Turk. J. Math. Comput. Sci.
16(2)(2024) 333–345
© MatDer
DOI : 10.47000/tjmcs.1505489

B-Fractional Integrals on Variable Lebesgue Spaces

Esra Kaya

Department of Mathematics, Faculty of Science, Bilecik Şeyh Edebali University, 11100 Bilecik, Turkey.

Received: 26-06-2024 • Accepted: 08-10-2024

Abstract. Here, the fractional integral operators which are generated by Laplace-Bessel differential operator will be examined. It will also be shown that M_ν^α, $I_\nu^\alpha : L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+}) \to L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})$ are bounded, where M_ν^α is B-fractional maximal operator, I_ν^α is B-Riesz potential and $\dfrac{1}{p(\cdot)} - \dfrac{1}{q(\cdot)} = \dfrac{\alpha}{Q}$.

2020 AMS Classification: 42B20, 42B25, 46E30

Keywords: Fractional maximal operator, generalized translation operator, Riesz potential, variable Lebesgue space.

1. Introduction

Fractional maximal and Riesz potential operators, indeed, fractional integral operators have big importance in harmonic analysis, PDEs and theory of functions. Fractional maximal operators, Riesz potentials have been introduced by Muckenhoupt and Wheeden [26], Riesz [27], respectively. Then, these operators acting from Lebesgue space and weighted Lebesgue space into itself have been studied by Stein [30], Muckenhoupt and Wheeden [26].

Nowadays, there is a big attention to Riesz potential and fractional maximal operators on variable Lebesgue spaces. On these spaces, the problem of boundedness of classical singular integral, maximal, fractional maximal and Riesz potential operators and related topics have been investigated in [2, 3, 6–10, 28].

As it is well-known Laplace-Bessel differential operator is defined by

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\nu_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \le k \le n,$$

and here we consider the fractional integral operators which are generated by Δ_B. On various function spaces, B-Riesz potential and B-fractional maximal operators have been studied by many researchers [1, 4, 5, 13, 15–17, 20, 29, 31, 32]. From this perspective, we generalize above results in variable Lebesgue spaces. We obtain that B-fractional maximal and B-Riesz potential operators are bounded from $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ to $L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})$. Our results on fractional integral operators in variable Lebesgue spaces are key tools to solve PDEs of mathematical physics and inverse problems. Therefore, they will make enough contribution to the existing literature.

Throughout the paper A will be a constant.

Email address: esra.kaya@bilecik.edu.tr (E. Kaya)

2. Preliminaries

We first give some basic concepts, notations and known results which are beneficial for us.

Let $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0, 1 \le k \le n\}$, $\nu = (\nu_1, \ldots, \nu_k)$, $\nu_1 > 0, \ldots, \nu_k > 0$, $|\nu| = \nu_1 + \ldots + \nu_k$, $(x')^\nu = x_1^{\nu_1} \cdots x_k^{\nu_k}$, $(x')^\nu dx = x_1^{\nu_1} \cdots x_k^{\nu_k} dx_1 \cdots dx_k$, and $B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. The measure of a measurable set $A \subset \mathbb{R}^n_{k,+}$ is defined as $|A|_\nu = \int_A (x')^\nu dx$. If $B_+(0, r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0, r)|_\nu = \int_{B_+(0,r)} (x')^\nu dx = \omega(n, k, \nu) r^Q,$$

where $\omega(n, k, \nu) = \dfrac{\pi^{\frac{n-k}{2}}}{2^k} \displaystyle\prod_{i=1}^{k} \dfrac{\Gamma\left(\frac{\nu_i+1}{2}\right)}{\Gamma\left(\frac{\nu_i}{2}\right)}$, $Q = n + |\nu|$. We denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n_{k,+}$ by $\mathcal{S}_+(\mathbb{R}^n_{k,+})$ and the dual space of all tempered distributions on $\mathbb{R}^n_{k,+}$ by $\mathcal{S}'_+(\mathbb{R}^n_{k,+})$. The Fourier-Bessel transform of a function $f \in \mathcal{S}_+(\mathbb{R}^n_{k,+})$ is defined as

$$\mathcal{F}_\nu(f)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) e^{-i(x', y')} \prod_{i=1}^{k} j_{\nu_i - \frac{1}{2}}(x'' y'') (y')^\nu dy,$$

where $(x', y') = x_1 y_1 + \ldots + x_k y_k$, j_ν is the normalized Bessel function of the first kind

$$j_\nu(t) = 2^\nu \Gamma(\nu + 1) \frac{J_\nu(t)}{t^\nu},$$

and $J_\nu(x)$ is the Bessel function of the first kind [21, 25].

The definition of generalized translation operator is as follows:

$$T^y f(x) := A_{\nu,k} \int_0^\pi \ldots \int_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \ldots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\nu(\alpha).$$

Here, $A_{\nu,k} = \pi^{-\frac{k}{2}} \Gamma(\frac{\nu_j+1}{2})[\Gamma(\frac{\nu_j}{2})]^{-1}$, $(x_j, y_j)_{\alpha_j} = (x_j^2 - 2x_j y_j \cos\alpha_j + y_j^2)^{\frac{1}{2}}$, $1 \le j \le k$, $1 \le k \le n$, and $d\nu(\alpha) = \prod_{j=1}^{k} \sin^{\nu_j - 1} \alpha_j \, d\alpha_j$ [22, 24]. Observe that generalized translation operator is related to Laplace-Bessel differential operator. The following is well known from [24]:

$$\int_{\mathbb{R}^n_{k,+}} |T^y f(x)| (y')^\nu dy \le \int_{\mathbb{R}^n_{k,+}} |f(y)| (y')^\nu dy. \tag{2.1}$$

B-convolution operator connected with T^y is given by

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\nu dy.$$

We will now recall the spaces $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ and their fundamental properties.

Let $\mathcal{P}(\mathbb{R}^n_{k,+}) = \left\{p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty] : p(\cdot) \text{ is measurable}\right\}$. Any element of $\mathcal{P}(\mathbb{R}^n_{k,+})$ is said to be a variable exponent function and also let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n_{k,+}} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} p(x).$$

If $p(\cdot)$ satisfies

$$|p(x) - p(y)| \le \frac{A_0}{-\log|x - y|}, \tag{2.2}$$

and

$$|p(x) - p_\infty| \le \frac{A_\infty}{\log(e + |x|)}, \tag{2.3}$$

for all $|x-y| \leq \frac{1}{2}$, $x, y \in \mathbb{R}^n_{k,+}$, then it is denoted by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, and $p(\cdot) \in \mathcal{P}^{\log}_{\infty}(\mathbb{R}^n_{k,+})$, respectively. Here, $p_\infty = \lim_{x\to\infty} p(x) > 1$. Moreover, if $p(\cdot)$ provides log-Hölder continuity conditions both locally and at infinity, then it is denoted by $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$, i.e $LH(\mathbb{R}^n_{k,+})$ consists of $p(\cdot)$ such that (2.2) and (2.3) are satisfied together. Conjugate exponent function is given by

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1, \quad x \in \mathbb{R}^n_{k,+},$$

for a given $p(\cdot)$.

Let f be a measurable function and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. Then, variable Lebesgue space is defined as follows:

$$L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+}) := \left\{ f : \ \|f\|_{L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})} = \inf \left\{ \mu > 0 : \ \varrho_{p(\cdot),\nu}(f/\mu) \leq 1 \right\} < \infty \right\},$$

where

$$\varrho_{p(\cdot),\nu}(f) := \int_{\mathbb{R}^n_{k,+} \setminus (\mathbb{R}^n_{k,+})_\infty} |f(x)|^{p(x)} (x')^\nu dx + \|f\|_{L_{\infty,\nu}((\mathbb{R}^n_{k,+})_\infty)} < \infty.$$

Note that $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ are Banach spaces for $1 < p_- \leq p(x) \leq p_+ < \infty$.

The following lemma is analog of Hölder's inequality for $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$.

Lemma 2.1 ([11]). *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$. Then,*

$$\int_{\mathbb{R}^n_{k,+}} |f(x)h(x)| (x')^\nu dx \leq A \|f\|_{p(\cdot),\nu} \|h\|_{p'(\cdot),\nu},$$

holds for all $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ and $h \in L_{p'(\cdot),\nu}(\mathbb{R}^n_{k,+})$, where $A = A(p(\cdot), p'(\cdot), \nu) > 0$ is a constant.

Now, we give the analogs of fundamental results dealing with variable $L_{p(\cdot),\nu}$ spaces. One can obtain their proofs in a similar way as in [23].

Lemma 2.2. *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ and $p_+ < \infty$. Then, the followings are equivalent:*

(i) $\|f\|_{p(\cdot),\nu} < A_1$,

(ii) $\int_{\mathbb{R}^n_{k,+}} |f(y)|^{p(y)} (y')^\nu dy < A_2$.

If one of the constant is 1, then we can pick the other one is also 1.

Lemma 2.3. *Let $p(\cdot)$, $q(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$. If $p(x) \leq q(x)$, then $\|f\|_{p(\cdot),\nu} \leq A \|f\|_{q(\cdot),\nu}$.*

Lemma 2.4 ([12]). *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$. Then, $p(\cdot)$ satisfying (2.2) is uniformly continuous if and only if $|B_+|_\nu^{p_- - p_+} \leq A$ for every open balls B_+.*

The next lemma is the analog lemma due to Diening [9].

Lemma 2.5. *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ be an exponent function satisfying (2.2) and $p_+ < \infty$. Then,*

$$\|\chi_{B_+}\|_{p(\cdot),\nu} \leq A\, |B_+|_\nu^{1/p(\cdot)}$$

holds for all open balls B_+ with $0 < |B_+|_\nu \leq 1$.

Proof. Let $p_+ < \infty$ and $|B_+|_\nu \leq 1$. Then, the $L_{p(\cdot),\nu}$ norm of χ_{B_+} is

$$\begin{aligned}
\|\chi_{B_+}\|_{p(\cdot),\nu} &= \inf \left\{ \mu > 0 : \int_{B_+} \mu^{-p(x)} (x')^\nu dx \leq 1 \right\} \\
&= \inf \left\{ 0 < \mu < 1 : \int_{B_+} \mu^{-p(x)} (x')^\nu dx \leq 1 \right\} \\
&\leq \inf \left\{ 0 < \mu < 1 : \int_{B_+} \mu^{-p_+} (x')^\nu dx \leq 1 \right\} = |B_+|_\nu^{1/p(\cdot)},
\end{aligned}$$

and using Lemma 2.4, we find

$$\begin{aligned}|B_+|_\nu^{\frac{1}{p_+}} &= |B_+|_\nu^{\frac{1}{p(x)}}|B_+|_\nu^{\frac{1}{p_+}-\frac{1}{p(x)}}\\ &\le |B_+|_\nu^{\frac{1}{p(x)}}|B_+|_\nu^{\frac{p_--p_+}{p_-^2}} \le A\,|B_+|_\nu^{\frac{1}{p(x)}}.\end{aligned}$$

Therefore, the proof is completed. □

The following lemma allows us to use the condition $LH(\mathbb{R}^n_{k,+})$ to replace a variable exponent function with a constant exponent.

Lemma 2.6. *Let F be a set, $p(\cdot)$ and $q(\cdot)$ be two nonnegative functions and let $\beta_t(x) = (e+|x|)^{-tn}$ for $t>0$. Suppose that*

$$|p(x)-q(x)| \le \frac{A}{\log(e+|x|)},$$

for each $x \in F$. Then, there exists a constant $A = A(t,\nu)$ such that

$$\int_F |f(x)|^{q(x)}(x')^\nu dx \le A\int_F |f(x)|^{p(x)}(x')^\nu dx + \int_F \beta_t(x)^{q_-(F)}(x')^\nu dx,$$

for every function f with $|f(x)| \le 1$ and $x \in F$.

Proof. Define $F^{\beta_t} = \{x \in F : |f(x)| \ge \beta_t(x)\}$. Then, we have

$$\int_F |f(x)|^{q(x)}(x')^\nu dx = \int_{F^{\beta_t}} |f(x)|^{q(x)}(x')^\nu dx + \int_{F\setminus F^{\beta_t}} |f(x)|^{q(x)}(x')^\nu dx.$$

Firstly, for $x \in F^{\beta_t}$,

$$|f(x)|^{q(x)} = |f(x)|^{p(x)}|f(x)|^{q(x)-p(x)} \le |f(y)|^{p(x)}\beta_t(x)^{\frac{A}{\log(e+|x|)}} \le A\,|f(x)|^{p(x)}.$$

Also, since $\beta_t \le 1$, we have

$$\int_{F\setminus F^{\beta_t}} |f(x)|^{q(x)}(x')^\nu dx \le \int_{F\setminus F^{\beta_t}} \beta_t(x)^{q(x)}(x')^\nu dx \le \int_F \beta_t(x)^{q_-(F)}(x')^\nu dx,$$

which is the desired result. □

In Theorem 2.7, the necessary condition for boundedness of the generalized translation operator has been recalled. For more details, one can see [12, Theorem 4.1].

Theorem 2.7. *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ with $1 < p_- \le p_+ < \infty$. Then,*

$$\|T^y f\|_{p(\cdot),\nu} \le A\|f\|_{p(\cdot),\nu}$$

holds for all $f \in L_{p(\cdot),\nu} \cap \mathcal{S}'_+(\mathbb{R}^n_{k,+})$ with $\operatorname{supp}\mathcal{F}_\nu f \subset \{\xi \in \mathbb{R}^n_{k,+} : |\xi| \le 2^{\eta+1}\}$, $\eta \in \mathbb{N}_0$, where $\mathcal{F}_\nu f$ is the Fourier-Bessel transform and $A>0$ is a constant.

3. On B-fractional Maximal Operator

Here, our aim is to show that B-fractional maximal operator from $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ to $L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})$ is bounded. First, let us recall definitions of B-maximal and B-fractional maximal operators.

B-maximal and B-fractional maximal operators are defined as follows:

$$M_\nu f(x) = \sup_{r>0}|B_+(0,r)|_\nu^{-1}\int_{B_+(0,r)} T^y|f(x)|(y')^\nu dy,$$

$$M_\nu^\alpha f(x) = \sup_{r>0}|B_+(0,r)|_\nu^{\frac{\alpha}{Q}-1}\int_{B_+(0,r)} T^y|f(x)|(y')^\nu dy, \quad 0 \le \alpha < Q,$$

for a given $f \in L^{\mathrm{loc}}_{1,\nu}(\mathbb{R}^n_{k,+})$. One can easily observe that $M_\nu^0 f = M_\nu f$ for $\alpha = 0$ (see [15]).

The next theorem deals with $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$-boundedness of B-maximal operator.

Theorem 3.1 ([11, 18, 19])**.** *Let $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$ with $1 < p_- \le p_+ < \infty$. Then, B-maximal operator is bounded in $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$.*

The following two propositions give us very important inequalities which allow us to obtain one of our main results.

Proposition 3.2. *Let* $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$, $1 \le p_- \le p_+ < \dfrac{Q}{\alpha}$ *satisfying (2.2),* $0 < \alpha < Q$ *and also let* $q(\cdot)$ *be* $\dfrac{1}{p(x)} - \dfrac{1}{q(x)} = \dfrac{\alpha}{Q}$. *Then,*

$$M^\alpha_\nu f(x) \le A\, M_\nu f(x)^{\frac{p(x)}{q(x)}},$$

holds for every $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ *such that* $\|f\|_{p(\cdot),\nu} \le 1$ *and* $|f(x)| \ge 1$ *or* $f(x) = 0$, $x \in \mathbb{R}^n_{k,+}$, *where* $A = A(n, p(\cdot), \alpha, \nu)$ *is a positive constant.*

Proof. Let $x \in \mathbb{R}^n_{k,+}$ be fixed and also let B_+ be a ball including x. Then, we have

$$\begin{aligned}
&|B_+|_\nu^{\frac{\alpha}{Q}-1} \int_{B_+(0,r)} T^y|f(x)|(y')^\nu dy \\
&= |B_+|_\nu^{\frac{\alpha}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{1-\frac{\alpha p(x)}{Q}} \\
&\le |B_+|_\nu^{\frac{\alpha}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} M_\nu f(x)^{\frac{p(x)}{q(x)}}.
\end{aligned}$$

It is necessary to illustrate that

$$|B_+|_\nu^{\frac{\alpha}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} \le A,$$

which allow us to obtain the desired result. Here, there are two cases: (i) $|B_+|_\nu \ge 1$, (ii) $|B_+|_\nu \le 1$. Firstly, for $|B_+|_\nu \ge 1$, by Lemma 2.2 and Chebyschev's inequality, we find

$$|\operatorname{supp} f| \le \int_{\mathbb{R}^n_{k,+}} |f(y)|^{p(y)}(y')^\nu dy \le \|f\|_{p(\cdot),\nu} \le 1.$$

Since $p(x) \ge 1$, and by (2.1), we have

$$\begin{aligned}
|B_+|_\nu^{\frac{\alpha}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} &\le \left(\int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} \\
&\le \left(\int_{B_+} |f(y)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} \\
&\le \|f\|_{1,\nu}^{\frac{\alpha p(x)}{Q}} \\
&\le \left(A\,(1 + |\operatorname{supp} f|)\|f\|_{p(\cdot),\nu}\right)^{\frac{\alpha p(x)}{Q}} \\
&\le A,
\end{aligned}$$

with the use of Lemma 2.3, Theorem 2.7 and (2.1). We now estimate for $|B_+|_\nu \le 1$. If $p_- > 1$, then $p'(\cdot)$ satisfies (2.2) and $p'_+ < \infty$. Hence, by Lemma 2.1 and Lemma 2.5,

$$\begin{aligned}
|B_+|_\nu^{\frac{\alpha}{Q}} \left(|B_+|_\nu^{-1} \int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p(x)}{Q}} &\le |B_+|_\nu^{\frac{\alpha}{Q}-\frac{\alpha p(x)}{Q}} \|\chi_{B_+}\|_{p'(\cdot),\nu}^{\frac{\alpha p(x)}{Q}} \|f\|_{p(\cdot),\nu}^{\frac{\alpha p(x)}{Q}} \\
&\le |B_+|_\nu^{\frac{\alpha}{Q}-\frac{\alpha p(x)}{Q}} \|\chi_{B_+}\|_{p'(\cdot),\nu}^{\frac{\alpha p(x)}{Q}} \\
&\le A\, |B_+|_\nu^{\frac{\alpha}{Q}-\frac{\alpha p(x)}{Q}} |B_+|_\nu^{\frac{\alpha p(x)}{Q p'(x)}} \\
&\le A.
\end{aligned}$$

From Lemma 2.3, we have

$$\|\chi_{B_+}\|^{\frac{\alpha p(x)}{Q}}_{L_{p'(\cdot),\nu}(\mathbb{R}^n_{k,+})} = \|\chi_{B_+}\|^{\frac{\alpha p(x)}{Q}}_{L_{p'(\cdot),\nu}(B_+)} \leq A\,(1+|B_+|_\nu)\,\|\chi_{B_+}\|^{\frac{\alpha p(x)}{Q}}_{\infty,\nu} \leq A.$$

Then, by Lemma 2.4, we get

$$|B_+|_\nu^{\frac{\alpha}{Q}-\frac{\alpha p(x)}{Q}} \leq |B_+|_\nu^{(p_- - p_+)\frac{\alpha}{Q}} \leq A,$$

which gives us the desired result. □

Proposition 3.3. *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$, $1 \leq p_- \leq p_+ < \dfrac{Q}{\alpha}$ satisfying (2.3), $0 < \alpha < Q$ and also let $q(\cdot)$ be $\dfrac{1}{p(x)} - \dfrac{1}{q(x)} = \dfrac{\alpha}{Q}$. Then,*

$$M_\nu^\alpha f(x) \leq A\, M_\nu f(x)^{\frac{p(x)}{q^*(x)}}$$

holds for all $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ such that $\|f\|_{p(\cdot),\nu} \leq 1$ and $|f(x)| \leq 1$, $x \in \mathbb{R}^n_{k,+}$, where $q^*(x) = \sup_{|y|\geq|x|} q(y)$.

Proof. Let $x \in \mathbb{R}^n_{k,+}$ be fixed and also let B_+ be a ball including x. Then, one can write $\dfrac{1}{p^*(x)} - \dfrac{1}{q^*(x)} = \dfrac{\alpha}{Q}$ and by using Lemma 2.1, we get

$$\begin{aligned}|B_+|_\nu^{\frac{\alpha}{Q}-1}\int_{B_+} T^y|f(x)|(y')^\nu dy &= |B_+|_\nu^{\frac{\alpha}{Q}}\left(|B_+|_\nu^{-1}\int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{\frac{\alpha p^*(x)}{Q}}\left(|B_+|_\nu^{-1}\int_{B_+} T^y|f(x)|(y')^\nu dy\right)^{1-\frac{\alpha p^*(x)}{Q}}\\ &\leq \left(\int_{B_+} T^y|f(x)|^{p^*(x)}(y')^\nu dy\right)^{\frac{\alpha}{Q}} M_\nu f(x)^{\frac{p^*(x)}{q^*(x)}}.\end{aligned}$$

Taking into account that $|f(x)| \leq 1$, $M_\nu f(x) \leq 1$ and therefore $M_\nu f(x)^{\frac{p^*(x)}{q^*(x)}} \leq M_\nu f(x)^{\frac{p(x)}{q^*(x)}}$. Then,

$$\int_{B_+} T^y|f(x)|^{p^*(x)}(y')^\nu dy = \int_{\{x\in B_+ : |y|\leq|x|\}} T^y|f(x)|^{p^*(x)}(y')^\nu dy + \int_{\{x\in B_+ : |y|>|x|\}} T^y|f(x)|^{p^*(x)}(y')^\nu dy.$$

We estimate the first integral by applying Lemma 2.6. If $y \in \{x \in B_+ : |y| \leq |x|\}$, then by (2.3),

$$|p(y) - p^*(x)| \leq \frac{A}{\log(e+|y|)}.$$

Thus, for any $t > 1$,

$$\begin{aligned}\int_{\{x\in B_+ : |y|\leq|x|\}} T^y|f(x)|^{p^*(x)}(y')^\nu dy &\leq A_{t,\nu}\int_{\{x\in B_+ : |y|\leq|x|\}} T^y|f(x)|^{p(y)}(y')^\nu dy + \int_{\{x\in B_+ : |y|\leq|x|\}} \beta_t(y)^{(p^*)_-}(y')^\nu dy\\ &\leq A.\end{aligned}$$

We now estimate the second integral. If $y \in \{x \in B_+ : |y| > |x|\}$, then we have $p^*(x) \geq p(y)$. Taking into account that $|f(y)| \leq 1$ and Lemma 2.2, we find

$$\begin{aligned}\int_{\{x\in B_+ : |y|>|x|\}} T^y|f(x)|^{p^*(x)}(y')^\nu dy &\leq \int_{\{x\in B_+ : |y|>|x|\}} |f(y)|^{p^*(x)}(y')^\nu dy\\ &\leq \int_{\{x\in B_+ : |y|>|x|\}} |f(y)|^{p(y)}(y')^\nu dy\\ &\leq \int_{\mathbb{R}^n_{k,+}} |f(y)|^{p(y)}(y')^\nu dy \leq 1,\end{aligned}$$

which gives us the desired result. □

The next theorem contains an important inequality for averages over balls.

Theorem 3.4. *Let $p(\cdot):\mathbb{R}^n_{k,+}\to[1,\infty)$ satisfy (2.2) and (2.3). Suppose that*

$$\int_{\mathbb{R}^n_{k,+}} |f(y)^{p(y)}(y')^{\nu}dy \le 1,$$

and $f(x)=0$ or $|f(x)|\ge 1$, $x\in\mathbb{R}^n_{k,+}$. Then, for all $x\in\mathbb{R}^n_{k,+}$,

$$\left(|B_+|_\nu^{-1}\int_{B_+}|f(y)|(y')^\nu dy\right)^{p(x)} \le A\left(|B_+|_\nu^{-1}\int_{B_+}|f(y)|^{\frac{p(y)}{p_-}}(y')^\nu dy\right)^{p_-} + A\beta_t(x)^{p_-}. \tag{3.1}$$

Proof. Suppose that $f\ge 0$ and set $f_1=f\chi_{\{x:\,f(x)\le 1\}}$ and $f_2=f\chi_{\{x:\,f(x)>1\}}$. Then

$$M_\nu f(x)^{p(x)} \le 2^{p_+}\left(M_\nu f_1(x)^{p(x)}+M_\nu f_2(x)^{p(x)}\right).$$

To complete the proof, we consider the cases: the first is $f(x)\ge 1$ or $f(x)=0$ and the second is $f(x)\le 1$. Throughout this section, fix

$$\beta(y)=\beta_1(y)=\frac{1}{(e+|y|)^n},$$

and set $\overline{p}(x)=\frac{p(x)}{p_-}$. Then $\overline{p}(x)\ge 1$, and (2.3) holds with p substituted by $\overline{p}$.

We first estimate (3.1) for $f(x)\ge 1$ or $f(x)=0$. Let $x\in\mathbb{R}^n_{k,+}$ be fixed and B_+ be a ball including x with $|B_+|_\nu>0$. We take into account three cases.

(1) $\underline{r<|x|/4}$. If $y_1,y_2\in B_+$, then we have $\log(e+|y_1|)\approx\log(e+|y_2|)$. Therefore, we get

$$|\overline{p}(y)-\overline{p}_-(B_+)|\le\frac{A}{\log(e+|y|)},$$

for all $y\in B_+$ and $y\in B_+$ implies $\beta(y)\le A\beta(x)$ since $r<|x|/4$. Then, by using Lemma 2.1 and Lemma 2.6, substituting $\overline{p}_-(B_+)$ for $v(\cdot)$, $\overline{p}(\cdot)$ for $u(\cdot)$, and with $t=1$, and since $\dfrac{p(x)}{\overline{p}_-(B_+)}\le p_+<\infty$, one can obtain

$$\begin{aligned}
\left(|B_+|_\nu^{-1}\int_{B_+}f(y)(y')^\nu dy\right)^{p(x)} &\le \left(|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}_-(B_+)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}\\
&\le\left(A\,|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy+|B_+|_\nu^{-1}\int_{B_+}\beta(y)^{\overline{p}_-(B_+)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}\\
&\le\left(A\,|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy+A\beta(x)^{\overline{p}_-(B_+)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}\\
&\le 2^{p_+}A\left(|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}+2^{p_+}A\beta(x)^{p(x)}.
\end{aligned}$$

Taking into account that $p_->1$, $\beta(x)^{p(x)}\in L_{1,\nu}(\mathbb{R}^n_{k,+})$. Therefore,

$$\begin{aligned}
|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy^{\frac{p(x)}{\overline{p}_-(B_+)}} &= \left(|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-}\left(|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}-p_-}\\
&=|B_+|_\nu^{-[(\frac{p(x)}{\overline{p}_-(B_+)})-p_-]/p_-}\left(\int_{B_+}f(y)^{p(y)}(y')^\nu dy\right)^{[(\frac{p(x)}{\overline{p}_-(B_+)})-p_-]/p_-}\\
&\quad\times\left(|B_+|_\nu^{-1}\int_{B_+}f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-}.
\end{aligned}$$

One can write

$$-\frac{1}{p_-}\left[\frac{p(x)}{\overline{p}_-(B_+)}-p_-\right]=p(x)\left[\frac{1}{p(x)}-\frac{1}{p_-(B_+)}\right]\le 0.$$

If $|B_+|_\nu \geq 1$, then $|B_+|_\nu^{-[\frac{p(x)}{\overline{p}_-(B_+)}-p_-]/p_-} \leq 1$. Also, if $|B_+|_\nu \leq 1$, we find

$$p(x)\left[\frac{1}{p(x)}-\frac{1}{p_-(B_+)}\right] \geq \frac{p_+}{p_-^2}(p_-(B_+)-p_+(B_+)),$$

and by Lemma 2.4,

$$|B_+|_\nu^{-[\frac{p(x)}{\overline{p}_-(B_+)}-p_-]/p_-} \leq |B_+|_\nu^{\frac{p_+}{p_-^2}(p_-(B_+)-p_+(B_+))} \leq A.$$

Again,

$$\frac{p(x)}{\overline{p}_-(B_+)}-p_- = \frac{p(x)}{p_-(B_+)}p_- - p_- \geq 0,$$

and since $\int_{\mathbb{R}^n_{k,+}} |f(y)|^{p(y)}(y')^\nu dy \leq 1$, we obtain

$$\left(\int_{B_+} f(y)^{p(y)}(y')^\nu dy\right)^{[\frac{p(x)}{\overline{p}_-(B_+)}-p_-]/p_-} \leq 1.$$

Taking into account of all these estimates, we obtain (3.1).

(2) $|x| \leq 1$ **and** $r \geq |x|/4$. Since $|x| \leq 1$, applying Lemma 2.6, we find

$$0 \leq \overline{p}(y)-\overline{p}_-(B_+) \leq \overline{p}_+ - \overline{p}_- \leq \frac{A}{\log(e+|x|)} \quad \text{for all } y \in B_+.$$

Then, with the use of Lemma 2.1, Lemma 2.6, substituting $\overline{p}_-(B_+)$ for $v(\cdot)$, $\overline{p}(\cdot)$ for $u(\cdot)$ with $t = 1$, and since $\frac{p(x)}{\overline{p}_-(B_+)} \leq p_+ < \infty$, one can see

$$\begin{aligned}\left(|B_+|_\nu^{-1}\int_{B_+} f(y)(y')^\nu dy\right)^{p(x)} &\leq \left(|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}_-(B_+)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}\\ &\leq \left(A\,|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy + |B_+|_\nu^{-1}\int_{B_+}\beta(x)^{\overline{p}_-(B_+)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}}\\ &\leq 2^{p_+}\left(A\,|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(B_+)}} + 2^{p_+}A\,\beta(x)^{p(x)}.\end{aligned}$$

As in **(1)**, desired inequality is obtained.

(3) $|x| \geq 1$ **and** $r \geq |x|/4$. Taking into account that $f(x) \geq 1$ or $f(x) = 0$, $p_- \geq 1$ and $\int_{\mathbb{R}^n_{k,+}} |f(y)|^{p(y)}(y')^\nu dy \leq 1$, we have

$$\begin{aligned}\left(|B_+|_\nu^{-1}\int_{B_+} f(y)(y')^\nu dy\right)^{p(x)} &\leq |B_+|_\nu^{-p(x)}\left(\int_{B_+} f(y)^{p(y)}(y')^\nu dy\right)^{p(x)}\\ &\leq |B_+|_\nu^{-p(x)} \leq A\,|x|^{-Q p(x)} \leq A\,\beta(x)^{p(x)}.\end{aligned}$$

We now estimate (3.1) for $f(x) \leq 1$. Let $x \in \mathbb{R}^n_{k,+}$ be fixed and $|x| \geq 1$. Therefore, we show that

$$\left(|B_+|_\nu^{-1}\int_{B_+} f(y)(y')^\nu dy\right)^{p(x)} \leq A\left(|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-} + A\,\beta(x)^{p_-}.$$

Since $p(x) \leq p_+ < \infty$, we have

$$\begin{aligned}\left(|B_+|_\nu^{-1}\int_{B_+} f(y)(y')^\nu dy\right)^{p(x)} &\leq 2^{p_+}\left(|B_+|_\nu^{-1}\int_{B_+\cap B_{|x|}(0)} f(y)(y')^\nu dy\right)^{p(x)} + 2^{p_+}\left(|B_+|_\nu^{-1}\int_{B_+\setminus B_{|x|}(0)} f(y)(y')^\nu dy\right)^{p(x)}\\ &= I_1 + I_2.\end{aligned}$$

We first estimate I_2. Let $E = B_+ \setminus B_{|x|}(0)$. Then, one can write

$$0 \le \overline{p}(y) - \overline{p}_-(E) \le \overline{p}_+(E) - \overline{p}_-(E) \le \frac{A}{\log(e+|x|)} \quad \text{for all } y \in E.$$

With the use of Lemma 2.1, Lemma 2.6, substituting $\overline{p}_-(E)$ for $v(\cdot)$, $\overline{p}(\cdot)$ for $u(\cdot)$ with $t = 1$, and since $E \subset B_+$ and $\frac{p(x)}{\overline{p}_-(E)} \le p_+ < \infty$, we have

$$\begin{aligned}\left(|B_+|_\nu^{-1}\int_E f(y)(y')^\nu dy\right)^{p(x)} &\le \left(|B_+|_\nu^{-1}\int_E f(y)^{\overline{p}_-(E)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(E)}}\\ &\le \left(A\,|B_+|_\nu^{-1}\int_E f(y)^{\overline{p}(y)}(y')^\nu dy + |B_+|_\nu^{-1}\int_E \beta(x)^{\overline{p}_-(E)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(E)}}\\ &\le 2^{p_+}\left(A\,|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}_-(E)}} + 2^{p_+}\beta(x)^{p(x)}.\end{aligned}$$

We also obtain that

$$I_2 = \left(|B_+|_\nu^{-1}\int_{B_+\setminus B_{|x|}(0)} f(y)(y')^\nu dy\right)^{p(x)} \le A\left(|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-} + A\,\beta(x)^{p(x)},$$

with the use of $\overline{p}(x) \ge \overline{p}_-(E)$. Let us obtain I_1 and $E^* = B_{|x|}(0) \cap B_+$. Then, by (2.3), we get

$$|\overline{p}(x) - \overline{p}(y)| \le \frac{A}{\log(e+|y|)} \quad \text{for all } y \in E^*.$$

Again, with the use of Lemma 2.1, Lemma 2.6 with $v(\cdot) = \overline{p}(x)$, $u(\cdot) = \overline{p}(\cdot)$, $t = 1$, and with the use of $r > |x|/4$, $|B_{|x|}(0)|_\nu \le A\,|B_+|_\nu$, we get

$$\begin{aligned}\left(|B_+|_\nu^{-1}\int_{E^*} f(y)(y')^\nu dy\right)^{p(x)} &\le \left(|B_+|_\nu^{-1}\int_{E^*} f(y)^{\overline{p}(x)}(y')^\nu dy\right)^{\frac{p(x)}{\overline{p}(x)}}\\ &\le \left(A\,|B_+|_\nu^{-1}\int_{E^*} f(y)^{\overline{p}(y)}(y')^\nu dy + |B_+|_\nu^{-1}\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)}(y')^\nu dy\right)^{p_-}\\ &\le A\left(|B_+|_\nu^{-1}\int_{E^*} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-} + A\left(|B_+|_\nu^{-1}\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)}(y')^\nu dy\right)^{p_-}\\ &\le A\left(|B_+|_\nu^{-1}\int_{B_+} f(y)^{\overline{p}(y)}(y')^\nu dy\right)^{p_-} + A\left(|B_+|_\nu^{-1}\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)}(y')^\nu dy\right)^{p_-}.\end{aligned}$$

Let $1 < v < p_-$. Then, by Lemma 2.1, we have

$$\left(|B_+|_\nu^{-1}\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)}(y')^\nu dy\right)^{p_-} \le |B_+|_\nu^{-p_-/v}\left(\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)v}(y')^\nu dy\right)^{p_-/v}.$$

Since $\overline{p}(x)v \ge \overline{p}_- v > 1$ and $\beta(y) \le 1$,

$$\int_{B_{|x|}(0)} \beta(y)^{\overline{p}(x)v}(y')^\nu dy \le \int_{B_{|x|}(0)} \beta(y)^{\overline{p}_- v}(y')^\nu dy \le A.$$

Also, since $|x| \ge 1$,

$$|B_{|x|}(0)|_\nu^{-p_-/v} \le A\,(e+|x|)^{-(Q)p_-/v} = A\,\beta(x),$$

and $\beta(x) \in L_{1,\nu}$ since $p_- > v$. Thus, we complete the proof. □

Now, we present the following main result on B-fractional maximal operator.

Theorem 3.5. *Let $0 \le \alpha < Q$, $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$ with $1 < p_- \le p_+ < \frac{Q}{\alpha}$ and $q(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ which is defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{Q}$. Then, B-fractional maximal operator $M^\alpha_\nu : L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+}) \to L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})$ is bounded.*

Proof. Let $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ be fixed. Without losing the generality, let $f \geq 0$ and $\|f\|_{p(\cdot),\nu} \leq 1$. With the use of Lemma 2.2 and Theorem 3.1, it is sufficient to illustrate that

$$\int_{\mathbb{R}^n_{k,+}} |M^\alpha_\nu f(y)|^{q(y)}(y')^\nu dy \leq A.$$

Define $f_1 = f\chi_{\{x:\, f(x)\geq 1\}}$ and $f_2 = f\chi_{\{x:\, f(x)<1\}}$. Then, $\|f_i\|_{p(\cdot),\nu} \leq \|f\|_{p(\cdot),\nu} = 1$, $i = 1, 2$. By Propositions 3.2 and 3.3 and since $q_+ < \infty$, we have

$$\int_{\mathbb{R}^n_{k,+}} M^\alpha_\nu f(x)^{q(x)}(x')^\nu dx \leq 2^{q_+} \int_{\mathbb{R}^n_{k,+}} M^\alpha_\nu f_1(x)^{q(x)}(x')^\nu dx + 2^{q_+} \int_{\mathbb{R}^n_{k,+}} M^\alpha_\nu f_2(x)^{q(x)}(x')^\nu dx$$

$$\leq A \int_{\mathbb{R}^n_{k,+}} M_\nu f_1(x)^{p(x)}(x')^\nu dx + A \int_{\mathbb{R}^n_{k,+}} M_\nu f_2(x)^{p(x)q(x)/q^*(x)}(x')^\nu dx.$$

We now estimate each integral. The proof is obvious for the first integral:

$$A \int_{\mathbb{R}^n_{k,+}} M_\nu f_1(x)^{p(x)}(x')^\nu dx \leq A \int_{\mathbb{R}^n_{k,+}} M_\nu f(x)^{p(x)}(x')^\nu dx \leq A.$$

For the second, let $\epsilon > 0$ be fixed. Then, one can find y, $|y| \geq |x|$ with $q^*(x) \leq (1+\epsilon)q(y)$. Then, we find

$$0 \leq q^*(x) - q(x) \leq (1+\epsilon)q(y) - q(x) \leq |q(y) - q(x)| + \epsilon q_+ \leq \frac{A}{\log(e+|x|)} + \epsilon q_+.$$

Since $q^*(x) \geq 1$ and $\epsilon > 0$ is arbitrary, we obtain

$$\left|1 - \frac{q(x)}{q^*(x)}\right| \leq \frac{A}{\log(e+|x|)}.$$

Let $v(x) = \frac{q(x)}{q^*(x)}$. Then $v_- \geq \frac{q_-}{q_+}$, so $R_{2q_+/q_-}(\cdot)^{v_-}$ is integrable. With the use of Lemma 2.6 for $u(x) = 1$, we have

$$\int_{\mathbb{R}^n_{k,+}} M_\nu f_2(x)^{p(x)\frac{q(x)}{q^*(x)}}(x')^\nu dx \leq A_t \int_{\mathbb{R}^n_{k,+}} M_\nu f(x)^{p(x)}(x')^\nu dx + \int_{\mathbb{R}^n_{k,+}} R_{2q_+/q_-}(\cdot)^{v_-}(x')^\nu dx \leq A.$$

This completes the proof. □

As an analog of the result in [16], we can write the following:

Corollary 3.6. *If $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ and $1 < p_- \leq p_+ \leq \infty$, then*

$$\lim_{r\to 0} |B_+(0,r)|^{-1}_\nu \int_{B_+(0,r)} T^y f(x)(y')^\nu dy = f(x), \quad a.a.\ x \in \mathbb{R}^n_{k,+}.$$

4. On B-Riesz Potentials

In this section, our aim is to show that B-Riesz potential operator from $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ to $L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})$ is bounded. Although, B-Riesz potential is well-known, it will be nice to recall it.

Let $0 < \alpha < Q$, then B-Riesz potential is introduced by

$$I^\alpha_\nu f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f(y)(y')^\nu dy.$$

Let us give the inequality that we will use to obtain our other main result.

Lemma 4.1 ([14]). *For any ϵ, $0 < \epsilon < \min(\alpha, Q - \alpha)$, there exists a constant $A = A(\alpha, n, \nu, \epsilon) > 0$ such that*

$$I^\alpha_\nu f(x) \leq A\sqrt{M^{\alpha-\epsilon}_\nu f(x) M^{\alpha+\epsilon}_\nu f(x)}, \tag{4.1}$$

for any nonnegative function $f : \mathbb{R}^n_{k,+} \to \mathbb{R}^n$ and for any $x \in \mathbb{R}^n_{k,+}$.

Now, we present the main result on B-Riesz potential operator.

Theorem 4.2. *Let* $0 < \alpha < Q$, $1 < p_- \leq p_+ < \dfrac{Q}{\alpha}$, $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$ *and* $q(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ *which is defined by* $\dfrac{1}{p(\cdot)} - \dfrac{1}{q(\cdot)} = \dfrac{\alpha}{Q}$. *Then,*

$$\|I^\alpha_\nu f\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \leq A\,\|f\|_{L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})}. \tag{4.2}$$

Proof. Let $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ be fixed. Without losing the generality, let $\|f\|_{p(\cdot),\nu} = 1$. Then, from Lemma 2.2, it is sufficient to illustrate that

$$\int_{\mathbb{R}^n_{k,+}} |I^\alpha_\nu f(y)|^{q(y)}(y')^\nu dy \leq A.$$

Let ϵ be fixed $0 < \epsilon < \max(\alpha, Q-\alpha)$ such that

$$\frac{2}{\frac{\epsilon q_+}{Q}+1} > 1, \tag{4.3}$$

and define $v(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty)$ as

$$v(x) = \frac{2}{\frac{\epsilon q(x)}{Q}+1}.$$

With the use of (4.3), we find $v_- > 1$. Also, for all $x \in \mathbb{R}^n_{k,+}$, one can write

$$\frac{1}{p(x)} - \frac{1}{\frac{v(x)q(x)}{2}} = \frac{\alpha-\epsilon}{Q}, \tag{4.4}$$

$$\frac{1}{p(x)} - \frac{1}{\frac{v'(x)q(x)}{2}} = \frac{\alpha+\epsilon}{Q}. \tag{4.5}$$

Take the power $q(x)$ of both side of (4.1) and by integrating over $\mathbb{R}^n_{k,+}$, then we get

$$\int_{\mathbb{R}^n_{k,+}} |I^\alpha_\nu f(x)|^{q(x)}(x')^\nu dx \leq A \int_{\mathbb{R}^n_{k,+}} \left[M^{\alpha-\epsilon}_\nu f(x)\right]^{q(x)/2} \left[M^{\alpha+\epsilon}_\nu f(x)\right]^{q(x)/2} (x')^\nu dx.$$

By Lemma 2.1, we get

$$\int_{\mathbb{R}^n_{k,+}} |I^\alpha_\nu f(x)|^{q(x)}(x')^\nu dx \leq A \left\|\left[M^{\alpha-\epsilon}_\nu f(\cdot)\right]^{q(\cdot)/2}\right\|_{v(\cdot),\nu} \left\|\left[M^{\alpha+\epsilon}_\nu f(\cdot)\right]^{q(\cdot)/2}\right\|_{v'(\cdot),\nu}.$$

In order to obtain the desired result, we calculate all of the norms on the RHS. Without losing the generality, each norm is greater than 1. Otherwise the proof is obvious. Therefore, in each norm we take the infimums over $\mu > 1$. Since for all $x \in \mathbb{R}^n_{k,+}$ and $\mu > 1$, $\mu^{2/q(x)} \geq \mu^{2/q_+}$, we get

$$\int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha-\epsilon}_\nu f(x)\right]^{q(x)/2}}{\mu}\right)^{v(x)} (x')^\nu dx = \int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha-\epsilon}_\nu f(x)\right]}{\mu^{2/q(x)}}\right)^{v(x)q(x)/2} (x')^\nu dx \leq \int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha-\epsilon}_\nu f(x)\right]}{\mu^{2/q_+}}\right)^{v(x)q(x)/2} (x')^\nu dx,$$

and by using (4.4) and Theorem 3.5,

$$\left\|\left[M^{\alpha-\epsilon}_\nu f(\cdot)\right]^{q(\cdot)/2}\right\|_{v(\cdot),\nu} \leq \left\|\left[M^{\alpha-\epsilon}_\nu f(\cdot)\right]\right\|^{q_+/2}_{\frac{v(\cdot)q(\cdot)}{2},\nu} \leq A\,\|f\|^{q_+/2}_{p(\cdot),\nu} \leq A.$$

On the other hand, the norm estimate for $\|\left[M^{\alpha+\epsilon}_\nu f(\cdot)\right]^{q(\cdot)/2}\|_{v'(\cdot),\nu}$ can be obtained in a similar manner. By (4.5), we have

$$\int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha+\epsilon}_\nu f(x)\right]^{q(x)/2}}{\mu}\right)^{v'(x)} (x')^\nu dx = \int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha+\epsilon}_\nu f(x)\right]}{\mu^{2/q(x)}}\right)^{v'(x)q(x)/2} (x')^\nu dx \leq \int_{\mathbb{R}^n_{k,+}} \left(\frac{\left[M^{\alpha+\epsilon}_\nu f(x)\right]}{\mu^{2/q_+}}\right)^{v'(x)q(x)/2} (x')^\nu dx.$$

Hence, we obtain

$$\left\|\left[M^{\alpha+\epsilon}_\nu f(\cdot)\right]^{q(\cdot)/2}\right\|_{v'(\cdot),\nu} \leq \left\|\left[M^{\alpha+\epsilon}_\nu f(\cdot)\right]\right\|^{q_+/2}_{\frac{v'(\cdot)q(\cdot)}{2},\nu} \leq A\,\|f\|^{q_+/2}_{p(\cdot),\nu} \leq A,$$

which gives us the desired result. □

Theorem 4.3. *Let* $0 < \alpha < Q$, $1 < p_- \le p_+ < \dfrac{Q}{\alpha}$*, then the following is necessary for (4.2) holds:*

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{Q}.$$

Proof. Let $1 < p_- \le p_+ < \dfrac{Q}{\alpha}$, $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ and suppose that the inequality (4.2) holds. Define $f_t := f(t^a x)$, then we obtain

$$\begin{aligned}
\|f_t\|_{L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})} &= \inf\left\{\mu > 0 : \int_{\mathbb{R}^n_{k,+}} \left(\frac{|f(tx)|}{\mu}\right)^{p(\cdot)} (x')^{\nu} dx \le 1\right\} \\
&= \inf\left\{\mu > 0 : \int_{\mathbb{R}^n_{k,+}} \left(\frac{|f(y)|}{\mu}\right)^{p(\cdot)} t^{-Q} (y')^{\nu} dy \le 1\right\} \\
&= t^{-Q}\|f\|_{L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})},
\end{aligned}$$

and

$$\begin{aligned}
\|I^{\alpha}_{\nu} f_t\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} &= \left\| \int_{\mathbb{R}^n_{k,+}} f_t(y) T^y |x|^{\alpha - Q} (y')^{\nu} dy \right\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \\
&= \left\| \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f_t(y) (y')^{\nu} dy \right\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \\
&= \left\| \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f(y) t^{-Q} (y')^{\nu} dy \right\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \\
&= \left\| \int_{\mathbb{R}^n_{k,+}} t^{Q-\alpha} T^y |tx|^{\alpha - Q} f(y) t^{-Q} (y')^{\nu} dy \right\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \\
&= t^{-Q}\|I^{\alpha}_{\nu} f\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})}.
\end{aligned}$$

By (4.2), we have

$$\|I^{\alpha}_{\nu} f\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} \le A\, t^{-2Q}\|f\|_{L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})},$$

where $A = A_(p(\cdot), q(\cdot), \nu) > 0$ is a constant. If $\dfrac{1}{p(\cdot)} > \dfrac{1}{q(\cdot)} + \dfrac{\alpha}{Q}$, then $\|I^{\alpha}_{\nu} f\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} = 0$ in the case $t \to 0$ and for all $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$. Thus, we obtain a contradiction. Similarly, if $\dfrac{1}{p(\cdot)} < \dfrac{1}{q(\cdot)} + \dfrac{\alpha}{Q}$, then $\|I^{\alpha}_{\nu} f\|_{L_{q(\cdot),\nu}(\mathbb{R}^n_{k,+})} = 0$ in the case $t \to \infty$, and for all $f \in L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$, Thus, we obtain a contradiction. Consequently, $\dfrac{1}{p(\cdot)} = \dfrac{1}{q(\cdot)} + \dfrac{\alpha}{Q}$. Hence, the proof is completed. □

5. Conclusions

Fractional maximal operators, Riesz potentials and boundedness of these operators on different function spaces are considerable problems of Harmonic Analysis. On variable Lebesgue spaces, for shortly $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$, fractional maximal and Riesz potential operators have been examined by many mathematicians. On the other hand, *B*-fractional maximal and *B*-Riesz potential operators have been studied on Lebesgue spaces. These results motivate us to investigate *B*-fractional maximal M^{α}_{ν} and *B*-Riesz potential operator I^{α}_{ν} on $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$. Here, *B*-fractional maximal and *B*-Riesz potential operators on $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ have been examined. We have obtained the mapping properties of *B*-fractional maximal operator in $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$. Finally, by using this result and $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$-boundedness of generalized translation operator, we have proved that *B*-Riesz potential on $L_{p(\cdot),\nu}(\mathbb{R}^n_{k,+})$ is bounded.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

References

[1] Aliev, I.A., Bayrakçı, S., *On inversion of B-elliptic potentials associated with the Laplace-Bessel differential operator*, Fract. Calc. Appl. Anal., **4**(1998), 365–384.

[2] Almeida, A., *Inversion of the Riesz potential operator on Lebesgue spaces with variable exponent*, Fract. Calc. Appl. Anal., **6**(2003), 311–327.

[3] Almeida, A., Samko, S., *Characterization of Riesz and Bessel potentials on variable Lebesgue spaces*, J. Funct. Spaces Appl., **4**(2006), 113–144.

[4] Aykol, C., Hasanov, J.J., *On the boundedness of B-maximal commutators, commutators of B-Riesz potentials and B-singular integral operators in modified B-Morrey spaces*, Acta Sci. Math. (Szeged), **86**(2020), 521–547.

[5] Aykol, C., Kaya, E., *B-maximal operators, B-singular integral operators and B-Riesz potentials in variable exponent Lorentz spaces*, Filomat, **37**(2023), 5765–5774.

[6] Capone, C., Cruz-Uribe, D., S.F.O., Fiorenza, A., *The fractional maximal operator and fractional integrals on variable L^p spaces*, Revista Matemática Iberoamericana, **23**(2007), 743–770.

[7] Cruz-Uribe, D., Fiorenza, A., Variable Lebesgue spaces, Foundations and Harmonic Analysis, Springer Science, Business Media, 2013.

[8] Cruz-Uribe, D., Shukla, P., *The boundedness of fractional maximal operators on variable Lebesgue spaces over spaces of homogeneous type*, arXiv preprint arXiv:1511.09456, (2015), 1–31.

[9] Diening, L., *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr., **268**(2004), 31–43.

[10] Edmunds, D.E., Meskhi, A., *Potential-type operators in $L^{p(x)}$ spaces*, Z. Anal. Anwendungen, **21**(2002), 681–690.

[11] Ekincioglu, I., Guliyev, V.S., Kaya, E., *B_n-maximal operator and B_n-singular integral operators on variable exponent Lebesgue spaces*, Mathematica Slovaca, **70**(2020), 893–902.

[12] Ekincioglu, I., Shishkina, E.L., Kaya, E., *On the boundedness of the generalized translation operator on variable exponent Lebesgue spaces*, Acta Appl. Math., **173**(2021), 1–14.

[13] Gadjiev, A.D., Guliyev, V.S., Şerbetçi, A., Guliyev, E.V., *The Stein-Weiss type inequalities for the $B-$Riesz potentials*, Journal of Math. Inequal., **5**(2011), 87–106.

[14] Guliyev, E.V., *Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **26**(2006), 71–80.

[15] Guliyev, V.S., *On maximal function and fractional integral, associated with the Bessel differential operator*, Math. Inequal. Appl., **6**(2003), 317–330.

[16] Guliyev, V.S., Garakhanova, N.N., Zeren, Y., *Pointwise and integral estimates for the B-Riesz potential in terms of B-maximal and B-fractional maximal functions*, Sib. Math. J., **49**(2008), 1008–1022.

[17] Hasanov, J.J., Ayazoglu, R., Bayrakçı, S., *B-maximal commutators, commutators of B-singular integral operators and B-Riesz potentials on B-Morrey spaces*, Open Mathematics, **18**(2020), 715–730.

[18] Kaya, E., *Maximal operators related to Laplace-Bessel differential operators on variable exponent Lebesgue spaces*, Open Mathematics, **19**(2021), 306–315.

[19] Kaya, E., *A different approach to boundedness of the B-maximal operators on the variable Lebesgue spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,**71**(2022), 710–719.

[20] Kaya, E., Aykol, C., *B-Riezs potential in B-local Morrey-Lorentz spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **73**(2024), 437–449.

[21] Kipriyanov, I.A., Singular Elliptic Boundary Value Problems, Nauka, Moscow, 1997.

[22] Klyuchantsev, M.I., *On singular integrals generated by the generalized shift operator I*, Sibirsk. Math. Zh., **11**(1970), 810–821. translation in Siberian Math. J., **11**(1970), 612–620.

[23] Kovacik, O., Rakosnik, J., *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41**(1991), 592–618.

[24] Levitan, B.M., *Bessel function expansions in series and Fourier integrals*, Uspekhi Mat. Nauk 6., **42**(1951), 102–143.

[25] Levitan, BM., *Expansion in Fourier series and integrals with Bessel functions*, Uspekhi Mat. Nauk., **6**(2)(1951), 2–143.

[26] Muckenhoupt, B., Wheeden, R.L., *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**(1974), 261–274.

[27] Riesz, M., *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math., **81**(1949), 1–223.

[28] Samko, S., *Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$*, Integral Transform. Spec. Funct., **7**(1998), 261–284.

[29] Sarıkaya, M.Z., Yıldırım, H., *On Sobolev type theorem for the generalized Riesz potential generated by the generalized shift operator on Morrey space*, Sarajevo Journal of Mathematics, **4**(2008), 207–214.

[30] Stein, E.M., Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, No 30 Princeton University Press, Princeton, N.J., 1970.

[31] Yıldırım, H., Riesz potentials generated by a generalized shift operator, Ankara University Graduate School of Natural and Applied Science, Department of Math. Ph.D. thesis, 1995.

[32] Yıldırım, H., Sarıkaya, M.Z., *(p, q) properties of a generalized Riesz potentials generated by the generalized shift operators*, Taiwanese J. Math., **12**(2008), 1201–1209.