2x3	0.0037	0.0141	0.0074	0.0662	0.5	1.9	1	8.9
9	2x3	0.0049	0.0128	0.0191	0.0741	0.3	0.7	1	3.9
10	2x3	0.0076	0.0066	0.0278	0.0773	0.3	0.2	1	2.8
Average		0.0056	0.0104	0.0106	0.0714	0.5	1	1	6.7
1	4x5	0.0147	0.012	0.0065	0.2199	2.3	1.8	1	33.8
2	4x5	0.0188	0.0162	0.0133	0.1479	1.4	1.2	1	11.1
3	4x5	0.0078	0.0175	0.0167	0.1161	0.5	1	1	7
4	4x5	0.0209	0.0185	0.0105	0.1354	2	1.8	1	12.9
5	4x5	0.0137	0.0167	0.0075	0.1018	1.8	2.2	1	13.6
6	4x5	0.0037	0.0099	0.0119	0.144	0.3	0.8	1	12.1
7	4x5	0.0206	0.0074	0.0101	0.2113	2	0.7	1	20.9
8	4x5	0.0359	0.009	0.0083	0.1382	4.3	1.1	1	16.7
9	4x5	0.0166	0.0075	0.0229	0.1258	0.7	0.3	1	5.5
10	4x5	0.0067	0.0078	0.0145	0.1216	0.5	0.5	1	8.4
Average		0.0159	0.0123	0.0122	0.1462	1.3	1	1	12
1	6x7	0.0357	0.0127	0.0231	0.3654	1.5	0.5	1	15.8
2	6x7	0.187	0.0171	0.0093	0.2758	20.1	1.8	1	29.7
3	6x7	0.0992	0.0109	0.0209	0.2788	4.7	0.5	1	13.3
4	6x7	0.0481	0.0181	0.0114	0.2593	4.2	1.6	1	22.7
5	6x7	0.0489	0.025	0.007	0.2382	7	3.6	1	34
6	6x7	0.0783	0.0065	0.0145	0.3185	5.4	0.4	1	22
7	6x7	0.0495	0.0105	0.0227	0.2721	2.2	0.5	1	12
8	6x7	0.0589	0.0178	0.0203	0.2874	2.9	0.9	1	14.2
9	6x7	0.1604	0.0161	0.0112	0.7377	14.3	1.4	1	65.9
10	6x7	0.0582	0.0224	0.0064	0.1988	9.1	3.5	1	31.1
Average		0.0824	0.0157	0.0147	0.3232	5.6	1.1	1	22
1	8x10	1.475	2.2918	0.0078	3.645	189.1	293.8	1	467.3
2	8x10	0.2461	0.0066	0.0074	0.8444	33.3	0.9	1	114.1
3	8x10	0.4781	0.0107	0.0129	1.0813	37.1	0.8	1	83.8

T1: Closed Path Solution Time (seconds) T2: Interval Optimization Solution Time (seconds) T3: Robust Optimization Solution Time (seconds) T4: Interval Optimization with Penalty Function Solution Time (seconds)

Problem	Problem Size	T1	T2	T3	T4	T1/T3	T2/T3	T3/T3	T4/T3
4	8x10	0.1551	0.0378	0.0366	0.9456	4.2	1	1	25.8
5	8x10	0.0689	0.0246	0.0271	0.6429	2.5	0.9	1	23.7
6	8x10	0.0451	0.0181	0.0118	0.6398	3.8	1.5	1	54.2
7	8x10	0.1862	0.0173	0.014	0.7765	13.3	1.2	1	55.5
8	8x10	0.1479	0.0083	0.0238	0.7431	6.2	0.3	1	31.2
9	8x10	0.2543	0.0276	0.0076	0.826	33.5	3.6	1	108.7
10	8x10	0.4399	0.0172	0.0207	0.9082	21.3	0.8	1	43.9
Average		0.3497	0.246	0.017	1.1053	20.6	14.5	1	65.1

T1: Closed Path Solution Time (seconds) T2: Interval Optimization Solution Time (seconds) T3: Robust Optimization Solution Time (seconds) T4: Interval Optimization with Penalty Function Solution Time (seconds)

Appendix 2. Behaviors of mathematical model parameters for the solutions of the test problem instances

Group	Size	Problem	α11	α12	α13	α21	α22	α23
S	2x3	1	1	0.5	1	0.5	1	0.5
S	2x3	2	0.5	1	1	1	0.5	0.5
S	2x3	3	1	1	0.5	0.5	0.5	1
S	2x3	4	1	0.5	0.5	0.5	1	1
S	2x3	5	1	1	0.5	0.5	0.5	1
S	2x3	6	0.5	0.5	0.5	1	1	1
S	2x3	7	0.5	1	0.5	1	0.5	1
S	2x3	8	1	1	0.5	0.5	0.5	1
S	2x3	9	0.5	0.5	0.5	1	1	1
S	2x3	10	1	1	0.5	0.5	0.5	1

Group	Size	Problem	α11	α12	α13	α14	α15	α21	α22	α23	α24	α25
M	4x5	1	0.5	1	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5
M	4x5	2	1	1	0.5	0.5	1	0.5	0.5	1	0.5	0.5
M	4x5	3	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
M	4x5	4	0.5	1	0.5	0.5	0.5	0.5	0.5	1	1	0.5
M	4x5	5	1	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5
M	4x5	6	0.5	0.5	1	0.5	0.5	0.5	1	0.5	1	0.5
M	4x5	7	1	1	0.5	1	0.5	0.5	0.5	0.5	1	0.5
M	4x5	8	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5
M	4x5	9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	1	0.5
M	4x5	10	0.5	0.5	0.5	0.5	0.5	1	0.5	1	0.5	0.5

Group	Size	Problem	α31	α32	α33	α34	α35	α41	α42	α43	α44	α45
M	4x5	1	1	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5
M	4x5	2	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5
M	4x5	3	0.5	0.5	1	0.5	1	0.5	1	0.5	1	0.5
M	4x5	4	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1	1
M	4x5	5	0.5	0.5	1	0.5	0.5	0.5	1	0.5	1	1
M	4x5	6	1	0.5	1	0.5	1	0.5	0.5	0.5	0.5	0.5

S: Small, M: Medium, L: Large, XL: Extra Large

M	4x5	7	0.5	0.5	1	0.5	0.5	0.5	1	0.5	0.5	1				
M	4x5	8	0.5	0.5	0.5	0.5	0.5	1	1	1	0.5	0.5				
M	4x5	9	0.5	1	0.5	0.5	1	1	0.5	0.5	0.5	0.5				
M	4x5	10	0.5	1	0.5	0.5	1	0.5	0.5	0.5	1	0.5				
Group	Size	Problem	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}	α_{21}	α_{22}	α_{23}	α_{24}	α_{25}	α_{26}	α_{27}
L	6x7	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
L	6x7	2	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1
L	6x7	3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
L	6x7	4	1	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5
L	6x7	5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5
L	6x7	6	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	1	1	0.5	1	1	0.5
L	6x7	7	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1
L	6x7	8	0.5	0.5	0.5	0.5	1	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5
L	6x7	9	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	1	0.5
L	6x7	10	1	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5
Group	Size	Problem	α_{31}	α_{32}	α_{33}	α_{34}	α_{35}	α_{36}	α_{37}	α_{41}	α_{42}	α_{43}	α_{44}	α_{45}	α_{46}	α_{47}
L	6x7	1	0.5	0.5	1	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
L	6x7	2	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
L	6x7	3	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5
L	6x7	4	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
L	6x7	5	0.5	1	0.5	0.5	0.5	1	1	0.5	0.5	0.5	1	0.5	0.5	0.5
L	6x7	6	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
L	6x7	7	0.5	1	0.5	1	1	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5
L	6x7	8	0.5	0.5	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
L	6x7	9	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1
L	6x7	10	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5
Group	Size	Problem	α_{51}	α_{52}	α_{53}	α_{54}	α_{55}	α_{56}	α_{57}	α_{61}	α_{62}	α_{63}	α_{64}	α_{55}	α_{66}	α_{67}
L	6x7	1	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5	1	0.5
L	6x7	2	0.5	1	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5
L	6x7	3	0.5	0.5	1	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5
L	6x7	4	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1
L	6x7	5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	1	0.5	0.5
L	6x7	6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5	1
L	6x7	7	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	1	0.5
L	6x7	8	0.5	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
L	6x7	9	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1
L	6x7	10	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1
Group	Size	Problem	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}	α_{18}	α_{19}	α_{110}				
XL	8x10	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1				
XL	8x10	2	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5	0.5				
XL	8x10	3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5				
XL	8x10	4	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	1				
XL	8x10	5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5				
XL	8x10	6	0.5	0.5	0.5	0.5	1	0.5	1	0.5	0.5	0.5				

S: Small, M: Medium, L: Large, XL: Extra Large

XL	8x10	7	1	0.5	1	0.5	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	8	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	9	1	0.5	1	0.5	1	0.5	0.5	0.5	0.5	0.5
XL	8x10	10	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1
Group	Size	Problem	α21	α22	α23	α24	α25	α26	α27	α28	α29	α210
XL	8x10	1	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	2	1	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5
XL	8x10	3	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	4	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	6	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	7	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	8	0.5	0.5	1	0.5	1	0.5	1	0.5	0.5	0.5
XL	8x10	9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	10	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5
Group	Size	Problem	α31	α32	α33	α34	α35	α36	α37	α38	α39	α310
XL	8x10	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5
XL	8x10	2	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	3	0.5	0.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5
XL	8x10	4	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	6	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5	0.5
XL	8x10	7	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	1	0.5
XL	8x10	8	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	9	0.5	0.5	1	0.5	0.5	1	0.5	1	0.5	0.5
XL	8x10	10	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	1
Group	Size	Problem	α41	α42	α43	α44	α45	α46	α47	α48	α49	α410
XL	8x10	1	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5
XL	8x10	2	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	3	0.5	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	4	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	5	0.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5
XL	8x10	6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	7	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1
XL	8x10	8	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	9	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	10	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5
Group	Size	Problem	α51	α52	α53	α54	α55	α56	α57	α58	α59	α510
XL	8x10	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	2	0.5	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5
XL	8x10	3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	4	1	0.5	0.5	0.5	0.5	1	0.5	1	1	0.5
XL	8x10	5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1
XL	8x10	6	0.5	0.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5

S: Small, M: Medium, L: Large, XL: Extra Large

XL	8x10	7	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	8	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	9	0.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5	1
XL	8x10	10	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5	0.5
Group	Size	Problem	α61	α62	α63	α64	α55	α66	α67	α68	α69	α610
XL	8x10	1	0.5	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	2	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5
XL	8x10	3	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	4	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	6	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	7	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5
XL	8x10	8	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	1	0.5
XL	8x10	9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1
XL	8x10	10	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
Group	Size	Problem	α71	α72	α73	α74	α75	α76	α77	α78	α79	α710
XL	8x10	1	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
XL	8x10	2	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	3	0.5	0.5	0.5	0.5	1	0.5	1	0.5	0.5	0.5
XL	8x10	4	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5
XL	8x10	5	1	1	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
XL	8x10	6	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	1	0.5
XL	8x10	7	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	8	0.5	0.5	0.5	0.5	0.5	0.5	1	1	0.5	0.5
XL	8x10	9	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5
XL	8x10	10	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5
Group	Size	Problem	α81	α82	α83	α84	α85	α86	α87	α88	α89	α810
XL	8x10	1	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	2	1	0.5	0.5	0.5	0.5	0.5	1	1	0.5	0.5
XL	8x10	3	0.5	0.5	0.5	1	0.5	1	0.5	0.5	1	0.5
XL	8x10	4	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
XL	8x10	5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.5
XL	8x10	6	1	0.5	0.5	0.5	1	0.5	0.5	1	0.5	1
XL	8x10	7	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5	0.5
XL	8x10	8	1	0.5	0.5	0.5	0.5	1	0.5	0.5	0.5	0.5
XL	8x10	9	0.5	0.5	0.5	0.5	0.5	1	0.5	0.5	1	0.5
XL	8x10	10	1	0.5	0.5	1	0.5	0.5	0.5	0.5	0.5	0.5
Group	Size	Problem	β1	β2	γ1	γ2	γ3					
S	2x3	1	0.5	0.5	1	1	1					
S	2x3	2	0.5	0.5	1	1	1					
S	2x3	3	0.5	0.5	1	1	1					
S	2x3	4	0.5	0.94	1	1	1					
S	2x3	5	0.5	0.5	1	1	1					
S	2x3	6	0.5	0.65	1	1	1					

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S	2x3	7	0.5	0.5	1	1	1													
S	2x3	8	0.81	0.5	1	1	1													
S	2x3	9	0.5	0.5	1	1	1													
S	2x3	10	0.5	0.5	1	1	1													
Group	Size	Problem	β1	β2	β3	β4	γ1	γ2	γ3	γ4	γ5									
M	4x5	1	0.5	0.5	0.5	0.5	1	1	1	1	1									
M	4x5	2	1	0.5	0.5	0.5	1	1	1	1	1									
M	4x5	3	0.5	0.5	0.5	0.5	1	1	1	1	1									
M	4x5	4	0.5	0.5	0.5	1	1	1	1	1	1									
M	4x5	5	0.59	0.5	0.5	1	1	1	1	1	1									
M	4x5	6	0.5	0.5	1	0.5	1	1	1	1	1									
M	4x5	7	1	0.5	0.5	1	1	1	1	1	1									
M	4x5	8	0.5	0.5	0.5	0.5	1	1	1	1	1									
M	4x5	9	0.5	0.5	0.5	0.5	1	1	1	1	1									
M	4x5	10	0.5	0.5	0.5	0.5	1	1	1	1	1									
Group	Size	Problem	β1	β2	β3	β4	β5	β6	γ1	γ2	γ3	γ4	γ5	γ6	γ7					
L	6x7	1	0.5	0.5	1	0.5	1	0.5	1	1	1	1	1	1	1					
L	6x7	2	0.5	1	0.5	0.5	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	3	0.5	0.5	0.5	0.5	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	4	0.5	0.5	0.5	0.5	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	5	0.5	1	0.5	0.5	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	6	0.5	1	1	0.5	0.5	1	1	1	1	1	1	1	1					
L	6x7	7	0.5	1	0.5	1	1	0.5	1	1	1	1	1	1	1					
L	6x7	8	0.5	0.5	1	0.5	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	9	0.5	1	0.5	1	0.5	0.5	1	1	1	1	1	1	1					
L	6x7	10	0.92	0.5	0.5	0.5	0.5	0.99	1	1	1	1	1	1	1					
Group	P. Size	Problem	β1	β2	β3	β4	β5	β6	β7	β8	γ1	γ2	γ3	γ4	γ5	γ6	γ7	γ8	γ9	γ10
XL	8x10	1	0.5	0.8	0.5	0.85	0.5	1	0.5	0.5	1	1	1	1	1	1	1	1	1	1
XL	8x10	2	1	0.5	0.5	1	1	1	0.5	1	1	1	1	1	1	1	1	1	1	1
XL	8x10	3	0.5	0.5	1	1	0.5	0.5	1	0.5	1	1	1	1	1	1	1	1	1	1
XL	8x10	4	0.5	0.5	0.5	1	1	1	1	0.5	1	1	1	1	1	1	1	1	1	1
XL	8x10	5	0.5	0.5	1	1	1	0.5	1	0.5	1	1	1	1	1	1	1	1	1	1
XL	8x10	6	0.5	0.5	1	0.5	1	0.5	1	1	1	1	1	1	1	1	1	1	1	1
XL	8x10	7	1	1	0.5	1	1	1	0.5	0.8	1	1	1	1	1	1	1	1	1	1
XL	8x10	8	0.5	1	1	0.5	0.5	0.5	1	0.71	1	1	1	1	1	1	1	1	1	1
XL	8x10	9	1	0.5	1	1	1	0.5	0.5	1	1	1	1	1	1	1	1	1	1	1
XL	8x10	10	1	1	1	0.5	1	0.5	0.5	0.5	1	1	1	1	1	1	1	1	1	1

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