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Brouwerian almost distributive lattices

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Abstract. This paper presents the idea of a Brouwerian almost distributive lattice, a generalization of an almost distributive lattice, and a Brouwerian algebra. We also derive some properties on Brouwerian almost distributive lattices. A set of equivalent conditions is provided for a Brouwerian almost distributive lattice to transform into a Brouwerian algebra.

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Keywords. Brouwerian algebra, almost distributive lattice, Brouwerian almost distributive lattice.

1. Introduction

Lattice [1, 2] and [4] is a mathematical structure constructed on a set of elements associated with two binary operations $\sqcap_\star$ (greatest lower bound) and $\sqcup_\star$ (least upper bound). These operations satisfy specific properties, such as associativity, commutativity, and idempotence. A distributive lattice is a type of lattice where meet and join operations distribute over each other. The inclusion of another unary operation on a distributive lattice paved the way to study a new concept known as Boolean algebra [9]. Heyting algebras [3] generalize the idea of Boolean algebras, with the implication operation $\rightarrow_\star$ playing a central role. Brouwerian algebras, also known as Kripke or topological algebras, are a specific subclass of Heyting algebras that incorporate topological structures. In addition to the algebraic operations of a Heyting algebra, Brouwerian algebras [10] include topological constraints, often represented by topological spaces or partial orders with additional topological properties.

Swamy and Rao studied almost distributive lattice [11] to understand the behavior of lattices when distributivity is nearly satisfied. In an almost distributive lattice, the distributive law holds almost everywhere, but a few exceptions may exist. The connection between lattices and almost distributive lattices lies in their relationship to distributivity. While distributive lattices strictly adhere to the distributive law for all elements, almost distributive lattices relax this requirement by allowing a few exceptions. This relaxation allows for a broader class of structures to be studied while retaining some distributive lattices' properties.

Almost distributive lattices were first studied under two binary operations, $\sqcap_\star$ and $\sqcup_\star$. The inclusion of another binary operation $\rightarrow_\star$ laid the foundation for studying many more algebras on almost distributive lattices [5,6] and [7]. Till now, the study of all these algebras on an almost distributive lattice where with

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the inclusion of both the least element 0 and maximal element ν_1. In this paper, we initiated to define an algebra named a Brouwerian almost distributive lattice, which is a generalization of a Brouwerian algebra and an almost distributive lattice, with the exclusion of the least element 0. In this context, we will go through the fundamental definition of a Brouwerian almost distributive lattice and provide some examples demonstrating the independence of the axioms stated in the definition and some properties related to the structure. Finally, we present a collection of equivalence conditions that enable the Brouwerian almost distributive lattice to transform into a Brouwerian algebra.

2. Preliminaries

Let us recall some beneficial, necessary results on an almost distributive lattice, semi-Brouwerian algebra, and semi-Brouwerian almost distributive lattice, which are frequently used in the paper.

Definition 1. *[11] An algebra $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ of type (2,2) is called an almost distributive lattice (ADL), if it assures the subsequent axioms;*

1. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_3 = (\chi_1 \sqcap_\star \chi_3) \sqcup_\star (\chi_2 \sqcap_\star \chi_3)$
2. $\chi_1 \sqcap_\star (\chi_2 \sqcup_\star \chi_3) = (\chi_1 \sqcap_\star \chi_2) \sqcup_\star (\chi_1 \sqcap_\star \chi_3)$
3. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_2 = \chi_2$
4. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_1 = \chi_1$
5. $\chi_1 \sqcup_\star (\chi_1 \sqcap_\star \chi_2) = \chi_1$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

Example 1. *[11] If $\mathcal{B}$ is a non-empty set, for any $\chi_1, \chi_2 \in \mathcal{B}$, define $\chi_1 \sqcap_\star \chi_2 = \chi_2, \chi_1 \sqcup_\star \chi_2 = \chi_1$, then $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an discrete ADL.*

Unless otherwise stated, $\mathcal{B}$ represents an almost distributive lattice $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ in this section. For any $\chi_1, \chi_2 \in \mathcal{B}$, $\chi_1 \leq_* \chi_2$ if $\chi_1 = \chi_1 \sqcap_\star \chi_2$ or equivalently $\chi_1 \sqcup_\star \chi_2 = \chi_2$, and it is noticed that $\leq_*$ is a partial order on $\mathcal{B}$.

Theorem 1. *[11] For any $\nu_1 \in S$, the following are equivalent,*

172. *ν_1 is a maximal element.*
173. *$\nu_1 \sqcup_\star \chi_1 = \nu_1$, for all $\chi_1 \in \mathcal{B}$.*
174. *$\nu_1 \sqcap_\star \chi_1 = \chi_1$, for all $\chi_1 \in \mathcal{B}$.*

Theorem 2. *[11] For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$,*

172. $\chi_1 \sqcup_\star \chi_2 = \chi_1 \iff \chi_1 \sqcap_\star \chi_2 = \chi_1$.
173. $\chi_1 \sqcup_\star \chi_2 = \chi_2 \iff \chi_1 \sqcap_\star \chi_2 = \chi_1$.
174. $\chi_1 \sqcap_\star \chi_2 = \chi_2 \sqcap_\star \chi_1 = \chi_1$ *whenever* $\chi_1 \leq_* \chi_2$.
175. $\wedge_*$ *is associative.*
176. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 = \chi_2 \sqcap_\star \chi_1 \sqcap_\star \chi_3$.
177. $(\chi_1 \sqcup_\star \chi_2) \sqcap_\star \chi_4 = (\chi_2 \sqcup_\star \chi_1) \wedge_* \chi_4$.
178. $\chi_1 \sqcap_\star \chi_2 \leq_* \chi_2$ *and* $\chi_1 \leq_* \chi_1 \vee_* \chi_2$.
179. $\chi_1 \sqcap_\star \chi_1 = \chi_1$ *and* $\chi_1 \sqcup_\star \chi_1 = \chi_1$.
180. *If* $\chi_1 \leq_* \chi_3$ *and* $\chi_2 \leq_* \chi_3$, *then* $\chi_1 \wedge_* \chi_2 = \chi_2 \sqcap_\star \chi_1$ *and* $\chi_1 \sqcup_\star \chi_2 = \chi_2 \sqcup_\star \chi_1$.

Theorem 3. *[11] Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \nu_1)$ be an ADL. Then the following are equivalent;*

172. *$\mathcal{B}$ is a distributive lattice.*
173. *$(\mathcal{B}, \leq_*)$ is directed above.*
174. *$\sqcup_\star$ is commutative.*
175. *$\sqcap_\star$ is commutative.*
176. *$\sqcup_\star$ is right distributive over $\sqcap_\star$.*
177. *The relation $\theta = \{(\chi_1, \chi_2) \in \mathcal{B} \times \mathcal{B} \mid \chi_2 \sqcap_\star \chi_1 = \chi_1\}$ on $\mathcal{B}$ is antisymmetric.*

Definition 2. *[10] An algebra $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, 1)$ of type (2,2,2,0) is said to be a Brouwerian algebra, if it assures the subsequent axioms;*

172. *The system $(\mathcal{B}, \sqcup_\star, \sqcap_\star, 1)$ is a lattice with a greatest element 1.*
173. *For all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}, \chi_1 \sqcap_\star \chi_3 \leq \chi_2$ if and only if $\chi_3 \leq \chi_1 \rightarrow_\star \chi_2$.*

Definition 3. *[8] $\mathcal{B}$ with a maximal element ν_1 is said to be a semi-Brouwerian almost distributive lattice (SBADL), if there is a binary operation $\rightarrow_\star$ on $\mathcal{B}$ with the subsequent axioms;*

(N_1) $(\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = \nu_1$
(N_2) $\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
(N_3) $\chi_1 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) = \chi_1 \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \sqcap_\star \chi_3)]$
(N_4) $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \nu_1) \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)]$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

3. Brouwerian Almost Distributive Lattices

In this section, we introduce Brouwerian almost distributive lattices and provide several counterexamples. We compare Brouwerian almost distributive lattices with semi-Brouwerian almost distributive lattices. We obtain several algebraic properties on Brouwerian almost distributive lattices. We derive some necessary and sufficient conditions for a Brouwerian almost distributive lattice to become a Brouwerian algebra.

Definition 4. *An almost distributive lattice $(\mathcal{B}, \sqcap_\star, \sqcup_\star)$ with a maximal element ν_1 is said to be a Brouwerian almost distributive lattice (abbreviated as BrADL), if there is a binary operation $\rightarrow_\star$ on $\mathcal{B}$, satisfying the following axioms;*

B_1. $(\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = \nu_1$
B_2. $\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
B_3. $\chi_2 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_2 \sqcap_\star \nu_1$
B_4. $\chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \chi_3) = (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$

for all $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.

In examples 2, 3, 4 and 5 we exhibit the independence of the axioms B_1,B_2,B_3 and B_4 of Definition 4.

Example 2. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	1	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element and the binary operation $\rightarrow_\star$ satisfies the axioms B_2,B_3 and B_4 of Definition 4 but B_1 fails for the pair $(1, 1)$.

$$(1 \rightarrow_\star 1) \sqcap_\star 5 = 5 \quad \Rightarrow \quad 1 \sqcap_\star 5 = 5$$
$$\Rightarrow \quad 1 \neq 5.$$

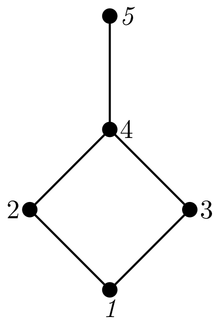
Example 3. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a five-element chain with binary operation $\rightarrow_\star$ as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	5	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is an ADL with 5 as its maximal element and the binary operation $\rightarrow_\star$ satisfies the axioms B_1,B_3 and B_4 of Definition 4 but B_2 fails for the pair $(2, 1)$.

$$2 \sqcap_\star (2 \rightarrow_\star 1) = 2 \sqcap_\star 1 \sqcap_\star 5 \quad \Rightarrow \quad 2 \sqcap_\star 5 = 1 \sqcap_\star 5$$
$$\Rightarrow \quad 2 \neq 1.$$

Example 4. *Let $\mathcal{B} = \{1, 2, 3, 4, 5\}$ be a set whose Hasse-diagram is*



with the binary operation $\rightarrow_\star$ *as illustrated in the following table;*

$\rightarrow_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	1	5	5
3	1	1	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\rightarrow_\star$ *satisfies the axioms* B_1,B_2 *and* B_4 *of Definition 4 but* B_3 *fails for the pares* $(2,3)$ *and* $(3,2)$. *For the pair* $(2,3)$

$$\begin{aligned} 3 \sqcap_\star (2 \rightarrow_\star 3) = 3 \sqcap_\star 5 &\Rightarrow 3 \sqcap_\star 1 = 3 \sqcap_\star 5 \\ &\Rightarrow 1 \neq 3. \end{aligned}$$

Example 5. *Let* $\mathcal{B} = \{1,2,3,4,5\}$ *be a five-element chain with binary operation* $\rightarrow_\star$ *as illustrated in the following table;*

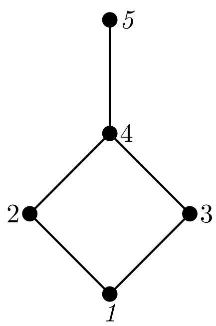
$\rightarrow_\star$	1	2	3	4	5
1	5	2	3	4	5
2	1	5	3	4	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\rightarrow_\star$ *satisfies the axioms* B_1,B_2 *and* B_3 *of Definition 4 but* B_4 *fails for the triplets* $(1,1,2), (1,1,3), (1,1,4), (1,2,1), (1,3,1), (1,4,1), (2,2,3), (2,2,4), (2,3,2)$ *and* $(2,4,2)$.
For the triplet $(1,1,2)$

$$\begin{aligned} 1 \rightarrow_\star (1 \sqcap_\star 2) = (1 \rightarrow_\star 1) \sqcap_\star (1 \rightarrow_\star 2) &\Rightarrow 1 \rightarrow_\star 1 = 5 \sqcap_\star 2 \\ &\Rightarrow 5 \neq 2. \end{aligned}$$

In examples 6 and 7 we define a binary operation $\rightarrow_\star$ on an ADL in such a way that it forms a Brouwerian almost distributive lattice.

Example 6. *Let* $\mathcal{B} = \{1,2,3,4,5\}$ *be a set whose Hasse-diagram is*



with the binary operation $\rightarrow_\star$ *as illustrated in the following table;*

$\to_\star$	1	2	3	4	5
1	5	5	5	5	5
2	3	5	3	5	5
3	2	2	3	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and the binary operation* $\to_\star$ *satisfies all the axioms* B_1,B_2,B_3 *and* B_4 *of Definition 4. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a Brouwerian almost distributive lattice.*

Example 7. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\to_\star$ *as illustrated in the following table;*

$\to_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element, and the binary operation* $\to_\star$ *satisfies all the axioms* B_1,B_2,B_3 *and* B_4 *of Definition 4. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a Brouwerian almost distributive lattice.*

In example 8 we demonstrate that the every binary operation $\to_\star$ defined on an ADL need not be a BrADL.

Example 8. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\sqcup_\star, \sqcap_\star$ *and* $\to_\star$ *as illustrated in the following tables;*

$\sqcup_\star$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	5	2	5
3	4	3	3	4	4
4	4	4	4	4	4
5	5	5	5	5	5

$\sqcap_\star$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	3	2
3	2	2	3	3	2
4	1	2	3	4	5
5	1	2	3	4	5

$\to_\star$	1	2	3	4	5
1	1	2	5	2	5
2	5	5	5	5	5
3	5	2	5	2	5
4	5	1	2	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element, and the binary operation* $\to_\star$ *does not satisfy the axioms* B_1, B_2 ,B_3 *and* B_4 *of Definition 4.*
B_1 *for the pair* $(1, 2)$.
B_2 *for the pares* $(1, 3), (1, 4), (4, 1), (4, 2), (4, 3)$.
B_3 *for the pares* $(1, 4), (3, 4)$.
B_4 *for the triplets* $(3, 2, 3), (3, 2, 4), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5), (3, 4, 1),$
$(3, 4, 3), (4, 2, 1), (4, 2, 4), (4, 2, 5), (4, 3, 1), (4, 3, 2), (4, 3, 5)$.
Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is not a BrADL.*

Every Brouwerian algebra is a Brouwerian almost distributive lattice. Vice versa is not possible. For, see Example 9.

Example 9. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\sqcup_\star, \sqcap_\star$ *and* $\to_\star$ *as illustrated in the following tables;*

$\sqcup_\star$	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5

$\sqcap_\star$	1	2	3	4	5
1	1	2	3	4	5
2	1	2	3	4	5
3	1	2	3	4	5
4	1	2	3	4	5
5	1	2	3	4	5

$\to_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	3	4	5
3	5	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is a discrete ADL also* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a BrADL but not a BA (since it is not a lattice).*

Example 10 shows that there is a binary operation $\to_\star$ on a five element chain which forms a BrADl but not a SBADL.

Example 10. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\to_\star$ *as illustrated in the following table;*

$\to_\star$	1	2	3	4	5
1	5	5	5	5	5
2	1	5	5	5	5
3	1	2	5	5	5
4	1	2	5	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element, and all the axioms* B_1,B_2,B_3 *and* B_4 *of Definition 4 are satisfied by the binary operation* $\to_\star$ *. As a result* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a BrADL.*
Furthermore, the triplet $(4, 5, 3)$ *does not satisfy the axiom* N_3 *of Definition 3 when using the binary operation* $\to_\star$*. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *not a SBADL. Hence* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a BrADL but not a SBADL.*

Example 11 shows that there is a binary operation $\to_\star$ on a five element chain which forms a SBADL but not a BrADl.

Example 11. *Let* $\mathcal{B} = \{1, 2, 3, 4, 5\}$ *be a five-element chain with binary operation* $\to_\star$ *as illustrated in the following table;*

$\to_\star$	1	2	3	4	5
1	5	2	3	4	5
2	1	5	3	4	5
3	1	2	5	5	5
4	1	2	3	5	5
5	1	2	3	4	5

Clearly, $(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ *is an ADL with* 5 *as its maximal element and that all the axioms* N_1,N_2,N_3 *and* N_4 *of Definition 3 are satisfied by the binary operation* $\to_\star$*. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is a SBADL.*
Here for the triplets $(1, 1, 2), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 1), (1, 3, 1), (1, 4, 1),$
$(2, 2, 3), (2, 2, 4), (2, 3, 2), (2, 4, 2)$*, the binary operation* $\to_\star$ *fails to satisfy the axiom* B_4 *of Definition 4. Therefore* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is not a BrADL. Hence* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, 5)$ *is not a BrADL but rather a SBADL.*

Here, we derive the primary characteristics on BrADL that are crucial for advancing the theory's development. Unless otherwise stated, $\mathcal{B}$ denotes a Brouwerian almost distributive lattice $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \to_\star, \nu_1)$, with ν_1 as its maximal element.

The properties that we derive in Theorem 4 and Theorem 5 plays a crucial role in developing the theory further.

Theorem 4. *For any $\chi_1, \chi_2 \in \mathcal{B}$, the following hold;*

172. $\nu_1 \rightarrow_\star \chi_1 = \chi_1 \sqcap_\star \nu_1$
173. $\chi_1 \rightarrow_\star \nu_1 = \nu_1$
174. $\chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3) = \chi_1 \rightarrow_\star \chi_3$.

Proof. Let $\chi_1, \chi_2 \in \mathcal{B}$. Consider,

172. $\nu_1 \rightarrow_\star \chi_1 = \nu_1 \sqcap_\star (\nu_1 \rightarrow_\star \chi_1)$ (by 174 of Theorem 1.)
$= \nu_1 \sqcap_\star \chi_1 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)
$= \chi_1 \sqcap_\star \nu_1$ (by 176 of Theorem 2.)

173. $\chi_1 \rightarrow_\star \nu_1 = \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$ (by 174 of Theorem 1.)
$= \nu_1 \sqcap_\star \nu_1$ (by B_3 of Definition 4.)
$= \nu_1$

174. $\chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3) = (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$ (by B_4 of Definition 4.)
$= (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$
$= \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)$ (by B_1 of Definition 4.)
$= \chi_1 \rightarrow_\star \chi_3$.

□

Theorem 5. *If $\chi_1 \leq \chi_2$ in $\mathcal{B}$ and $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, then the following holds;*

172. $\chi_3 \rightarrow_\star \chi_1 \leq \chi_3 \rightarrow_\star \chi_2$
173. $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Consider,

172. $(\chi_3 \rightarrow_\star \chi_1) \sqcap_\star (\chi_3 \rightarrow_\star \chi_2) = \chi_3 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2)$ (by B_4 of Definition 4.)
$= \chi_3 \rightarrow_\star \chi_1$ (since $\chi_1 \leq \chi_2$).

Therefore $\chi_3 \rightarrow_\star \chi_1 \leq \chi_3 \rightarrow_\star \chi_2$.

173. $\chi_1 \leq \chi_2 \Rightarrow \chi_1 \rightarrow_\star \chi_1 \leq \chi_1 \rightarrow_\star \chi_2$ (by 172)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$ (by B_1 of Definition 4.)
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \leq \nu_1$
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$

□

Corollary 3.15, is the consequence of B_2 and B_3 of Definition 4.

Corollary 1. *For any $\chi_1, \chi_2 \in \mathcal{B}$, the following holds;*

172. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
173. $\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.

Theorem 6. *For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, $\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ if and only if $\chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.*

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Then, $\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1) \leq \chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1) \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow \nu_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Corollary 1.)
$\Rightarrow \chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$.

On the other hand,

$\chi_3 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \quad \Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$.

□

Theorem 7. *For any* $\chi_1, \chi_2 \in \mathcal{B}$ *the following hold,*
$\chi_1 \sqcap_\star \nu_1 \leq [(\chi_1 \rightarrow_\star \chi_2) \rightarrow_\star \chi_2] \sqcap_\star \nu_1$.

Proof. Let $\chi_1, \chi_2 \in \mathcal{B}$. Then, by B_2 of Definition 4.
$\chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ (by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \leq [(\chi_1 \rightarrow_\star \chi_2) \rightarrow_\star \chi_2] \sqcap_\star \nu_1$ (by Theorem 6.).

□

Theorem 8. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$*, the following holds;*

172. $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$ *if and only if* $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$.
173. $\chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$ *if and only if* $\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
174. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ *if and only if* $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$

Proof. Let $\chi_1, \chi_2, \nu_1 \in \mathcal{B}$. Consider,
172. $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \nu_1) \leq \chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1) \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow \nu_1 \leq (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \leq \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1$

On the other hand,
$(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \nu_1 \quad \Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)

Therefore $\chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \nu_1$.
173. $\chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
$\Rightarrow \chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by Theorem 6.)

On the other hand,
$\chi_2 \sqcap_\star \nu_1 \leq (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow \chi_2 \sqcap_\star \chi_1 \sqcap_\star \nu_1 \leq \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow \chi_1 \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by Theorem 6.)

174. $\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1) = \chi_1 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$
(by 172 of Theorem 5.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
$= (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \nu_1)$
(by B_4 of Definition 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by 173 of Theorem 4.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
On the other hand, $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 = \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$\Rightarrow \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)

□

Theorem 9. *For any $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, the following holds;*

172. $[\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
173. $[(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 = [(\chi_2 \sqcap_\star \chi_1) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
174. $[\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = [\chi_2 \rightarrow_\star (\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$.
172. $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$ (by B_2 of Definition 4.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_1 \sqcap_\star \chi_2 \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1]$
$= \chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$ (by 172 of Theorem 5.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_2 \sqcap_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]]$
$= \chi_2 \rightarrow_\star (\chi_2 \sqcap_\star \chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)$ (by 176 of Theorem 2.)
$\Rightarrow \chi_2 \rightarrow_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]$
$= \chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1) - - - -(I)$
(by 174 of Theorem 4.)
Consider,
$[\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star [\chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)] \sqcap_\star \nu_1$
$= [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star [\chi_2 \rightarrow_\star [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1]] \sqcap_\star \nu_1$
(by I)
$= [\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1] \sqcap_\star \nu_1.$ (by B_3 of Definition 4.)
Therefore $\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq [\chi_2 \rightarrow_\star (\chi_1 \sqcap_\star \chi_3 \sqcap_\star \nu_1)] \sqcap_\star \nu_1$.
Hence by B_4 of Definition 4,
$\chi_1 \sqcap_\star (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$.
Thus from Theorem 6,
$(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3) \sqcap_\star \nu_1 \leq [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$.
On the other hand,
$\chi_1 \sqcap_\star \chi_2 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$
$= \chi_2 \sqcap_\star \chi_1 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$= \chi_2 \sqcap_\star \chi_1 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.).
Therefore $\chi_1 \sqcap_\star \chi_2 \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$.

Now, $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
$\Rightarrow (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star [(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1]$
$= (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star [\chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1]$
$\Rightarrow (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star [(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1]$
$= (\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_3 \sqcap_\star \nu_1)$
(by 174 of Theorem 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \sqcap_\star \chi_2)] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3))] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \nu_1]$
$= [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \nu_1]$
(by B_4 of Definition 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \sqcap_\star \chi_2)] \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3))] \sqcap_\star \nu_1$
$= [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
(by 174 of Theorem 4.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star (\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3))] \sqcap_\star \nu_1$
$= [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
(by B_1 of Definition 4.)
$\Rightarrow [(\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3))] \sqcap_\star \nu_1 \leq [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
(by 173 of Corollary 1.)
Therefore $[\chi_1 \rightarrow_\star (\chi_2 \rightarrow_\star \chi_3)] \sqcap_\star \nu_1 = [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$.
173. $(\chi_1 \sqcap_\star \chi_2) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 = \chi_1 \sqcap_\star \chi_2 \sqcap_\star \chi_3 \sqcap_\star \nu_1$
(by B_2 of Definition 4.)
$\Rightarrow (\chi_2 \sqcap_\star \chi_1) \sqcap_\star [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 \leq \chi_3 \sqcap_\star \nu_1$
(by 176 of Theorem 2.)
$\Rightarrow [(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 \leq [(\chi_2 \sqcap_\star \chi_1) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
(by Theorem 6.).
By symmetry, $[(\chi_1 \sqcap_\star \chi_2) \rightarrow_\star \chi_3] \sqcap_\star \nu_1 = [(\chi_2 \sqcap_\star \chi_1) \rightarrow_\star \chi_3] \sqcap_\star \nu_1$
174. Follows from 172 and 173. □

Theorem 10. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, $(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \chi_3 = [(\chi_1 \sqcap_\star \chi_3) \rightarrow_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \chi_3$

Proof. Let $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$. Then
$[(\chi_1 \sqcap_\star \chi_3) \rightarrow_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \chi_3$
$= [((\chi_1 \sqcap_\star \chi_3) \rightarrow_\star \chi_2) \sqcap_\star ((\chi_1 \sqcap_\star \chi_3) \rightarrow_\star \chi_3)] \sqcap_\star \chi_3 \sqcap_\star \chi_3$
(by B_4 of Definition 4.)
$= [((\chi_1 \sqcap_\star \chi_3) \rightarrow_\star \chi_2) \sqcap_\star \chi_3 \sqcap_\star ((\chi_1 \sqcap_\star \chi_3) \rightarrow_\star \chi_3)] \sqcap_\star \chi_3$
(by 176 of Theorem 2.)
$= [((\chi_1 \sqcap_\star \chi_3) \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by B_3 of Definition 4.)
$= [((\chi_3 \sqcap_\star \chi_1) \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 173 of Theorem 9.)
$= [\chi_3 \rightarrow_\star (\chi_1 \rightarrow_\star \chi_2)] \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 172 of Theorem 9.)
$= \chi_3 \sqcap_\star [\chi_3 \rightarrow_\star (\chi_1 \rightarrow_\star \chi_2)] \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by 176 of Theorem 2.)
$= \chi_3 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_1 \sqcap_\star \chi_3$
(by B_2 of Definition 4.)
$= (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \chi_3$
(by 174 of Theorem 1 and 176 of Theorem 2.)
□

Corollary 2. *For any* $\chi_1, \chi_2, \chi_3 \in \mathcal{B}$, *if* $\chi_1 \sqcap_\star \chi_3 = \chi_2 \sqcap_\star \chi_3$, *then*

172. $(\chi_1 \rightarrow_\star \chi_3) \sqcap_\star \chi_3 = (\chi_2 \rightarrow_\star \chi_3) \sqcap_\star \chi_3$
173. $(\chi_3 \rightarrow_\star \chi_1) \sqcap_\star \chi_3 = (\chi_3 \rightarrow_\star \chi_2) \sqcap_\star \chi_3$.

Theorem 11. *If* $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, \nu_1)$ *is a BrADL, then for any maximal element* ν_2 *in* $\mathcal{B}$, $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_{\nu_2}, \nu_2)$ *is a BrADL where* $\chi_1 \rightarrow_{\nu_2} \chi_2 = (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2$ *for* $\chi_1, \chi_2 \in \mathcal{B}$.

Proof. Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_\star, \nu_1)$ be a BrADL and $\rightarrow_{\nu_2}$ is a maximal element in $\mathcal{B}$. For $\chi_1, \chi_2 \in \mathcal{B}$, define $\chi_1 \rightarrow_{\nu_2} \chi_2 = (\chi_1 \rightarrow_\star \chi_2) \wedge \nu_2$.
Then, for any $\chi_1, \chi_2 \in \mathcal{B}$,

172. $(\chi_1 \rightarrow_{\nu_2} \chi_1) \sqcap_\star \nu_2 \quad = (\chi_1 \rightarrow_\star \chi_1) \sqcap_\star \nu_2 \sqcap_\star \nu_2$

$= \nu_1 \sqcap_\star \nu_2$

$= \nu_2.$

173. $\chi_1 \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_2) \quad = \chi_1 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2$

$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_2$

$= \chi_1 \sqcap_\star \chi_2 \sqcap_\star \nu_2.$

174. $\chi_2 \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_2) \quad = \chi_2 \sqcap_\star (\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2$

$= \chi_2 \sqcap_\star \nu_1 \sqcap_\star \nu_2$

$= \chi_2 \sqcap_\star \nu_2.$

175. $\chi_1 \rightarrow_{\nu_2} (\chi_2 \sqcap_\star \chi_3) \quad = [\chi_1 \rightarrow_\star (\chi_2 \sqcap_\star \chi_3)] \sqcap_\star \nu_2$

$= [(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star (\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_2$

$= [(\chi_1 \rightarrow_\star \chi_2) \sqcap_\star \nu_2] \sqcap_\star [(\chi_1 \rightarrow_\star \chi_3)] \sqcap_\star \nu_2]$

$= (\chi_1 \rightarrow_{\nu_2} \chi_2) \sqcap_\star (\chi_1 \rightarrow_{\nu_2} \chi_3).$

Therefore $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \rightarrow_{\nu_2}, \nu_2)$ is a BrADL. ☐

We give several equivalent conditions for a BrADL to become a Brouwerian algebra as we wrap up the paper.

Theorem 12. *[11] Let $(\mathcal{B}, \sqcup_\star, \sqcap_\star, \nu_1)$ be an ADL. Then the subsequent statements are comparable;*

172. *$\mathcal{B}$ is a Brouwerian algebra.*
173. *$(\mathcal{B}, \leq_*)$ is directed above.*
174. *$(\mathcal{B}, \sqcup_\star, \sqcap_\star)$ is a distributive lattice.*
175. *$\sqcup_\star$ is commutative.*
176. *$\sqcap_\star$ is commutative.*
177. *$\sqcup_\star$ is right distributive over $\sqcap_\star$.*
178. *The relation $\theta = \{(\chi_1, \chi_2) \in \mathcal{B} \times \mathcal{B} \mid \chi_2 \sqcap_\star \chi_1 = \chi_1\}$ is antisymmetric.*

4. Conclusion

The concept of a Brouwerian almost distributive lattice is presented in this paper with several examples and counter-examples, and some of its primary and necessary properties are studied. We derive a few identities and inequalities in a Brouwerian almost distributive lattice. Also, we provided a set of equivalence conditions required for transforming the Brouwerian almost distributive lattice into a Brouwerian algebra.

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