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Research Article

On strongly deferred Cesaro summability and deferred statistical convergence of the sequences

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Abstract

In this paper, it is shown that, if a sequence is strongly $\alpha-$ deferred by the Cesaro summable for any α, $(0<\alpha<\infty)$ then it must be deferred by a statistically convergent and the inverse is also satisfied when the sequence is bounded.

Keywords: Summability of sequences, strongly summable sequence, statistical convergence

1. Introduction

The theory of statistical convergence was first introduced by Steinhaus (1951) and Fast (1951) independently in the same year. Since then, this subject has become one of the most active research areas for many mathematicians such as Erdös & Tenenbaum (1989), Freedman et al. (1978), Connor (1989), Fridy (1985), Fridy & Miller (1991), Schoenberg (1959).

The statistical convergence is also closely related to the subject of asymptotic density of the subset of natural numbers (see, Buck (1953) and Zygmund (1979).

Recall that a sequence $x=(x_k)$ is said to be statistically convergent to $L\in N$ if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k:k\le n,\left|x_k-L\right|\ge\varepsilon\right\}\right|=0,$$

where the vertical bars indicate the cardinality of the elements inside the set and this limit is denoted by $\lim_{n\to\infty}x_n=L(S)$.

Agnew (1932) defined deferred Cesaro mean of real valued sequence as $x=(x_k)$ by

$$(D_{p,q}x)_n:=\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}x_k,\ n=1,2,3,...$$

where

$p=\{p(n):n\in N\}$ and $q=\{q(n):n\in N\}$ are the sequences of natural numbers satisfying $p(n)<q(n)$ and $\lim_{n\to\infty}q(n)=\infty$.

It is clear that deferred Cesaro mean is a regular summability method complementing the Silverman Toeplitz theorem (see, Maddox (1970)).

Agnew (1932) showed that this method has some important properties besides regularity.

Definition 1.1. A sequence $x=(x_k)$ is said to be $\alpha-$ strongly deferred Cesaro summable to $L\in N$ if

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|x_k-L\right|^{\alpha}=0,$$

exist and it is denoted by $\lim_{n\to\infty}x_n=L(\alpha-D[p,q])$.

Definition 1.2. A sequence $x=(x_k)$ is said to be deferred statistically convergent to $L\in N$, if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{\left|\left\{k:p(n)+1\le k\le q(n),\left|x_k-L\right|\ge\varepsilon\right\}\right|}{q(n)-p(n)}=0,$$

and it is denoted by $\lim_{n\to\infty}x_n=L(DS[p,q])$.

In the above definition, the $\alpha-$ strongly deferred Cesaro summability and the deferred statistical convergence coincide with the $\alpha-$ strongly Cesaro summability and the statistical convergence respectively when $q(n)=n$ and $p(n)=0$.

There is a natural relation between the statistical convergence and the $\alpha-$ strongly Cesaro summability. This relation has been investigated by some authors including Connor (1989), Maddox (1967), Mursaleen (2000), Nuray (2010).

In this work the main aim is to investigate the relation between the deferred statistical convergence and the $\alpha-$ strongly deferred Cesaro summability.

2. Main Results and Their Proof

Thorough this work $p=\{p(n):n\in N\}$ and $q=\{q(n):n\in N\}$ denote the sequences of positive

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natural numbers and α is the real number such as $0<\alpha<\infty$. The obtained theorems are as follows:

Theorem 2.1. Let $0<\alpha<\infty$ be a real number. If a sequence $x=(x_k)$ is $\alpha-$strongly $D[p,q]$ convergent to L, then it is deferred by the statistical convergent to L.

Proof. Assume that $x=(x_k)$ is $\alpha-$strongly $D[p,q]$ convergent to L and denote the set

$$\{k: p(n)+1\le k\le q(n), |x_k-L|\ge\varepsilon\}$$

by $K(\varepsilon)$. Therefore, the inequality

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-L|^\alpha \ge$$

$$\ge \frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\ k\in K(\varepsilon)}}^{q(n)}|x_k-L|^\alpha$$

$$\ge \varepsilon^\alpha\frac{1}{q(n)-p(n)}|K(\varepsilon)|$$

holds. After taking limits when $n\to\infty$ the proof is obtained.

Corollary 2.1. Let $q(n)=n$. If a sequence $x=(x_k)$ is $\alpha-$strongly $D[p,n]$ convergent to L, then it is statistically convergent to L.

Proof. If we consider Theorem 2.1 and Theorem 2.2.5 in Kucukaslan & Yılmaztürk (2012), the proof is obtained.

Theorem 2.2. If a sequence $x=(x_k)$ is bounded and deferred by the statistical convergence to L for an arbitrary $p=\{p(n): n\in N\}$ and $q=\{q(n): n\in N\}$, then it is $\alpha-$strongly $D[p,q]$ convergent to L.

Proof. Suppose that $x=(x_k)$ is bounded and deferred by the statistical convergence to L for an arbitrary $p=\{p(n): n\in N\}$ and $q=\{q(n): n\in N\}$. Also, let us denote the complement of $K(\varepsilon)$ by $K^c(\varepsilon):=\{k: p(n)\le k\le q(n), |x_k-L|<\varepsilon\}$. It is clear from the assumption that there is a positive real number M such that $|x_k-L|\le M$ for all $n\in N$. Therefore, we have

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-L|^\alpha =$$

$$=\frac{1}{q(n)-p(n)}\left[\sum_{\substack{k=p(n)+1\\ k\in K(\varepsilon)}}^{q(n)}|x_k-L|^\alpha+\sum_{\substack{k=p(n)+1\\ k\notin K(\varepsilon)}}^{q(n)}|x_k-L|^\alpha\right]$$

$$\le\frac{1}{q(n)-p(n)}\left[\sum_{\substack{k=p(n)+1\\ k\in K(\varepsilon)}}^{q(n)}M^\alpha+\sum_{\substack{k=p(n)+1\\ k\notin K(\varepsilon)}}^{q(n)}\varepsilon^\alpha\right]$$

$$\le\frac{1}{q(n)-p(n)}\left[M^\alpha|K(\varepsilon)|+\varepsilon^\alpha|K^c(\varepsilon)|\right].$$

The limit relation gives the proof since

$$\lim_{n\to\infty}\frac{|K(\varepsilon)|}{q(n)-p(n)}=0 \quad\text{and}\quad \lim_{n\to\infty}\frac{|K^c(\varepsilon)|}{q(n)-p(n)}=1.$$

Corollary 2.2. Let $q(n)$ be an arbitrary strictly increasing sequence and $\frac{q(n)}{q(n)-p(n)}$ be a bounded sequence. If a sequence $x=(x_k)$ is bounded and statistically convergent to L, then it is $\alpha-$strongly $D[p,q]$ convergent to L.

Corollary 2.3. Let $q(n)$ be an arbitrary sequence such as $q(n)<n$ for all $n\in N$ and $\frac{n}{q(n)-p(n)}$ be a bounded sequence. If a sequence $x=(x_k)$ is bounded and statistically convergent to L, then it is an $\alpha-$strongly $D[p,q]$ convergent to L.

When Theorem 2.2 is taken into consideration together with Theorem 2.2.1 and Theorem 2.2.2 in Kucukaslan and Yılmaztürk (2012) then the proof of Corollary 2.2 and Corollary 2.3 is obtained.

Theorem 2.3. The $\alpha-$strongly $D[p,q]$ convergent sequence $x=(x_k)$ is an $\alpha-$strongly Cesaro convergent only if $\frac{p(n)}{q(n)-p(n)}$ is bounded.

Proof. The technique that was used by Agnew R.P. in Agnew (1932) can be applied for this purpose. Let us assume that $x=(x_k)$ is an $\alpha-$strongly Cesaro convergent to L.
In this case, the following equality

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-L|^\alpha =$$

$$=\frac{1}{q(n)-p(n)}\left[\sum_{k=1}^{q(n)}-\sum_{k=1}^{p(n)}\right]|x_k-L|^\alpha$$

$$=-\left(\frac{p(n)}{q(n)-p(n)}\right).\frac{1}{p(n)}\sum_{k=1}^{p(n)}|x_k-L|^\alpha+$$

$$+\left(\frac{q(n)}{q(n)-p(n)}\right).\frac{1}{q(n)}\sum_{k=1}^{q(n)}|x_k-L|^\alpha$$

hold. It can be said that the $\alpha-$strongly $D[p,q]$ convergence of $x=(x_k)$ is the linear combination of the $\alpha-$strongly Cesaro convergence of $x=(x_k)$. We can consider this linear combination as a matrix transformation. For the regularity of this matrix transformation the sequence

$$\left\{\frac{q(n)+p(n)}{q(n)-p(n)}\right\} \tag{2.1}$$

must be bounded.

For the boundedness of (2.1) (if and only if)

$$\frac{p(n)}{q(n)-p(n)}$$

must be bounded since

$$\frac{q(n)+p(n)}{q(n)-p(n)}=\frac{q(n)-p(n)+2p(n)}{q(n)-p(n)}=1+\frac{2p(n)}{q(n)-p(n)}$$

This assertion completes the proof.

Now, in the following Theorems $\alpha-$strongly $D[p,q]$ convergence and $\alpha-$strongly $D[p',q']$ convergence of the sequence $x=(x_k)$ are compared under the restriction

$$p(n)\le p'(n)<q'(n)\le q(n) \tag{2.2}$$

for all $n\in N$.

Theorem 2.4. Let $p'=\{p'(n)\}$ and $q'=\{q'(n)\}$ be sequences of positive natural numbers satisfying (2.2) and the sets $\{k: p(n)<k\le p'(n)\}$, $\{k: q'(n)<k\le q(n)\}$ are finite for all $n\in N$. Then, $\alpha-$strongly $D[p',q']$ convergence of bounded sequence implies $\alpha-$strongly $D[p,q]$ convergence.

Proof. There is a positive real number M in the assumption such that $|x_k-L|\le M$ which holds for all $n\in N$. Therefore, we have

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-L|^\alpha=\frac{1}{q(n)-p(n)}\cdot$$
$$\cdot\left(\sum_{k=p(n)+1}^{p'(n)}+\sum_{k=p'(n)+1}^{q'(n)}+\sum_{k=q'(n)+1}^{q(n)}+\right)|x_k-L|^\alpha$$

$$\le\frac{2}{q'(n)-p'(n)}M^\alpha O(1)+$$
$$+\frac{1}{q'(n)-p'(n)}\sum_{k=p'(n)+1}^{q'(n)}|x_k-L|^\alpha$$

If we take the limit, we obtain the sequence $x=(x_k)$ which is an $\alpha-$strongly $D[p,q]$ convergence.

Theorem 2.5. Let $p'=\{p'(n)\}$ and $q'=\{q'(n)\}$ be sequences of positive natural numbers satisfying (2.2) and

$$\lim_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}=d>0.$$

Then, the $\alpha-$strongly $D[p',q']$ convergence of the sequence of $x=(x_k)$ implies a $\alpha-$strongly $D[p,q]$ convergence.

Proof. It is easy to see that the inequality

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_k-L|^\alpha\ge$$
$$\ge\frac{1}{q(n)-p(n)}\sum_{k=p'(n)+1}^{q'(n)}|x_k-L|^\alpha\ge$$
$$\ge\frac{q'(n)-p'(n)}{q(n)-p(n)}\frac{1}{q'(n)-p'(n)}\sum_{k=p'(n)+1}^{q'(n)}|x_k-L|^\alpha$$

holds. After taking limit when $n\to\infty$, we understand that sequence $x=(x_k)$ is an $\alpha-$strongly $D[p,q]$ convergent to L.

At this point the inclusion relationship between the $\alpha-$strongly C_λ summability and the $\alpha-$strongly $D[p,q]$ summability need to be examined. The method C_λ is obtained by deleting a set of rows from the Cesaro matrix when $\{\lambda(n)\}_{n\in N}$ is a strictly increasing sequence of positive natural numbers (see, Armitage at al. (1989)). Let us denote that for the method $D[p,q]$ by D_λ for $p(n)=\lambda(n-1)$ and $q(n)=\lambda(n)$.

Theorem 2.6. Let $\{\lambda(n)\}_{n\in N}$ be an increasing sequence of positive integers and $\lambda(0)=0$. If a sequence $x=(x_k)$ is a $\alpha-$strongly D_λ convergent to L, then it is a $\alpha-$strongly C_λ convergent to L.

Proof. We are going to use the same technique used by Agnew (1932). Assume that the sequence $x=(x_k)$ is a $\alpha-$strongly D_λ convergent to L. So, for any $n\in N$ we have

$$\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}|x_k-L|^\alpha=\sum_{i=0}^{n-1}\left(\frac{1}{\lambda(n)}\sum_{k=\lambda(i)+1}^{\lambda(i+1)}|x_k-L|^\alpha\right)$$

$$= \sum_{i=0}^{n-1} \frac{\lambda(i+1)-\lambda(i)}{\lambda(n).\left(\lambda(i+1)-\lambda(i)\right)} \sum_{k=\lambda(i)+1}^{\lambda(i+1)} |x_k - L|^{\alpha} =$$

$$= \sum_{i=0}^{n-1} b_{n,i} \left(D_\lambda x\right)_i ,$$

where

$$(D_\lambda x)_i = \frac{1}{\lambda(i+1)-\lambda(i)} \sum_{k=\lambda(i)+1}^{\lambda(i+1)} |x_k - L|^{\alpha}$$

and

$$b_{n,i} = \begin{cases} \dfrac{\lambda(i+1)-\lambda(i)}{\lambda(n)}, & i = 1,2,3,\dots,n-1, \\ 0, & otherwise. \end{cases}$$

Since, the matrix is regular and $\lim_{i\to\infty}(D_\lambda x)_i = 0$, then

$$\lim_{n\to\infty} \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} |x_k - L|^{\alpha} = 0 .$$

It means that, the sequence $x = (x_k)$ is a $\alpha-$ strongly C_λ convergent to L.

Corollary 2.7. Let $\{\lambda(n)\}_{n\in N}$ be an increasing sequence of positive integers and $\lambda(0) = 0$. The $\alpha-$ strongly D_λ convergent sequence can be an $\alpha-$ strongly C_λ convergent only if

$$\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1 .$$

If we accept that $p(n) = \lambda(n-1)$ and $q(n) = \lambda(n)$ in Theorem 2.3 then we have

$$\frac{\lambda(n-1)}{\lambda(n)-\lambda(n-1)} = \frac{1}{\dfrac{\lambda(n)}{\lambda(n-1)} - 1} .$$

The proof of Corollary 2.7 becomes clear with this fact. So, it is omitted here.

Remark 2.1. If we consider the case $\alpha = 1$, then Theorem 2.6 and Theorem 2.7 coincide with Theorem 2.14 and Theorem 2.15 in Osikiewicz (1997), respectively.

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