Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 24-49, Doi: https://doi.org/10.47087/mjm.1467436

ON A FAMILY OF DISCRETE ND LADDER-TYPE OPERATORS CONSTRUCTED IN TERMS OF THE HERMITIAN TOEPLITZ COMMUTATOR OPERATOR Z_N

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Abstract. A development of an algebraic system with N-dimensional ladder-type operators associated with the discrete Fourier transform is described, following an analogy with the canonical commutation relations of the continuous case. It is found that a Hermitian Toeplitz matrix Z_N, which plays the role of the identity, is sufficient to satisfy the Jacobi identity and, by solving some compatibility relations, a family of ladder operators with corresponding Hamiltonians can be constructed. The behaviour of the matrix Z_N for large N is elaborated. It is shown that this system can be also realized in terms of the Heun operator W, associated with the discrete Fourier transform, thus providing deeper insight on the underlying algebraic structure.

1. Introduction

The study of discrete structures is significant for the theory of signal processing, entanglement, quantum computation and more [1], and it serves as a source of interesting and surprising considerations worth studying. The problem of the construction of a system of eigenvectors for the discrete Fourier transform (DFT) is still open and has been approached from several directions since J. H. McClellan and T. W. Parks [2]. Recent results of M. K. Atakishiyeva and N. M. Atakishiyev (AA) [3]–[5] aim to enrich the resulting eigensystem with the quality of being canonical. Techniques for associating eigensystems with the DFT include the use of an uncertainty principle associated with cyclic groups of prime order [6], as well as commutative matrices construction for a matrix that commutes with the DFT, thus ensuring that both matrices share the same set of eigenvectors, which provides an orthonormal eigenbasis for the DFT [7]–[9]. This last method is employed in [3],

2020 *Mathematics Subject Classification.* Primary: 47A75 ; Secondaries: 11C20, 39A70, 39B42, 65T50.

Key words and phrases. ladder operators; Hermitian Toeplitz operator; discrete Fourier transform.

Submitted on April 10th, 2024. Accepted on May 24th, 2024.
Communicated by Huseyin CAKALLI and Nazlım Deniz ARAL.

where raising and lowering operators has been found to construct a number operator $\mathcal{N}$ commuting with the DFT, in complete analogy with the linear quantum harmonic oscillator of the continuous case. It is this analogy that motivates the present work targeting the establishment of a broader framework to deal with this problem even more systematically. Can we extend this analogy of the harmonic oscillator for the DFT to mimic the continuous case? To what extent can we make an analogy of the canonical commutation relations (CCR) for finite dimensional Hilbert spaces? This is the common thread that guides us to put forward an algebraic system through some compatibility relations that admit operator solutions of ladder type, beginning with the use of a Hermitian Toeplitz matrix Z_N, which plays the role of the identity. We establish a remarkable relationship between this treatment and the AA-approach, by using the Heun operator W of the latter, thus leading naturally to the proposal of a complete realization of the algebraic system.

The paper is organized as follows: In Section 2, we briefly review some quantum foundations about the CCR. In Section 3, we present the mathematical background necessary for discrete structures in finite dimensional Hilbert space. In Section 4, we establish the discrete commutation relations and show they satisfy the Jacobi identity. In Section 5, we give an explicit matrix representation for Z_N and conduct a brief investigation into the nature of this operator for large N as well. Section 6 is devoted to the main results of this work, namely, the proposal of an algebraic system in terms of the Hermitian Toeplitz operator Z_N and through compatibility relations, whose solutions consist of ladder-type operators. From these operators a family of Hamiltonians $\mathscr{H}$ can be obtained for each N; this solutions, however, fulfill at least two of the four requirements of the proposed algebraic system and not necessarily the other two. In Section 7 we show then how the operators Q and P, which generate Z_N, are related to the raising and lowering operators, as well as the Heun operator W of the AA-approach, through the exponential map. Thus we conclude that this connection could provide a complete realization of the algebraic system; that is, the fulfillment of the four requirements which comprise it. Finally, Section 8 offers concluding remarks on the outstanding issues.

2. Quantum foundations

Canonical commutation relations.

We seek a suitable analogy between continuous and discrete realizations of the canonical commutation relations that underlie the Heisenberg algebra, briefly analyzing the parallels between Classical Mechanics (CM) and Quantum Mechanics (QM)[10].

- Observables in CM are smooth functions on $\mathbb{R}^{2n}$.
- Hermitian operators in QM are regarded as infinitesimal canonical transformations or *infinitesimal automorphisms*, the vector fields are used to obtain (local) canonical transformations by integrating Hamilton's equations. Similarly, Hermitian operators A are employed to derive skew-adjoint operators $2\pi \mathrm{i} A$, which upon the exponentiation yield a one-parameter unitary group.
- The automorphisms of the underlying set $\mathbb{R}^{2n+1}$ are considered, where $\mathbb{R}^{2n+1}$ corresponds to a Lie algebra or a Lie group, depending on whether a bracket operation or a group law is defined.

- The Fourier transform arises naturally from the very wavy nature of QM in an idealized basis of basic plane wave packets (eigenfunctions of the momentum) $e_\xi(x) = e^{2\pi \mathrm{i} x\xi}$, the momentum of which is $h\xi$.
- By constructing the j-th component of momentum through the correspondence of Borel measures with its Hermitian operators, one finds that the Fourier transform intertwines it with the position operator

$$P_j = h\mathcal{F}Q_j\mathcal{F}^{-1} \implies P_j = \frac{h}{2\pi\mathrm{i}}\frac{\partial}{\partial x_j} = hD_j. \tag{2.1}$$

- It has been proven that the action of the exponentials of momentum and position operators is on the functions on L^2 and represents a translation in momentum space and a translation in position space, respectively.
- The basic observables Q_j and P_j satisfy the *canonical cammutation relations* (CCR) (see [10], p.15)

$$[P_j, P_k] = [Q_j, Q_k] = 0, \qquad [P_j, Q_k] = \frac{h\delta_{jk}}{2\pi\mathrm{i}}I. \tag{2.2}$$

Following Seligman's treatment of representation theory [11], the Heinsenberg-Weyl Lie algebra of QM, denoted by $\mathfrak{h}$, with elements Q, P and I over the field of complex numbers is considered. This algebra is defined by the following commutation relations:

$$[Q, P] = \mathrm{i}I, \qquad [Q, I] = 0, \qquad [P, I] = 0. \tag{2.3}$$

The elements Q, P and I form a basis for the algebra $\mathfrak{h}$, so we can express any element E in $\mathfrak{h}$ as:

$$E = xQ + yP + zI, \qquad x, y, z \in \mathbb{C}.$$

Taking Q and P is sufficient for an algebraic basis of $\mathfrak{h}$, as I can be derived from the Lie bracket in the first of equations (2.3). The elements of interest in this abstract scheme are

$$R = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P), \qquad L = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P), \tag{2.4}$$

which satisfy:

$$[L, R] = I, \qquad [L, I] = 0, \qquad [R, I] = 0. \tag{2.5}$$

These operators also form a basis for $\mathfrak{h}$ and thus define $\mathfrak{h}$ as well. The primary goal in this approach is to construct concrete models through representations of $\mathfrak{h}$ with sets of linear operators representable by matrices; this is always feasible since every Lie algebra over $\mathbb{C}$ is isomorphic to some matrix algebra. It may be interesting to note that our search for different bases for representations of the algebra $\mathfrak{h}$ is intimately related with the fact that different bases for the same representations of the Lie group lead to different special functions and provide a group theoretical underpinning for all of these functions (see [12] for a more detailed discussion of this point).

The Heisenberg-Weyl group

For the Heisenberg-Weyl algebra $\mathfrak{h}$ defined in (2.3), we can use its faithful representation and subalgebra of $gl(3, \mathbb{C})$, denoted by $\mathfrak{h}^f$, to build its corresponding Lie Group H^f, which is a subgroup of $GL(3, \mathbb{C})$, through the exponential map

$\exp : \mathfrak{h}^f \to H^f$, such that $\mathfrak{h}^f$ constitutes the tangent space of H^f at the identity. The exponential map

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

exists since the exponential of an arbitrary matrix with finite elements is an absolutely convergent series that yields an invertible matrix; this map sends the zero element in $\mathfrak{h}^f$ to the identity element in H^f. As a manifold, the parameters which form a canonical coordinate system of H^f are given by the Lie group elements of H^f:

$$G(x,y,z) = \exp \mathrm{i}(xQ^f + yP^f + zI^f) = \exp \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ y & \mathrm{i}z & 0 \end{pmatrix}. \tag{2.6}$$

From the properties of the exponential map, one can derive the composition law of the abstract corresponding group H:

$$g(x_1,y_1,z_1)g(x_2,y_2,z_2) = g\Big(x_1+x_2, y_1+y_2, z_1+z_2+\frac{1}{2}[y_1x_2 - x_1y_2]\Big), \tag{2.7}$$

$$e = g(0,0,0), \qquad g(x,y,z)^{-1} = g(-x,-y,-z). \tag{2.8}$$

This group is named the Heisenberg-Weyl (or simply Heisenberg) group. All parameters range over $\mathbb{R}$ so the group manifold is isomorphic to $\mathbb{R}^3$, non-compact and simply connected.

3. Mathematical background for discrete structures

In this section, we explore the discrete structure in finite dimensions, considering the intricacies involved. To address specific nuances, the action of a group G on a set X is represented by a function $f : G \times X \longrightarrow X$ such that for all x in X, $f(e,x) = x$, where e is the identity element of G. An N-dimensional representation of a group G over a field K is a group homomorphism $\phi : G \longrightarrow GL(V)$, where V is an N-dimensional vector space on K, and $GL(V)$ is the group of linear operators on V. If $\phi(g)$ is a unitary operator for every g in G, and its corresponding conjugate transpose satisfies $\phi(g)^\dagger = \phi(g)^{-1}$, we say the representation ϕ is unitary. We consider discrete groups to be Lie groups endowed with the discrete topology; a finite group G acts on itself by automorphisms and can be embedded in some permutation group S_N, which admit a representation on K^N, with K a field.

The Fourier transform arises naturally from the harmonic periodic behaviour of quantum systems, where periodicity is somehow fundamental. Thus, we focus on the finite cyclic abelian group Z_N which is isomorphic to the additive group of integers modulo N, denoted $\mathbb{Z}/N\mathbb{Z}$. We also consider the geometric series:

$$1 + z + z^2 + \ldots + z^{N-1} = \begin{cases} (1-z^N)/(1-z), & \text{if } z \neq 1 \\ N, & \text{if } z = 1 \end{cases}. \tag{3.1}$$

Define $\omega = e^{2\pi \mathrm{i}/N}$, with $\mathrm{i} = \sqrt{-1}$, to be the N-th primitive root of unity, then the set of N-th roots of unity, $\{\omega^k\}$, $k = 0, \ldots, N-1$, is a group and satisfies $1 + \omega + \omega^2 + \ldots + \omega^{N-1} = 0$, since $e^{2\pi \mathrm{i}} - 1 = 0$. Such a group is isomorphic to Z_N and is denoted by C_N. Let's consider $\mathrm{Hom}(\mathbb{Z}/N\mathbb{Z})$ as the set of homomorphisms of $\mathbb{Z}/N\mathbb{Z}$ into C_N. A vector in $\mathbb{C}^N$ is denoted by v and its components by v_j, the canonical basis of $\mathbb{C}^N$ is represented as $e_k = \{(\delta_{k,0}, \ldots, \delta_{k,N-1}) : k = 0, \ldots, N-1\}$, where δ_{kj} is the Kronecker delta function. Operators are denoted by capital letter

T and their matrix entries by T_{lm}. With an abuse of notation, operators and matrix representations of them are denoted with the same letter, unless otherwise specified.

Most of what will be mentioned in this section without proof, can be found in [13].

As discussed in Section 2, the position and momentum operators generate translations in the underlying group, thus we consider a notion of translation in our space of complex-valued functions on $\mathbb{Z}/N\mathbb{Z}$. First we endow it with an inner product to turn it into a Hilbert space $L^2(\mathbb{Z}/N\mathbb{Z})$ by means of

$$\langle f, g\rangle = \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} f(\alpha)\overline{g(\alpha)},$$

where $\bar{x}$ denotes the complex conjugate of x in $\mathbb{C}$. A translation operator $T_a : L^2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow L^2(\mathbb{Z}/N\mathbb{Z})$, for every $a \in \mathbb{Z}/N\mathbb{Z}$, is defined by the action of $\mathbb{Z}/N\mathbb{Z}$ on $L^2(\mathbb{Z}/N\mathbb{Z})$ given by

$$T_a f(b) := f(b-a), \quad \forall a, b \in \mathbb{Z}/N\mathbb{Z}.$$

It can be shown that an orthonormal basis for $L^2(\mathbb{Z}/N\mathbb{Z})$ is $\{f_\alpha\}$, $0 \le \alpha \le (N-1)$, where

$$f_\alpha(b) = \begin{cases} 1 & \text{if } \alpha = b \\ 0 & \text{if } \alpha \neq b \end{cases} \quad \alpha, b \in \mathbb{Z}/N\mathbb{Z}.$$

If we look for a matrix representation V of T in the $\{f_\alpha\}$ basis, we can take the range of $f \in L^2(\mathbb{Z}/N\mathbb{Z})$, ordered by its argument from $0,\ldots,N-1$, as a vector $(f(0), f(1), \ldots, f(N-1)) \in \mathbb{C}^N$. This means that the set $\{f_\alpha\}$ is represented in $\mathbb{C}^N$ by the canonical basis $\{e_l\}$; therefore the matrix representation V of $T_{a=1} := T$ in the $\{f_\alpha\}$ basis is the $N \times N$ matrix given by

$$V = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix};$$

note that $V = C^{\intercal}$, where C is the circulant permutation matrix with entries $C_{kl} = \delta_{k,l-1}$. Thus the full set of matrices, defined as $V_j = V^j$, are unitary and $V^N = I$, $0 \le j \le (N-1)$. These matrices V_j are well known in the literature as the shift matrices and are a basis for the algebra of circulant matrices [14].

On the other hand, the regular representation $\rho : Z_N \to GL(L^2(\mathbb{Z}/N\mathbb{Z}))$ of the cyclic group Z_N is given by $\rho(a_j) = V_j,\ 0 \le j \le (N-1)$ ([15], p.4), this implies that the matrix representation of the translation operators on the function space $L^2(\mathbb{Z}/N\mathbb{Z})$ is the regular representation of the N-cyclic group, which in turn is completely reducible. Moreover, since $\mathbb{Z}/N\mathbb{Z}$ is abelian, it can be decomposed into a direct sum of one-dimensional irreducible representations. This simple decomposition is the source of a rather intricate structure which gives rise to the Fourier analysis, structure that is employed in this work.

Let C^1_N be the multiplicative group of complex numbers of absolute value 1, a character on $\mathbb{Z}/N\mathbb{Z}$ is a group homomorphism $\lambda : \mathbb{Z}/N\mathbb{Z} \longrightarrow C^1_N$. Taking characters in $\lambda_1, \lambda_2 \in Hom(\mathbb{Z}/N\mathbb{Z})$ and defining $(\lambda_1+\lambda_2)(a) := \lambda_1(a)\lambda_2(a)\ \ \forall a \in \mathbb{Z}/N\mathbb{Z}$, it can be concluded that $\mathbb{Z}/N\mathbb{Z} \cong Hom(\mathbb{Z}/N\mathbb{Z}) \cong C_N$. It can be shown that $Hom(\mathbb{Z}/N\mathbb{Z})$ consists of elements of the form

$$\lambda_l := \frac{1}{\sqrt{N}}\, e^{-2\pi \mathrm{i} l/N}, \quad l = 0, \dots, N-1,$$

and the set $\{\lambda_l\}$, $l = 0, \dots, N-1$, is an orthonormal basis of $L^2(\mathbb{Z}/N\mathbb{Z})$, which is a consequence of the geometric series (3.1). We call this basis $\{\lambda_l\}$, $0 \le l \le (N-1)$, the normalized character basis, NCB, for $L^2(\mathbb{Z}/N\mathbb{Z})$. Because of this, we can expand $f \in L^2(\mathbb{Z}/N\mathbb{Z})$ as a linear combination of the $\{\lambda_l\}$ basis, namely

$$f = \sum_{l=0}^{N-1} \hat{f}_l \lambda_l, \qquad \hat{f}_l \in \mathbb{C};$$

such expansion is the (Nth partial sum of the discrete) Fourier series of f. The coefficients $\hat{f}_l$ can be obtained applying orthonormality of the NCB as usual:

$$\begin{aligned} \langle f, \lambda_m \rangle &= \langle \sum_{l=0}^{N-1} \hat{f}_l \lambda_l, \lambda_m \rangle = \sum_{l=0}^{N-1} \hat{f}_l \langle \lambda_l, \lambda_m \rangle \\ &= \sum_{l=0}^{N-1} \hat{f}_l \delta_{lm} = \hat{f_m}, \quad 0 \le m \le (N-1); \end{aligned}$$

whereby

$$\hat{f_m} = \langle f, \lambda_m \rangle = \sum_{n=0}^{N-1} f(n)\overline{\lambda_m(n)} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{2\pi \mathrm{i} mn/N}, \quad 0 \le m \le (N-1).$$

This is clearly the mth component of a matrix multiplication with the vector $(f(0), f(1), \dots, f(N-1))$. We give a name to the underlying linear transformation.

Definition 3.1. *Let* $\{\lambda_m\}$, $l = 0, \dots, N-1$, *be the normalized character basis and define* $\hat{f_m}$ *as* $(\Phi_N f)(m)$. *The discrete Fourier transform (DFT) is the linear operator* $\Phi_N : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z})$, *defined by*

$$(\Phi_N f)(m) := \langle f, \lambda_m \rangle = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} f(n) \exp\left(\frac{2\pi \mathrm{i}}{N} mn\right).$$

The DFT is known to satisfy the following properties:

(1) Φ_N is a unitary operator,
(2) $\Phi_N^4 = I$, where I is the identity operator,
(3) the matrix representation of Φ_N in $\{f_\beta\}$ is

$$(\Phi_N)_{mn} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi \mathrm{i}}{N} mn\right), \quad 0 \le m, n \le (N-1);$$

this implies that the columns of the DFT matrix are an orthonormal basis $\{\epsilon_k\}$ for $\mathbb{C}^N$. We call this basis the normalized Fourier basis (NFB).

Next we get back to the shift matrices to connect them with the DFT. Since the V_j are unitary, they must be unitarily similar to a diagonal matrix because of the *spectral theorem*. Thus the DFT plays a fundamental role in this work because the following holds:

Theorem 3.1. *The DFT Φ_N simultaneously diagonalizes the V_j matrices, with eigenvalues*

$$\alpha_{jl} = exp\left(\frac{2\pi \mathrm{i}}{N} jl\right), \qquad 0 \le j, l \le (N-1),$$

and the columns of the DFT as eigenvectors.

Let $\rho_1 : G \longrightarrow U(\mathcal{H}_1)$ and $\rho_2 : G \longrightarrow U(\mathcal{H}_2)$ be unitary representations on G over the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We say that the operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ intertwines ρ_1 and ρ_2 if $A\rho_1(g) = \rho_2(g)A$, $\forall g \in G$; thus ρ_1 and ρ_2 are said to be unitarily equivalent if A is unitary and the operator A is called an intertwining operator. If U is the diagonal operator, obtained through diagonalization of V by the DFT, then U_j and V_j are unitarily equivalent representations of $\mathbb{Z}/N\mathbb{Z}$ with the DFT as intertwining operator. Namely, $V_j = \Phi_N U_j \Phi_N^{\dagger}$, $j = 0, \ldots, N-1$. With $\omega = \exp(2\pi \mathrm{i}/N)$ the Nth primitive root of unity, $\{\epsilon_k\}$ the NFB and $\{e_k\}$ the canonical basis, the previous theorem implies that

(1) The eigenvectors of U and V satisfy

$$V_j \epsilon_k = \omega^{kj} \epsilon_k \text{ and } U_j e_k = \omega^{kj} e_k,$$

(2) V acts as a shift on the eigenvectors of U and viceversa

$$V_j e_k = e_{k+j} \text{ and } U_j \epsilon_k = \epsilon_{k+j},$$

(3) $U_N = I$;

due to these properties, the matrices U_j are called the clock matrices.

Finally, by applying these properties, $UVe_k = Ue_{k+1} = \omega^{k+1} e_{k+1} = \omega\, \omega^k e_{k+1}$; $VUe_k = V\omega^k e_k = \omega^k V e_k = \omega^k e_{k+1}$. Combining these results, we get $(UV - \omega VU)e_k = 0, \forall k \in \mathbb{Z}/N\mathbb{Z}$, whereof we conclude that the shift and clock matrices V and U satisfy the so-called Weyl commutation relation, namely, $UV = e^{\frac{2\pi \mathrm{i}}{N}} VU$.

4. An algebra with discrete commutation relations

Because of the very definition of translation, the operator V generates translations in the underlying space $L^2(\mathbb{Z}/N\mathbb{Z})$, thus, we look for Hermitian solutions P, whose exponentiation yields the unitary T in complete analogy with eq.(2.6). This request establishes the connection with the quantum picture. This idea is not new; it was previously explored by Santhanam and Tekumalla [16] using a different approach.

Theorem 4.1. *Let j be an element of $\mathbb{Z}/N\mathbb{Z}$. Then a hermitian operator $P_j \in GL(L^2(\mathbb{Z}/N\mathbb{Z}))$, which is a solution of the equation $V_j = exp(i\eta P_j)$, where η is a real parameter, is given by*

$$P_j = \frac{2\pi}{\eta N}\, \Phi_N\, diag\,(0, j, \ldots, (N-1)j) \Phi_N^{-1}. \tag{4.1}$$

Proof. Due to the unitarity of Φ_N,

$$\begin{aligned} P_j^{\dagger} &= \frac{2\pi}{\eta N} \left(\Phi_N \mathrm{diag}(0, j, \ldots, (N-1)j) \Phi_N^{\dagger}\right)^{\dagger} \\ &= \frac{2\pi}{\eta N}\, \Phi_N \mathrm{diag}(0, j, \ldots, (N-1)j) \Phi_N^{\dagger} = P_j, \end{aligned}$$

thus confirming that P_j is Hermitian. Now we show that under exponentiation, we certainly recover V_j. As said before, exponentiation of P_j is well defined since P_j is a matrix with finite elements, it is a series of P_j which converges absolutely and yields a matrix which is invertible; moreover, when multiplied by the imaginary unit i, this matrix becomes a skew-adjoint matrix, which upon exponentiation produces a unitary matrix. Then let $\eta \in \mathbb{R}$ and let's compute the matrix representation of $\exp(\mathrm{i}\eta P_j)$ in the canonical basis $\{e_l\}$; using the unitarity of Φ_N

$$\begin{aligned}\exp(\mathrm{i}\eta P_j) &= \exp\left(\mathrm{i}\eta\frac{2\pi}{\eta N}\Phi_N \mathrm{diag}(0, j, \ldots, (N-1)j)\Phi_N^{-1}\right)\\ &= \Phi_N \exp\left(\frac{2\pi \mathrm{i} j}{N}\mathrm{diag}(0, 1, \ldots, N-1\right)\Phi_N^{-1},\end{aligned}$$

multiplying by Φ_N on the right and applying e_l, we get

$$\begin{aligned}e^{\mathrm{i}\eta P_j}\Phi_N e_l &= \Phi_N I e_l + \Phi_N \frac{2\pi \mathrm{i} j}{N}\mathrm{diag}(0, 1, \ldots, N-1)e_l\\ &\quad + \Phi_N \frac{(2\pi \mathrm{i} j)^2}{2!N^2}\mathrm{diag}^2(0, 1, \ldots, N-1)e_l + \cdots\\ &= \Phi_N\left(1 + \frac{2\pi \mathrm{i} j}{N}l + \frac{(2\pi \mathrm{i} j)^2}{2!N^2}l^2 + \cdots\right)e_l\\ &= \Phi_N e^{2\pi \mathrm{i} j l/N}e_l = \Phi_N U_j e_l, \qquad \forall j, l \in \{0, 1, \ldots, N-1\}.\end{aligned}$$

Whereby, using the intertwining property of Φ_N, it follows

$$e^{\mathrm{i}\eta P_j}\Phi_N e_l = \Phi_N \Phi_N^{\dagger} V^j \Phi_N e_l = V^j \Phi_N e_l, \ \forall j, l \in \{0, 1, \ldots, N-1\}.$$

Thus,

$$\left(e^{\mathrm{i}\eta P_j}\Phi_N - V^j \Phi_N\right)e_l = 0, \quad \forall j, l \in \{0, 1, \ldots, N-1\},$$

and consequently

$$e^{\mathrm{i}\eta P_j}\Phi_N = V^j \Phi_N, \quad \forall j \in \{0, 1, \ldots, N-1\},$$

from which it follows that

$$e^{\mathrm{i}\eta P_j} = V^j, \quad \forall j \in \{0, 1, \ldots, N-1\}.$$

□

Remark. *Clearly, the eigenvalues of P_j are $2\pi jl/(\eta N)$ with the NFB as eigenvectors. Similarly, by exponentiating the U_j, we obtain the same eigenvalues.*

Let the diagonal operators be denoted as

$$Q_j := \frac{2\pi}{\eta N}\mathrm{diag}(0, j, \ldots, (N-1)j);$$

thus the DFT intertwines the operators P_j and Q_j as in eq.(2.1). Thereby the DFT intertwines operators at the group level (the V_j's) and at some algebra level as well (the P_j's); this is a direct consequence of the exponential map and the unitarity of the DFT. Thus intertwining is a necessary condition to build possible algebras at the Schrödinger realization level, but not sufficient.

Next, we consider commutators of skew-adjoint operators,

$$\frac{1}{\mathrm{i}}[\mathrm{i}Q, \mathrm{i}P] = \mathrm{i}[Q, P],$$

with

$$Q := \frac{2\pi}{\eta N}\mathrm{diag}(0, 1, \ldots, N-1), \qquad P := \Phi_N Q \Phi_N^{-1}. \tag{4.2}$$

Remark. *The real number η is a coupling parameter which will take appropriate values according to the finite discrete (DFT), infinite discrete (Fourier series) or continuous (integral Fourier transform) cases we deal with.*

Since $[Q, P]$ remains constant for the unitary transformation uniparametric group $\{V_j\}_{j=0,\dots,N-1}$, this suggest the following definition.

Definition 4.1. *Let the commutator of $Q : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ and $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be defined by Z_N through*

$$Z_N := \mathrm{i}[Q, P].$$

Thus we get

Corollary 4.2. *The operators Q, P, Z_N satisfy the Jacobi identity*

$$[Q, [P, Z_N]] + [P, [Z_N, Q]] + [Z_N, [Q, P]] = 0.$$

Proof. Direct computation of the commutators yields

$$[Q, [P, Z_N]] = \mathrm{i}(2(QP)^2 - Q^2P^2 - 2(PQ)^2 + P^2Q^2),$$

$$[P, [Z_N, Q]] = \mathrm{i}(2(PQ)^2 - P^2Q^2 - 2(QP)^2 + Q^2P^2),$$

$$[Z_N, [Q, P] = 0;$$

therefore

$$[Q, [P, Z_N]] + [P, [Z_N, Q]] + [Z_N, [Q, P]] = 0.$$

□

In addition, the following relationships are established.

Corollary 4.3. *The operators Q_j, P_k and Z_N, $j, k = 0, \dots, N-1$, satisfy the discrete commutation relations*

$$[P_j, P_k] = 0, \quad [Q_j, Q_k] = 0, \quad [P_j, Q_k] = \mathrm{i}jkZ_N.$$

Proof. Since

$$Q = \frac{2\pi}{\eta N}\mathrm{diag}(0, 1, \dots, N-1),$$

then, by Theorem 4.1 and eqs.(4.2), $Q_j = jQ$, thereby $P_j = \Phi_N Q_j \Phi_N^{-1} = \Phi_N jQ\Phi_N^{-1} = jP$, so that $[P_j, Q_k] = P_jQ_k - Q_kP_j = jk(PQ - QP) = -\mathrm{i}^2 jk[P, Q] = -\mathrm{i}jk\mathrm{i}[P, Q] = \mathrm{i}jkZ_N$. The other commutators are trivially satisfied. □

Therefore, Z_N plays the role of the identity in this discrete algebra.

5. About the nature of Z_N

As discussed in the previous section, the operator Z_N plays the role of the identity in this context. In this section we explore more the operator Z_N and suggest that in the limit when $N \to \infty$, $Z_N \to \delta$, where δ is the Dirac delta distribution, which is the identity in distributions under convolution.

5.1. **The explicit form of Z_N.** First we need to know the explicit form of Z_N. So we compute the action of the operators Q_j and P_j on the NFB, to get the explicit form of the matrix entries of $[Q_j, P_k]$. It is clear that P_j is in the canonical representation; since the DFT diagonalizes it, its inverse acts as a transition matrix from $\{e_l\}$ to $\{\epsilon_l\}$, resulting Q_j, which is in the NFB representation (note that the definition we are using for the DFT is with positive sign in the exponent). Since eigenvectors are preserved under the exponential map, the eigenvectors of P_j are the NFB, and because Q_j is diagonal, those of it are the $\{e_l\}$. Also, the corresponding eigenvalues are the exponents of those of V_j, U_j, namely,

$$P_j\epsilon_k = \frac{2\pi jk}{\eta N}\epsilon_k, \qquad Q_j e_k = \frac{2\pi jk}{\eta N} e_k, \tag{5.1}$$

and

$$\epsilon_k = \sum_m \Phi_{mk} e_m, \qquad e_k = \sum_m \Phi^{-1}_{mk}\epsilon_m, \tag{5.2}$$

where Φ_{mk} stands for the matrix entries of Φ_N.

We now compute the transformation of the NFB by the operator P_kQ_j and express it in terms of itself.

Lemma 5.1. *The operator P_kQ_j transforms the NFB as*

$$P_kQ_j\epsilon_m = \frac{4\pi^2 jk}{\eta^2 N^3}\sum_n\sum_{n'} nn'\omega^{n(m-n')}\epsilon_{n'}, \quad 0 \le k, j, m, n, n' \le (N-1).$$

Proof. First we compute $Q_j\epsilon_m$, $0 \le j, m \le (N-1)$ by using the first of eqs.(5.2), to apply the eigenvalue property of Q_j; then the second of eqs.(5.1) to obtain the corresponding eigenvalues; finally the second of eqs.(5.2) to express the result in the NFB basis:

$$\begin{aligned} Q_j\epsilon_m &= Q_j\sum_n \frac{1}{\sqrt{N}}\omega^{nm}e_n = \sum_n \frac{1}{\sqrt{N}}\omega^{nm}Q_j e_n \\ &= \frac{1}{\sqrt{N}}\sum_n \omega^{nm}\frac{2\pi j}{\eta N} n e_n = \frac{2\pi j}{\eta N^2}\sum_n \omega^{nm} n\sum_{n'}\omega^{-nn'}\epsilon_{n'} \\ &= \frac{2\pi j}{\eta N^2}\sum_n\sum_{n'} n\omega^{n(m-n')}\epsilon_{n'}. \end{aligned} \tag{5.3}$$

Next we get

$$P_kQ_j\epsilon_m = \frac{2\pi j}{\eta N^2}\sum_n\sum_{n'} n\omega^{n(m-n')}\epsilon_{n'}\frac{2\pi k n'}{\eta N}\epsilon_{n'} \quad = \quad \frac{4\pi^2 jk}{\eta^2 N^3}\sum_n\sum_{n'} nn'\omega^{n(m-n')}\epsilon_{n'}.$$

□

Therefore the matrix elements of the commutator satisfy the following

Theorem 5.2. *The matrix elements of $[Q_j, P_k]$ in the NFB are given by*

$$[Q_j, P_k]_{lm} = \frac{4\pi^2 jk}{\eta^2 N^3}\sum_n (m-l)n\omega^{n(m-l)},$$

where $j, k, l, m, n = 0, \ldots, N-1$.

Proof. Let $[Q_j, P_k]_{lm}$ be the (l,m) entry of the matrix representation of $[Q_j, P_k]$ in the NFB, then, because of eq.(5.3), Lemma 5.1, and orthonormality, it follows that

$$\begin{aligned}
[Q_j, P_k]_{lm} &= \langle \epsilon_l, [Q_j, P_k]\epsilon_m \rangle = \langle \epsilon_l, \frac{2\pi k m}{\eta N} Q_j \epsilon_m - P_k Q_j \epsilon_m \rangle \\
&= \langle \epsilon_l, \frac{2\pi k m}{\eta N} \frac{2\pi j}{\eta N^2} \sum_n \sum_{n'} n \omega^{n(m-n')} \epsilon_{n'} \\
&\quad - \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} n n' \omega^{n(m-n')} \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \langle \epsilon_l, \sum_n \sum_{n'} (m-n') n \omega^{n(m-n')} \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} (m-n') n \omega^{n(m-n')} \langle \epsilon_l, \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n (m-l) n \omega^{n(m-l)},
\end{aligned}$$

where bilinearity of $\langle , \rangle$ has been used. □

To estimate the behaviour of Z_N for large N, it is necessary to recenter the matrix elements of $[Q_j, P_k]$ with respect to its entries lm. Set

$$r := \begin{cases} N = 2L+1, & \frac{N-1}{2} = L \\ N = 2M, & \frac{N}{2} = M \end{cases};$$

we define new centered indices l', m', n' the by means of $n' = n - r$, $m' = m - r$, $n' = n - r$, then

$$[Q,P]_{lm} = \frac{4\pi^2}{\eta^2 N^3} \omega^{(m'-l')r} \sum_{n'} (m'-l')(n'-r) \omega^{n'(m'-l')},$$

thereby

$$[Q,P]_{lm} = \omega^{(m-l)r} [Q,P]^C_{l'm'}, \tag{5.4}$$

where $[Q,P]^C_{l'm'}$ is given as in the following definition.

Definition 5.1. *$[Q,P]^C_{l'm'}$ is called the centered version of $[Q,P]_{lm}$, and is given by*

$$[Q,P]^C_{l'm'} = \frac{4\pi^2}{\eta^2 N^3} \sum_{n'} (m'-l')(n'+r) \omega^{n'(m'-l')},$$

where

$$l', m', n' := \begin{cases} -\frac{N-1}{2}, \ldots, \frac{N-1}{2} & \text{if } N \text{ is odd} \\ -\frac{N}{2}, \ldots, \frac{N}{2} - 1 & \text{if } N \text{ is even} \end{cases}.$$

The centered version can be simplified.

Proposition 5.3. *The centered version of $[Q,P]$ satisfies*

$$[Q,P]^C_{l'm'} = \frac{4\pi^2}{\eta^2 N^3} \sum_{n'} (m'-l') n' \omega^{n'(m'-l')}.$$

Proof. We take from Definition 5.1 the sum with factor $1/N$ and split it into two terms; so on account that $m'-l'=m-l$ one can write

$$\frac{1}{N}\sum_{n'}(m'-l')(n'+r)\omega^{n'(m'-l')} = \frac{1}{N}\sum_{n'}(m'-l')n'\omega^{n'(m'-l')} + \frac{1}{N}\sum_{n'}(m'-l')r\omega^{n'(m'-l')},$$

where the second term vanishies,

$$\begin{aligned}\frac{1}{N}\sum_{n'}(m'-l')r\omega^{n'(m'-l')} &= \frac{m-n}{N}r\sum_{n=0}^{N-1}\omega^{(n-r)(m-l)}\\ &= (m-l)r\omega^{-r(m-l)}\frac{1}{N}\sum_{n=0}^{N-1}\omega^{n(m-l)}\\ &= (m-l)r\omega^{-r(m-l)}\delta_{ml}=0,\ \forall\ m,l=0,\dots,N-1.\end{aligned}$$

The Kronecker delta δ_{ml} appears after using the geometric series in eq.(3.1); therefore, putting this in the Definition 5.1, we get the assertion. □

So the centered version is the non-centered times the phase factor $\omega^{-(m-l)r}$ for each matrix entry.

Corollary 5.4. $Z_N = \mathrm{i}[Q,P]$ *is a traceless Hermitian Toeplitz operator.*

Proof. This clearly follows from Theorem 5.2. □

5.2. **On the behaviour of Z_N for large N.** We are now in a position to roughly estimate the behaviour of $Z_N=\mathrm{i}[Q,P]$ for large N, provided that a restriction on the η parameter is given. In this subsection, we relax formality and rigor to gain intuition on Z_N. We are going to deal a little with tempered distributions as continuous linear functionals on the space of Schwartz functions and also with the Fourier transform of distributions. A gentle treatment of this concepts can be found in [17] and a rigorous one in [18].

Let $\mathcal{F}$ be the set of square integrable periodic functions $f:[-\pi,\pi]\longrightarrow\mathbb{C}$ with convergent Fourier series in the basis $\{\lambda_n(t)=e^{-\mathrm{i}nt} : n=0,1,...,N-1\}$. Then

$$f(t)=\sum_{-\infty}^{\infty}\hat{f}_n e^{-\mathrm{i}nt},$$

with

$$\langle f(t),\lambda_m(t)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{\mathrm{i}nt}dt =: \hat{f}_n; \tag{5.5}$$

with $\hat{f}_n$ the Fourier transform of f defined through the inner product $\langle , \rangle$ on the corresponding Hilbert space $\mathcal{H}$. Then, introducing the Nth partial sum of f,

$$\begin{aligned} f(t) &= \sum_{-\infty}^{\infty} \hat{f}_n e^{-int} = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}_n e^{-int} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t') e^{int'} dt' e^{-int} = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t') e^{in(t'-t)} dt' \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} f(t') \sum_{n=-N}^{N} e^{in(t'-t)} dt = \lim_{N\to\infty} \int_{-\infty}^{\infty} f(t') D_N(t'-t) dt', \end{aligned} \tag{5.6}$$

where D_N is a well known kernel:

Definition 5.2. *The sequence of functions $\{D_N\}, N \in \mathbb{N}$, is called Nth Dirichlet kernel,*

$$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$

It is also well known ([19]) that, using the geometric series (3.1), the Dirichlet kernel obeys, after inserting a factor of $2\pi/N$in the variable t

(1) $D_N(t) = \frac{\sin[(N+\frac{1}{2})t]}{\sin(t)}$,
(2) $\int_{-\pi}^{\pi} D_N(t)dt = 1$,
(3) $D_N(0) = 2N + 1$.

Due to these facts, the Dirichlet kernel is taken as an approximating sequence of functions for the Dirac delta distribution δ. So in the sense of distributions we can write

$$\lim_{N\to\infty} D_N = \delta. \tag{5.7}$$

Definition 5.3. *If f and g are two integrable 2π periodic functions, then*

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y) dy \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y) dy$$

is the convolution of f and g.

Hence, it inmediately follows that the partial sums in eq.(5.6) satisfy

$$S_N(f)(t) \quad := \quad \sum_{-N}^{N} \hat{f}_n e^{int} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t') D_N(t'-t) dt' = (f * D_N)(t);$$

thus $f(t) = \lim_{N\to\infty} (D_N * f)(t),\ \forall\ t \in (-\pi, \pi),\ \forall\ f \in \mathcal{F}$, so, using eq.(5.7), we can write

$$\delta * f = f \qquad \forall f \in \mathcal{F}, \tag{5.8}$$

because f determines a well defined tempered distribution through its action on test functions ϕ under the integral sign and, with an abuse of notation, it is common to write f instead of its induced distribution T_f. This means we can consider δ as the identity under convolution in the distributional sense.

Unfortunately, the Dirichlet kernel is known to be problematic as a kernel function and is not a good kernel in the sense of [19], pp. 99, 102, since as $N \to \infty$ the area between the curves and the x-axis measured in absolute value diverges. This makes the kernel very difficult to work with and one has to be very careful when dealing with integrals with absolute value because they may not converge. To avoid this difficulty, it is usual to use the so-called Fejér kernel instead, obtained through arithmetic mean of $D_0, D_1, \ldots, D_{N-1}$, which is named a *Cesàro mean.* This remedy the problems because averaging frequently make things better behaved; thus, a non-negative good kernel is obtained (cf.[19], Remark 5, p.103). But convergence of the partial sums $S_N(f)(t)$ is guaranteed however, if f is Lipschitz or differentiable, then they converge pointwise everywhere.

Let us take the derivative of D_N:

$$D'_N(t'-t) := \frac{\mathrm{d}D_N(t'-t)}{\mathrm{d}t'} = \sum_{n=-N}^{N} \mathrm{i}ne^{\mathrm{i}n(t'-t)},$$

so that

$$(t'-t)D'_N(t'-t) = \sum_{n=-N}^{N} \mathrm{i}n(t'-t)e^{\mathrm{i}n(t'-t)}.$$

Now we take a sample of N data given by $t' = \frac{2\pi m}{N}, t = \frac{2\pi l}{N} \in [-\pi, \pi]$. When coupling continuous and discrete treatments, l, m and n are taken according to Definition 5.3. This avoids changing the 2π factor in the exponential. Therefore, using Lemma 5.3 and completing necessary factors we get

$$\begin{aligned}
(m-l)D'_N(m-l) &= \sum_{n=-N}^{N} n(m-l)\exp\left(\frac{2\pi\mathrm{i}}{N} n(m-l)\right) \\
&= \frac{4\pi^2}{\eta^2 N^3}\frac{N^3\eta^2\mathrm{i}}{4\pi^2} \sum_{n=-N}^{N} (m-l)n\omega^{n(m-l)} = \frac{N^3\eta^2}{4\pi^2}\mathrm{i}[Q,P]_{lm} \\
&= Z_{lm}, \quad \text{provided } \eta = \frac{2\pi}{N^{3/2}};
\end{aligned}$$

that is,

$$Z_{lm} = (m-l)D'_N(m-l). \tag{5.9}$$

A refined computation can be made if we consider a continuum of frequencies; then the Fourier series is replaced by the integral Fourier transform and η would take the value $\sqrt{2\pi/N}$.

The eq.(5.9) suggests the following claim: in the limit as $N \to \infty$, it is expected that $Z_{lm} \to (t'-t)\delta'(t'-t)$. Then, since $f(t)\delta'(t) = -f'(t)\delta(t)$, taking $f(t) = t'-t$, it is found that $Z_{lm} \to -(-1)\delta(t'-t) = \delta(t'-t)$ as $N \to \infty$. Thus, in accordance with eq.(5.8), the limit of Z_N is conjectured to be the identity under convolution in the vector space of tempered distributions. In addition, since Z_N is represented by a Toeplitz matrix, it is natural to ask if its entries are the coefficients of the Fourier series of a real valued function, as it happens for Toeplitz matrices when their entries are the Fourier coefficients of an L^1 function or of a Radon function, converging to zero as $N \to \infty$ for the former and remaining bounded for the latter. Thus, as $N \to \infty$ we wonder what is this real valued function whose Fourier transform is the Dirac δ. It is known that such a function is 1; that is to say, to be precise, the Fourier transform of the distribution T_1 induced by 1 is the distribution δ (see [17], p.203). In the study of algebras of Toeplitz operators, such a real valued function is called the *symbol* of the Toeplitz operator; thus, in our context, the symbol of the limit of the Toeplitz Z_N is the constant 1 as a distribution. The issues about boundedness and compactness is analyzed in [20] for symbols as functions and in [21] for symbols as distributions. Finally, the formal analysis about the veracity of these claims requires the rigorous application of distribution theory and it is the topic for future work.

6. An algebraic system associated with the DFT

Now we proceed to the construction of operators with the ladder property, starting from their generic definition and giving to Z_N its role as a form of identity in its own right. We do not use the traditional definitions $(1/\sqrt{2})(Q \pm \mathrm{i}P)$ since there is no a priori reason to assume they should be valid in the context of the discrete commutation relations. This implies relaxing the condition of the "canonical commutation relations (CCR) equal to a constant" ([22], [23]) and give enough freedom to look for solutions associated to the DFT, involving new operators $\mathscr{H}$, L^- and L^+, but taking a sort of compatibility relations as elemental and even lift them to a fundamental postulate in the context we are dealing with. These compatibility relations are also named compatibility of Hamilton's equations with the Heisenberg equations in [24], p.5 and [25], p.3., as well as Heisenberg-Schrödinger consistency relations. A similar study can be found in [26].

6.1. **Discrete algebraic system from Z_N.** To obtain the form of the compatibility relations in the context we are dealing with, let us take for a while the standard definitions $L = (1/\sqrt{2})(Q + \mathrm{i}P)$, $R = (1/\sqrt{2})(Q - \mathrm{i}P)$ and notice that they imply that

$$\mathscr{H} = \frac{1}{2}\{R, L\}, \qquad Z_N = [R, L].$$

From this it follows that the commutation relations

$$[\mathscr{H}, R] = -\frac{1}{2}\{R, Z_N\}, \qquad [\mathscr{H}, L] = \frac{1}{2}\{L, Z_N\},$$

are valid and one gets

$$[\{R, L\}, R] = -\{R, Z_N\}, \qquad [\{R, L\}, L] = \{L, Z_N\}.$$

We call the above relations the discrete compatibility relations (DCR) and refer to the operator $\mathscr{H}$ as the Hamiltonian operator. Now we leave the standard definitions of L and R and at the same time, leverage such relations as a fundamental postulate in the context of the discrete commutation relations, for their continuous counterpart is not guaranteed to be so in quantum mechanics as is mentioned by Wigner in [27]. Further, the fact that $Z_N, N \in \mathbb{N}$, determines a sequence of Hermitian Toeplitz matrices according to Corollary 5.4, allows to naturally connect those relations with the DFT, in the sense that the entries of Z_N are linked to an approximating sequence of a distribution, whose Fourier transform is the Dirac δ, namely, the distribution T_1 induced by the constant 1, as discussed above.

On the other hand, Toeplitz matrices can be split into two parts, so that

$$Z_N = \sum_{k=1}^{N-1} z_{-k}(B^T)^k + \sum_{k=0}^{N-1} z_k B^k,$$

where B is the backward shift matrix satisfying $B^T = KBK$ with K the reversal matrix defined by ones in the antidiagonal and zero otherwise.

Taking the parity operator, used in the AA-approach, as $S = KV$ with V the basic circulant matrix, there seems to be an enriched structure which provides a possible framework that we believe it is worthwhile analyzing.

Summarizing, the following algebraic system is proposed (note that we will also refer to it as an algebraic scheme in this work).

Construct operators $L^- : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ and $L^+ : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ such that

(1) The operator $\mathscr{H} := (1/2)\{L^+, L^-\}$ is Hermitian (semi-)positive definite;
(2) The discrete compatibility relations (DCR) must be satisfied,

$$[\{L^+, L^-\}, L^+] = -\{L^+, Z_N\},\ [\{L^+, L^-\}, L^-] = \{L^-, Z_N\}; \tag{6.1}$$

(3) Parity–point reflection–condition: to split the eigenvectors in even and odd parts,

$$[\{L^+, L^-\}, S] = 0; \tag{6.2}$$

(4) Commutation with the DFT: to obtain an eigensystem for Φ_N from the Hamiltonian $\mathscr{H}$,

$$[\{L^+, L^-\}, \Phi_N] = 0. \tag{6.3}$$

In this scheme, non-equally spaced eigenvalues of the Hamiltonian are allowed, for this property plays a key role in quantum information processes. To see if there exist models for this system, the general form for a discrete Hermitian operator is used

$$\mathscr{H} = \sum_{n=0}^{N-1} \xi_n \phi_n \langle \phi_n, \cdot \rangle,$$

where ϕ_n is an eigenbasis of $\mathscr{H}$. Then use the ladder-type generic form to construct creation and annihilation operators through

$$L^+ := \sum_{n=0}^{N-1} r_n \phi_{n+1} \langle \phi_n, \cdot \rangle, \qquad L^- := \sum_{n=0}^{N-1} l_n \phi_{n-1} \langle \phi_n, \cdot \rangle.$$

Sufficient conditions to solve this scheme are provided in what follows.

6.2. **Solving the discrete compatibility relations.** Following the algebraic system proposed in the last paragraph, we establish now sufficient criteria to find solutions for the first two requirements, at least. We aim to establish the conditions under which Hamiltonians can be constructed and to understand their relationship with the DFT within the framework of the DCR. In what follows, bra-ket notation is used.

Let $\mathscr{H}$ be a (semi-)positive definite Hermitian operator on $L^2(\mathbb{Z}/N\mathbb{Z})$ and $\{|\xi_j\rangle | j = 0, \ldots, N-1\}$ be a complete set of eigenvectors of $\mathscr{H}$, such that

$$\mathscr{H}|\xi_j\rangle = \xi_j|\xi_j\rangle, \tag{6.4}$$

also

$$\sum_{n=0}^{N-1} |\xi_n\rangle\langle\xi_n| = I,$$

therefore

$$\mathscr{H} = \sum_{n=0}^{N-1} \xi_n |\xi_n\rangle\langle\xi_n|.$$

Definition 6.1. *The creation and annihilation linear operators $L^+, L^- : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, are defined by*

$$L^+ := \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n|, \qquad L^- := \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|.$$

We establish some results to determine under which conditions solutions $L^\pm$ can be found, such that

$$[\{L^+, L^-\}, L^+] = -\{L^+, Z_N\}, \quad [\{L^+, L^-\}, L^-] = \{L^-, Z_N\}$$

holds. Thus we have the following

Lemma 6.1. *$|\xi_j\rangle$ is an eigenvector of $\{L^+, L^-\}$ with eigenvalue $l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-$.*

Proof. We compute directly, using definitions and orthonormality, that

$$\begin{aligned}
\{L^+, L^-\}|\xi_j\rangle &= (L^+L^- + L^-L^+)|\xi_j\rangle \\
&= L^+ \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|\xi_j\rangle + L^- \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n|\xi_j\rangle \\
&= L^+ l_j^- |\xi_{j-1}\rangle + L^- l_j^+ |\xi_{j+1}\rangle \\
&= \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n| l_j^- |\xi_{j-1}\rangle + \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n| l_j^+ |\xi_{j+1}\rangle \\
&= (l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-)|\xi_j\rangle.
\end{aligned}$$

□

Lemma 6.2. *The matrix representations for $[\{L^+, L^-\}, L^\pm]$ in the $\{|\xi_j\rangle\}$ basis, are given by*

$$[\{L^+, L^-\}, L^-]_{kj}^\xi = (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^+ \delta_{k,j-1},$$

$$[\{L^+, L^-\}, L^+]_{kj}^\xi = (l_{j+1}^+ l_{j+2}^- - l_{j-1}^+ l_j^-) l_j^+ \delta_{k,j+1},$$

respectively.

Proof. Computing the commutator with L^- using the previous Lemma 6.1,

$$\begin{aligned}
[\{L^+, L^-\}, L^-]|\xi_j\rangle &= (\{L^+, L^-\}L^- - L^-\{L^+, L^-\})|\xi_j\rangle \\
&= \{L^+, L^-\} \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|\xi_j\rangle - L^-(l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-)|\xi_j\rangle \\
&= \{L^+, L^-\} l_j^- |\xi_{j-1}\rangle - (l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-) l_j^- |\xi_{j-1}\rangle \\
&= (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- |\xi_{j-1}\rangle,
\end{aligned}$$

whereby, the orthonormality of $\{|\xi_j\rangle\}$ implies

$$\begin{aligned}
\langle\xi_k|[\{L^+, L^-\}, L^-]|\xi_j\rangle &= \langle\xi_k|(l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- |\xi_j\rangle \\
&= (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- \delta_{k,j-1}.
\end{aligned}$$

The second equation is obtained similarly. □

Next, we obtain the matrix representation of $\{L^\pm, Z_N\}$. Let $Z^\xi_{\alpha\beta}$ be the matrix representation of Z_N in $\{|\xi_j\rangle\}$ basis and recall that $\{|e_j\rangle\}$ is the canonical basis. It is clear then that, if $C_{\alpha\beta} := \langle\xi_\alpha|e_m\rangle$ represents the transition matrix from $\{|e_j\rangle\}$ to $\{|\xi_j\rangle\}$, we have

$$Z^\xi_{\alpha\beta} = \sum_{m,n} C_{\alpha m} Z^e_{mn} C^{-1}_{n\beta} = \sum_{m,n} C_{\alpha m} Z^e_{mn} \bar{C}_{\beta n} = \sum_{m,n} \langle\xi_\alpha|e_m\rangle Z^e_{mn} \langle e_n|\xi_\beta\rangle,$$

since C is a unitary matrix.

Lemma 6.3. *The matrix representations of $\{L^\pm, Z_N\}$ in the $\{|\xi_j\rangle\}$ basis, are given by*

$$\{L^\pm, Z_N\}^\xi_{kj} = l^\pm_{k\mp1} Z^\xi_{k\mp1,j} + l^\pm_j Z^\xi_{k,j\pm1},\ k, j = 0, 1, \dots, N-1,$$

where $Z^\xi_{\alpha\beta}$ are the entries of the matrix representation of Z_N in the $\{|\xi_j\rangle\}$ basis.

Proof. Since $Z^\xi_{\alpha\beta}$ is the representation of Z_N in the $\{|\xi_j\rangle\}$ basis, then the action of Z_N on a basis vector can be expanded as $Z_N|\xi_j\rangle = \sum_\alpha Z^\xi_{\alpha j}|\xi_\alpha\rangle$, so that

$$\begin{aligned} \{L^\pm, Z_N\}^\xi_{kj} &= \langle\xi_k|L^\pm Z_N + Z_N L^\pm|\xi_j\rangle = \langle\xi_k|L^\pm \sum_\alpha Z^\xi_{\alpha j}|\xi_\alpha\rangle + \langle\xi_k|Z_N l^\pm_j|\xi_{j\pm1}\rangle \\ &= \langle\xi_k|\sum_\alpha Z^\xi_{\alpha j} l^\pm_\alpha |\xi_{\alpha\pm1}\rangle + l^\pm_j Z^\xi_{k,j\pm1} = l^\pm_{k\mp1} Z^\xi_{k\mp1,j} + l^\pm_j Z^\xi_{k,j\pm1}. \end{aligned}$$

□

Therefore, putting all this together, one can readily see that the DCR are equivalent to

$$(l^\pm_{j\pm1} l^\mp_{j\pm2} - l^\pm_{j\mp1} l^\mp_j) l^\pm_j \delta_{k,j\pm1} = \mp l^\pm_{k\mp1} Z^\xi_{k\mp1,j} \mp l^\pm_j Z^\xi_{k,j\pm1},\ \forall k, j = 0, 1, \dots, N-1. \quad (6.5)$$

Now conjugate transposition between the $L^\pm$ operators is imposed, and the Hermiticity of Z_N becomes essential to employ.

Proposition 6.4. *Let's suppose that the DCR hold. If $(L^-)^\dagger = L^+$, then*

(1) *the DCR are equivalent, and*
(2) *they reduce to*

$$(|l^-_{j-1}|^2 - |l^-_{j+1}|^2) l^-_j \delta_{k,j-1} = l^-_{k+1} Z^\xi_{k+1,j} + l^-_j Z^\xi_{k,j-1},\ \forall k, j = 0, 1, \dots, N-1. \quad (6.6)$$

Proof. First consider that

$$\begin{aligned} (L^-)^\dagger|\xi_j\rangle &= \left(\sum_{k=0}^{N-1} l^-_k |\xi_{k-1}\rangle\langle\xi_k|\right)^\dagger |\xi_j\rangle = \left(\sum_{k=0}^{N-1} \overline{l^-_k} |\xi_k\rangle\langle\xi_{k-1}|\right)|\xi_j\rangle \\ &= \sum_{k=0}^{N-1} \overline{l^-_k} |\xi_k\rangle \delta_{k-1,j},\ k-1 = j, \\ &= \overline{l^-_{j+1}} |\xi_{j+1}\rangle; \end{aligned}$$

on the other hand $L^+|\xi_j\rangle = l^+_j|\xi_{j+1}\rangle$, thus, if $(L^-)^\dagger = L^+$, we necessarily have $(\overline{l^-_{j+1}} - l^+_j)|\xi_{j+1}\rangle = 0,\ \forall\ j = 0, \dots, N-1$, whereby $l^+_j = \overline{l^-}_{j+1}$. Therefore, using this and eq.(6.5) we get for the DCR

$$(|l^-_{j+2}|^2 - |l^-_j|^2)\overline{l^-_{j+1}} \delta_{k,j+1} = -\overline{l^-_k} Z^\xi_{k-1,j} - \overline{l^-_{j+1}} Z^\xi_{k,j+1},$$

$$(|l^-_{j-1}|^2 - |l^-_{j+1}|^2) l^-_j \delta_{k,j-1} = l^-_{k+1} Z^\xi_{k+1,j} + l^-_j Z^\xi_{k,j-1}.$$

So to probe part 1 of the proposition, it is enough to conjugate transpose anyone of the last equations to obtain the other using the Hermiticity of Z_N (Corollary 5.4) and a change of index. Part 2 clearly follows as a consequence of the equivalence in part 1 picking the second expression. □

Thus, when imposing the conjugate transposition condition, we can deal with only one of the DCR contained in eq.(6.5), namely that of part 2 of the last proposition; it deploys into the following two equations after applying the definition of the Kronecker delta:

$$|l^-_{j-1}|^2 - |l^-_{j+1}|^2 = Z_{jj} + Z_{j-1,j-1},\ k = j-1, \tag{6.7}$$

$$l^-_{k+1} Z^\xi_{k+1,j} + l^-_j Z^\xi_{k,j-1} = 0,\ k \neq j-1. \tag{6.8}$$

This means that we have a set of N^2 equations, which we would like to solve for $l^- := \{l^-_0, l^-_1, \ldots, l^-_{N-1}\}$; or equivalently, to solve $2N-1$ recurrence relations: the first one given by eq.(6.7) with $k = j-1$ and the other remaining ones correspond to $k = j-N+1, j-N, j-N-1, \ldots, j-2, j, j+1, j+2, \ldots, j+N-2$ given by eq.(6.8). We make the following correspondence about the indices: $N \to 0$ and $-1 \to N-1$, for instance, $l_N = l_0$ and $l_{-1} = l_{N-1}$. Expanding the first recurrence relation, we deploy N of the N^2 equations as

$$\begin{aligned}
|l^-_{N-1}|^2 - |l^-_1|^2 &= Z^\xi_{00} + Z^\xi_{N-1,N-1},\ j=0, k=N-1\\
|l^-_0|^2 - |l^-_2|^2 &= Z^\xi_{11} + Z^\xi_{00},\ j=1, k=0,\\
|l^-_1|^2 - |l^-_3|^2 &= Z^\xi_{22} + Z^\xi_{11},\ j=2, k=1,\\
&\vdots\\
|l^-_{N-2}|^2 - |l^-_0|^2 &= Z^\xi_{N-1,N-1} + Z^\xi_{N-2,N-2},\ j=N-1, k=N-2.
\end{aligned} \tag{6.9}$$

To solve this recurrence relation for a given N, we need to express everything in terms of an initial $|l^-_0|^2$; it turns out that this is possible for N odd only, whereas for N even, l^-_1 is additionally required, whence we have to treat the even and odd cases separately. The solution is summarized in the following theorem, in which another of the remarkable properties of Z_N is required, namely, its tracelessness.

Theorem 6.5. *The solutions for the recurrence relation (6.7) satisfy the following hyperbolic relations:*

(1) *for N odd,*

$$\begin{aligned}
|l^-_{N-1}|^2 - |l^-_0|^2 &= Z^\xi_{N-1,N-1},\\
|l^-_1|^2 - |l^-_0|^2 &= -Z^\xi_{00};
\end{aligned}$$

(2) *for N even,*

$$\begin{aligned}
|l^-_{N-2}|^2 - |l^-_0|^2 &= Z^\xi_{N-2,N-2} + Z^\xi_{N-1,N-1},\\
|l^-_{N-1}|^2 - |l^-_1|^2 &= Z^\xi_{00}.
\end{aligned}$$

Proof. Note the l^-_j are interrelated by even and odd indices in (6.9), so we separate the equations in sets of even and odd indices, which means we will have one no interrelating equation when N is even. Then we solve the even and odd cases for N separately.

Case 1: N odd. Lets separate the indices of $j = \{0, 1, \ldots, N-1\}$ in even and odd integers. Solving eqs.(6.9) for $j = 2m, m = 1, 2, \ldots, (N-1)/2$ in terms of l_0^- and substituting recursively,

$$\begin{aligned} |l_2^-|^2 &= |l_0^-|^2 - Z_{00}^\xi - Z_{11}^\xi, \\ |l_4^-|^2 &= |l_0^-|^2 - Z_{00}^\xi - Z_{11}^\xi - Z_{22}^\xi - Z_{33}^\xi, \\ &\vdots \\ |l_{N-1}^-|^2 &= |l_0^-|^2 + Z_{N-1,N-1}^\xi - Tr(Z_N). \end{aligned} \tag{6.10}$$

Similarly for $j = 2m-1$ in reverse order,

$$\begin{aligned} |l_{N-2}^-|^2 &= |l_0^-|^2 + Z_{N-2,N-2}^\xi + Z_{N-1,N-1}^\xi, \\ |l_{N-4}^-|^2 &= |l_0^-|^2 + Z_{N-4,N-4}^\xi + Z_{N-3,N-3}^\xi + Z_{N-2,N-2}^\xi + Z_{N-1,N-1}^\xi, \\ &\vdots \\ |l_1^-|^2 &= |l_0^-|^2 - Z_{00}^\xi + Tr(Z_N). \end{aligned} \tag{6.11}$$

Case 2: N even. This case can be handled similarly, just that it is not possible to express everything in terms of only $|l_0^-|^2$, but $|l_1^-|^2$ becomes also necessary.

For $j = 2m$ in (6.9)

$$|l_{N-2}^-|^2 = |l_0^-|^2 - \sum_{m=0}^{N-3} Z_{mm}^\xi,$$

for $j = 2m-1$

$$|l_{N-1}^-|^2 = |l_1^-|^2 - \sum_{m=1}^{N-2} Z_{mm}^\xi;$$

which are equivalent to

$$|l_{N-2}^-|^2 = |l_0^-|^2 + Z_{N-2,N-2}^\xi + Z_{N-1,N-1}^\xi - \mathrm{Tr}(Z_N),$$

$$|l_{N-1}^-|^2 = |l_1^-|^2 + Z_{00}^\xi - \mathrm{Tr}(Z_N).$$

Finally we get the result, stated in this theorem, remembering that Z_N is traceless, in accordance with Corollary 5.4. □

Note that the odd case requires only three parameters l_0^-, l_1^-, l_{N-1}^-, whereas the even case requires four. The fact that only the modules of the l_j^- are involved, anticipates the existence of many operators $L^\pm$ and so, many hamiltonians $\mathscr{H}$.

Hitherto two main properties of Z_N have been used: its Hermiticity and its tracelessness; now its diagonalizability is needed.

Corollary 6.6. *An annihilation operator L^- which is a solution of the DCR, exists if the parameters $l_0^-, l_1^-, l_{N-1}^-, l_{N-2}^-$ satisfy the hyperbolic relations (6.5) and $\{|\xi_j\rangle\}$ is a complete basis of eigenvectors of Z_N.*

Proof. Since $l_0^-, l_1^-, l_{N-1}^-, l_{N-2}^-$ satisfies the relations (6.5), then the recurrence (6.7) is fulfilled. As for the remaining $2N-2$ recurrences in (6.8), the fact that $|\xi_j\rangle$ are eigenvectors of Z_N, implies Z_N^ξ is diagonal, therefore $Z_{k,j-1}^\xi = Z_{k+1,j}^\xi = 0 \; \forall \; k = j-N+1, j-N, \ldots, j-2, j, j+1, \ldots, j+N-2$; that is, the off-diagonal elements of Z_N^ξ vanish, thus, eq.(6.8) is satisfied. □

Therefore, the construction of the operators $L^{\pm}$ is recursively given by (6.10) and (6.11) starting with solutions of the hyperbolic relations.

Example 6.1. *We can obtain a numeric example running code to compute $Z_N = i[Q,P]$ for $N = 13$ in Wolfram-Mathematica, this program yields the following eigenvalues for Z_{13}*

$$Z^{\xi}_{00} = 11.1582, \quad Z^{\xi}_{1212} = -0.371055,$$

whereby the corresponding hyperbolic relations of Theorem 6.5 which solve the recurrence relation are

$$|l^-_{12}|^2 - |l^-_1|^2 = Z^{\xi}_{00} + Z^{\xi}_{1212} = 10.7872,$$
$$|l^-_1|^2 - |l^-_0|^2 = -11.1582.$$

This implies that $|l^-_1|^2 = |l^-_0|^2 - 11.1582$, and therefore $|l^-_0|^2 > 11.1582$. So if we take $|l^-_0| = 12$, then the circle $C_0 = \{l_0 \in \mathbb{C} | |l^-_0| = 12\}$ determines an infinite set of solutions. We choose building on the imaginary axis, so we take $l^-_0 = 12i$; hence $|l^-_1|^2 = 144 - 11.1582, or |l^-_1| = 11.5257$, thus we take $l^-_1 = 11.5257i$; also, $|l^-_{12}|^2 = |l^-_1|^2 + 10.7872$ implies that $l^-_{12} = 11.9845i$. The remaining l^-_j are given recursively by eqs.(6.10) and (6.11).

Corollary 6.7. *Let $\mathbf{h}_N$ be the set of Hamiltonians $\mathscr{H} = \frac{1}{2}\{L^+, L^-\}$, such that $[\{L^+, L^-\}, L^-] = \{L^-, Z_N\}$, then $[\mathscr{H}, Z_N] = 0$, $\forall\, \mathscr{H} \in \mathbf{h}_N$.*

Proof. Now this is clear because of construction since we are taking as $\{|\xi_j\rangle\}$ the eigenvectors of Z_N, which in turn are also eigenvectors of $\mathscr{H}$ in accordance with eq. (6.4); this is, they have the same set of eigenvectors, so they must commute. □

Definition 6.2. *The number operator $\mathscr{N} : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is defined as $\mathscr{N} := L^+L^-$.*

Corollary 6.8. $[\mathscr{N}, \mathscr{H}] = [\mathscr{N}, Z_N] = 0$.

Proof. It is enough to probe that the eigenvectors of $\mathscr{N}$ are those of the set $\{|\xi_j\rangle\}$. Using the definitions of L^+ and L^-, we have

$$\begin{aligned}
\mathscr{N} &= L^+L^- = \sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} l^+_j |\xi_{j+1}\rangle\langle\xi_j| l^-_{j'} |\xi_{j'-1}\rangle\langle\xi_{j'}| \\
&= \sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} l^+_j l^-_{j'} |\xi_{j+1}\rangle \delta_{j,j'-1} \langle\xi_{j'}| = \sum_{j=0}^{N-1} l^+_j l^-_{j+1} |\xi_{j+1}\rangle\langle\xi_{j+1}| \\
&= \sum_{j=0}^{N-1} |l^-_{j+1}|^2 |\xi_{j+1}\rangle\langle\xi_{j+1}|,
\end{aligned}$$

since $l^+_j = \overline{l^-}_{j+1}$ (see the proof of Proposition 6.4); therefore

$$\mathscr{N}|\xi_\alpha\rangle = \sum_{j=0}^{N-1} |l^-_{j+1}|^2 |\xi_{j+1}\rangle\langle\xi_{j+1}|\xi_\alpha\rangle \quad = \quad |l^-_\alpha|^2 |\xi_\alpha\rangle.$$

Thus $\mathscr{N}$ has the same eigenvectors as $\mathscr{H}$, as desired. □

With these results requirements 1 and 2 of the proposed algebraic scheme are satisfied.

7. On the AA-approach

The purpose of this work is to provide a framework for analysing eigensystems, associated with the DFT for arbitrary N, either by using an operator not necessarily commuting with the DFT, as in the realization in terms of the operator Z_N, or by employing an operator W, which does commute with the DFT. In this section we propose a possible realization for the algebraic scheme 6.1, by using the Heun operator W from the Atakishiyeva and Atakishiyev (AA) approach, which could provide the fulfilment of its four requirements. This can be done due to the existence of a significant connection between the operators Q and P and the raising and lowering operators A and $A^\dagger$ of the AA-approach through the exponential map. Thus, we show how these operators are related and establish corresponding DCR which naturally follow in this approach.

The A and $A^\dagger$ operators, also called *intertwining operators*, are linear transformations $A, A^\dagger : \mathbb{C}^N \to \mathbb{C}^N$ such that (see [3, 4] for more on the subject)

$$A = X + \mathrm{i}Y, \quad A^\dagger = X - \mathrm{i}Y,$$

where $X = \mathrm{diag}(S_0, S_1, \ldots, S_{N-1})$, $S_n := 2\sin(2\pi n/N), n \in \mathbb{Z}_N$, and $Y = i(V^\dagger - V)$. Since X and Y are Hermitian they can be identified as position and momentum operators, respectively. Note that the operators A and $A^\dagger$ satisfy the *intertwining relations*

$$A\Phi_N = \mathrm{i}\Phi A, \qquad A^\dagger\Phi = -\mathrm{i}\Phi A^\dagger.$$

By using them it is possible to prove the following important result: if the discrete number operator is defined as $\mathcal{N} := A^\dagger A$, then

$$[\mathcal{N}, \Phi_N] = 0,$$

hence they have the same eigenvectors.

On the other hand, it is not hard to see that the operator X satisfies the relation (see [4], p. 89)

$$X = \frac{1}{2\mathrm{i}}\left(U - U^\dagger\right),$$

from which it follows that

$$A = \frac{1}{4}\sqrt{\frac{N}{\pi}}\left[V^\dagger - V + \mathrm{i}\left(U^\dagger - U\right)\right],$$

$$A^\dagger = \frac{1}{4}\sqrt{\frac{N}{\pi}}\left[V - V^\dagger - \mathrm{i}\left(U - U^\dagger\right)\right],$$

where, as we know from Section 3, the operators U and V are intertwined by the DFT.

In [4], Theorem 3.1, p.86, it was shown that the set of unitary operators defined as

$$\mathfrak{u}(l; m, n) := \sqrt{N} q^l V^n U^m, \quad 0 \le l, m, n \le N-1,$$

form an irreducible unitary representation $\mathcal{U}(N)$ on $\mathbb{C}^N$ of the finite Heisenberg group H. Additionally, as is proved in [15], Theorem 1.5, p.19, the matrix elements of the unitary, irreducible representations of H are a complete orthonormal set for the vector space of the regular representation $\mathbb{C}[H]$ of H, which in turn is obtained by a basis indexed by the elements of H. It is possible to turn $\mathbb{C}[H]$ into an algebra by means of the product $v_g v_h = v_{gh}$, $g, h \in H$ and extending it linearly; the resulting algebra is the group algebra $\mathbb{C}H$ of H on the field $\mathbb{C}$. Therefore, the matrix elements of A and $A^\dagger$ are elements of the group algebra $\mathbb{C}H$. Recall that, by virtue of Theorem 4.1,

$$V^j = \exp(i\eta P_j),$$

which implies that the raising and lowering operators A and $A^\dagger$ can also be interpreted as linear combinations of exponentiations of the operators Q and P, with matrix elements belonging to the group algebra $\mathbb{C}H$. These arguments probe the deep connection between Q, P, A and $A^\dagger$ as desired. Shortly, we can say that the operators Q, P, L^+, L^- and Z_N *lie at the level* of representations of the Heisenberg algebra $\mathfrak{h}$, whereas the operators X, Y, A, $A^\dagger$ and W *lie at the level* of representations of the Heisenberg group H – turned into a group algebra $\mathbb{C}H$ – and connected through the exponential map, with underlying finite-dimensional Hilbert space $L^2(\mathbb{Z}/N\mathbb{Z})$.

Thus, we have gained insight on the mathematical structure underlying the DFT and now we are in a position to lay down a possible realization for 6.1 lying naturally in the AA-approach. It would be desirable, not mandatory though, that $\mathcal{N}$ was Hermitian; this condition is not fulfilled, however, since A and $A^\dagger$ generate a cubic algebra (for more details see [28]). Fortunately, it has been possible to find a transformation T, which turns $\mathcal{N}$ into an Hermitian matrix by means of a symmetrization with respect to the parity operator S extendable to arbitrary N; this implies that $\mathcal{N}$ is diagonalizable and so there exists an orthonormal eigenbasis of $\mathcal{N}$ and it has been already constructed [5] (at least for $N = 5$ up to the writing of this paper). This means we could use such basis to build an associated family of ladder operators apart from the AA's raising and lowering operators. The natural candidate that plays the role of Z_N is the Heun operator

$$W := -2i\,[X, Y] = [A, A^\dagger],$$

defined in [28], p.7, which additionally can be chosen to commute with Φ_N. So despite the fact that W may not have the Toeplitz property, we could remarkably have in exchange, the fulfilment of the whole four requirements of the algebraic scheme in 6.1. The Toeplitz property will only be relevant when we formally study the intrinsic nature of Z_N, so its absence here is not detrimental to the scheme under analysis, for the existence of solutions of the DCR only required hermiticity, tracelessness and diagonalizability, as we learned in the previous section. Therefore it is important to determine whether or not it is possible to construct operators L^+ and L^-, such that

$$[\{L^+, L^-\}, L^+] = -\{L^+, W\}, \quad [\{L^+, L^-\}, L^-] = \{L^-, W\}.$$

If solutions exist for these equations, then the proposed algebraic scheme will have been solved completely, since

$$[W, \Phi_N] = 0.$$

These observations undoubtedly help to deeply understand the mathematical structure underlying the behaviour of Q, P, A and $A^\dagger$, and we think this can be useful for further developments on the subject.

8. Concluding remarks

In this paper we have established a set of discrete commutation relations obtained from a Hermitian Toeplitz operator Z_N, which plays the role of the identity. Thus, by means of the formulation of the discrete compatibility relations, the algebraic system 6.1 was proposed and it has been shown to admit solutions featuring ladder-type operators that lead to the construction of families of Hamiltonians $\mathscr{H}$ for each N. Also, it was established a remarkable relationship between the operators Q and P, and those of the AA-approach, A and $A^\dagger$, in the sense that these are linear combinations of exponentiations of the former, with matrix elements belonging to the group algebra $\mathbb{C}H$. Then we proposed that the condition $[W, \Phi_N] = 0$ can naturally lead to the fulfilment of the four requirements of the algebraic system and thereby provide a complete model of it, obtaining insight on the underlying algebraic structure of the DFT. We believe this can lead to a well formulated framework for systematically studying discrete systems in finite dimensional Hilbert spaces.

Further analysis is needed to formally deal with the conjectured relationship between Z_N and the δ distribution, as well as to determine whether analytic expressions for the eigenvectors of Z_N can be obtained. It is also important the study of the asymptotics –i.e. upper and lower bounds– of the extreme eigenvalues, if any, trying to follow procedures similar to those in [30]. The establishment of three-term recurrence relations for eigenvectors of the DFT, leading to associated polynomials through Rodrigues-type difference formulas, as well as the extension of the results to the multivariate case still remains pending.

The possible recovery of the continuous case withing the scope of the *Limit Central Theorem*, would be desirable and lies on the veracity of the limit of Z_N as the Dirac distribution, and the knowledge of the explicit form of its eigenvectors. If this limit holds, then Z_N has Schwartz distributional behaviour. Therefore one needs to find an appropriate analog of the identity operator in the CCR (2.2), instead of employing the identity operator under convolution in the distributional sense of the discrete conmutation relations (4.3). This is due to the fact that Schwartz distributions are a bit different notion than the usual probability distributions. Nevertheless, this is not detrimental to the algebra of the discrete commutation relations because in our study under discussion we prioritize a *correspondence principle* from Tarasov. This principle states (see [31] p.4) that the correspondence between discrete and continuous quantum theories lies not so much in the limiting agreement when the step of discretization tends to zero, as in the fact that mathematical operations on the two theories obey, in many cases, the same laws. Finally, the problem of the recovery of the continuous case, associated with the Heun operator, or if the limit of Z_N turns out to be a bounded operator, is left open.

Acknowledgments. NMA is grateful to Alexei Zhedanov for illuminating discussions.

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