Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education
Vol. 18, Issue 2, December 2024, pp. 291-323. ISSN: 1307-6086

Research Article

Mathematical Reasoning Activity: Compare, Generalize and Justify

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Received : 28.06.2024 Accepted : 23.08.2024

Doi: https://doi.org/10.17522/balikesirnef.1506921

Abstract – The significance of mathematical reasoning skills is often highlighted in national and international curricula. In recent years, the process aspect of mathematical reasoning has been examined through comparison, generalization, and justification. Emphasizing these process abilities is crucial for creating learning settings that develop mathematical thinking and enhance teacher's understanding. This study assessed middle school students' comparation, generalization, and justification within reasoning activities. The participants were 27 sixth-grade students engaged in a mathematical reasoning workshop. The research data were gathered via a reasoning activity including three open-ended sub-problems addressed by the students. The data were analyzed using content analysis. The results showed that middle school students were capable of comparison, although they had difficulties in generalization and justification. Upon comprehensive evaluation, it was concluded that the number of students who completed these three steps cohesively was considerably low.

Keywords: Mathematical reasoning skill, comparing, generalizing, justification.

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Introduction

Mathematical reasoning is essential for encouraging creative thinking and enhancing students' comprehension in the mathematics teaching process (Carpenter et al., 2003). Through the advancement of mathematical thinking, students will comprehend that mathematics is logical and comprehensible (Pengmanee, 2016). Mathematical reasoning necessitates procedural approaches to mathematical issues and the provision of explanations for answers (Waluyo et al., 2021). This ability is incorporated into the mathematics curriculum of several countries, interwoven with mathematics instruction at all grade levels (Brodie, 2010; Carpenter et al., 2003; Hunter, 2006; Kilpatrick et al., 2001; Visnovska & Cobb, 2009). Mathematical reasoning can be described as a universal mode of thought (Lithner, 2008). Mathematical reasoning entails selecting a strategy and implementing it to solve a problem (Säfström et al., 2024). Moreover, mathematical thinking is occasionally characterised as a general aptitude and a problem-solving instrument (Hjelte et al., 2020). Jeannotte and Kieran (2017) proposed a "concept blur" in the definition and conceptual framework of mathematical reasoning. In fact, mathematical reasoning is a concept that is sometimes difficult for teachers to recognize, let alone teach (Herbert & Williams, 2023). This uncertainty complicates the comprehension of how students might be supported in mathematical reasoning and the scientific research process associated with the idea (Hjelte et al., 2020). Jeannotte and Kieran (2017) defined mathematical reasoning as a communicative activity, either with others or internally, that facilitates the derivation of new mathematical statements from existing ones. They also pointed out that mathematical thinking has both structural and process aspects. Deductive, inductive, and abductive approaches characterize the structural dimension of mathematical reasoning, while the processes of identifying similarities and differences and verification methods represent the process dimension of mathematical reasoning (Jeannotte & Kieran, 2017). The structural component of mathematical reasoning includes more static situations, whereas the process component includes cognitive actions aimed at generating outcomes through inference. Consequently, the process dimension of mathematical thinking has garnered more attention in recent years (Widjaja et al., 2021; Geteregechi, 2020). Nevertheless, this process dimension of thinking is less examined in the literature (Jeannotte & Kieran, 2017). Numerous scholars have indicated that various mathematical reasoning processes, including comparing and contrasting, generalizing and justifying, conjecturing, persuading, and debating, are interconnected (Ellis, 2007; Jeannotte & Kieran, 2017; Lannin et al., 2011 as cited in Santos et al., 2022;

Stylianides, 2007). Jeannotte and Kieran (2017) characterized the process aspect of mathematical reasoning in two groups as "the process of investigating similarities and differences" and "the process of investigating justification ".They asserted that the process of investigating similarities and differences encompasses generalization, pattern recognition, hypothesis formulation, classification, and comparative analysis. They asserted that justification and evidentiary acts comprise the process of investigative verification, while exemplification serves as the action encompassing both processes.

Despite the inclusion of mathematical reasoning in the mathematics education standards of many countries, its application in classrooms remains seldom (Smit et al., 2023). Educators should comprehend their cognitive processes and inferential reasoning while addressing mathematical problems to gain insights into their students' learning (Yeşildere & Türnüklü, 2007). Consequently, evaluating students' mathematical reasoning processes is essential for comprehending learning and teaching contexts (Güler Baran, 2023). In this study, we will analyze the reasoning processes of "comparing," "generalizing," and "justification" by focusing on the process aspect of mathematical reasoning.

Theoretical Background

Comparing

The process dimension of reasoning involves the search for similarities and differences, which encompasses generalizing, making assumptions, identifying patterns, categorizing, and comparing (Widjaja et al., 2021). Comparing, as a reasoning process, is associated with the identification of similarities and differences within mathematical reasoning. This process involves the investigation of common and distinct features among mathematical objects, followed by the establishment of connections (Jeannotte & Kieran, 2017). Comparing is defined by Vale et al. (2017b) as the process of comparing and contrasting to identify a common aspect by recalling past information. These characteristics allow for comparisons to occur in various processes of mathematical reasoning (Jeannotte & Kieran, 2017). For example, consider the processes of pattern identification and conjecture formulation when investigating the similarities and differences among mathematical objects. Conjectures require the comparison of specific examples; likewise, identifying patterns necessitates the comparison of situations or examples (Pedomente, 2002). Pattern identification represents a progression beyond mere comparison. Comparison is confined to the construction of a narrative regarding similarities and differences (Jeannotte & Kieran, 2017).

Comparing is closely linked to generalizing during the examination of similarities and differences (Jeannotte & Kieran, 2017). Identifying patterns and similarities within the context during the comparison process serves as the foundation for generalization (Melhuish et al., 2020). The enquiries "What is the same?" and "What is different?" regarding a mathematical scenario necessitate a comparison and contrast of actions. The ability to distinguish critical aspects from non-critical ones is essential for addressing these enquiries and is crucial for facilitating generalization (Lo & Marton, 2012).

Generalizing

Generalizing is one of the indicators evaluated in the process dimension of mathematical reasoning skills (Jeannotte & Kieran, 2017). Generalizing is employed by mathematicians and mathematics educators to denote both a process and an outcome (Harel & Tall, 1991). A formal rule created from a generalizing job is termed a generalizing, which is a result (Chua, 2013); conversely, when a common or unifying quality is sought among a class of objects, it is seen as a process (Venenciano & Heck, 2016). Nevertheless, when a generalizing statement emerges from the generalizing process, both the method and the output may be regarded collectively (Yerushalmy, 1993, as cited in Oflaz, 2017). Ellis (2007) posits that generalization is a dynamic process wherein learners participate in at least one of the following activities: discovering commonalities across examples, broadening their thinking, or deriving overarching conclusions from particular instances. Numerous academics have characterized generalizing by highlighting its inferential and extensional dimensions. Jeannotte and Kieran (2017) describe generalizing as the process of deriving a link between a collection of mathematical entities or elements from a subset of that collection. Mason et al. (2010) define generalizing as the extension of outcomes derived from mathematical reasoning and problem-solving to a broader context. Lannin (2005) describe generalizing as the activity of contemplating analogous and ongoing occurrences within the broadest context. Kaput (1999) described generalizing as the identification of common characteristics among sample scenarios and the organization of communication and reasoning into a coherent pattern, structure, or connection (Kaput, 2008). The process of generalization is founded on recognizing patterns, delineating commonalities, and correlating analogous materials. The crucial aspect of this approach is not to identify the parallels between occurrences but to broaden and modify these similarities (Ayber, 2017). For instance, asserting that the elements in the sets 1, 3, 5, 7, and 9 are odd and increment by two exemplifies generalization (Hargreaves et al., 1998). The crucial aspect is to provide an explanation that transcends the dataset concerning the regularity of numerical features.

Mason's (1996) assertion that courses devoid of generalizations and assumptions do not constitute mathematics lessons, irrespective of their designation, underscores the significance of generalization in mathematical education. The capacity to generalize allows pupils to engage in systematic thinking and apply principles to specific scenarios (Venenciano & Heck, 2016). Moreover, generalization is a mathematical cognitive process that facilitates students' comprehension of symbolic representations and the establishment of connections with their existing arithmetic knowledge (Lannin, 2005). Generalizing involves utilizing mathematical meanings and relationships to construct accurate assumptions regarding mathematical structures (Melhuish et al., 2020). Generalization prompts the individual to address the enquiries: "What is probable (assumption), why is it valid (justification), and in what context is it applicable (general framework)?" Mason et al. (2010). Consequently, generalization is inseparable from the validation of the derived assertion (Lannin, 2005). While the formulation of a forecast, whether verbal or symbolic, serves as adequate evidence for generalization (Chua, 2013), it remains only a prediction until its correctness is substantiated (Watson, 1980). This research defines generalization as the extrapolation of observed similarities (relations or qualities) from a sample context to a broader context.

Justifying

Justifying, substantiation, and formal proof fall under the category of verification (Widjaja et al., 2021). Verification pertains to critical functions including systematization, communication, integration, creation, and dissemination of new knowledge (Staples et al., 2012). Hanna (2000) identifies two primary functions of validation: to demonstrate truth and to elucidate the reasons for its truth. While it is often straightforward to ascertain "what," understanding "why" is considerably more complex. Addressing the why question necessitates a compelling justification (Mason et al., 2010). Mathematical justification necessitates an examination of existing knowledge and an assessment of the validity of assertions (Staples et al., 2012). Justification, which encompasses the arguments employed to validate and persuade, extends beyond mere explanation (Carpenter et al., 2003; Stebbing, 1952).

"Justifying" refers to the process of validating the truthfulness of information without engaging in a comprehensive proof process (Jeannotte & Kieran, 2017). While justifying serves the same purpose as proof and proving—assessing the truth of a statement—it diminishes the emphasis on the necessary level of formality and specificity that proof entails. This approach facilitates access to pertinent concepts while ensuring that no ideas are overlooked (Staples & Newton, 2016). A key distinction between mathematical proof and mathematical justification is that justifications do not require logical completeness (Jaffe, 1997). Justifying entails employing mathematics to convince oneself or others, irrespective of the completeness of the argument or its acceptance as incontrovertible evidence by the mathematical community (Lesseig, 2016). Melhuish et al. (2020) redirected the focus on student arguments from their completeness or correctness to the process of justification, encouraging greater student engagement with justification. Justification need not be formal or accurate; however, it remains a mathematical reasoning process (Lannin et al., 2011, as cited in Santos et al., 2022).

A strong mathematical justification should effectively address the question of "Why?". Addressing the why question elucidates the background of students' knowledge (Özmusul, 2018). The act of "justifying" enhances students' comprehension of mathematical concepts and aids them in uncovering the rationale behind mathematical principles, as well as substantively articulating their disagreements (Hanna, 2000). Justification serves as an effective learning practice and pedagogical instrument, enhancing students' comprehension of mathematics and facilitating mathematical processes (Staples et al., 2012). Justification allows students to comprehend mathematical concepts and to persuade others of the validity of the procedures, strategies, assumptions, or generalizations they employ (Carpenter et al., 2003; Dreyfus, 1999; Lannin et al., 2011 as cited in Santos et al., 2022; Lannin, 2005; Pedemonte, 2007).

This study defines justifying as the process of persuading the researcher by elucidating the validity of generalizations derived from observed commonalities (relationships or shared characteristics) within the sample context.

The Objective and Significance of the Research

Mathematical reasoning skills are crucial for attaining mathematics learning objectives (Putra et al., 2020). The achievement of mathematics learning objectives has elevated the significance of mathematical literacy. A primary objective of the mathematics curriculum is to cultivate students capable of 'developing and successfully utilizing mathematical literacy abilities' (Ministry of National Education [MoNE], 2018). As stated in the OECD 2022 report, the Program for International Student Assessment (PISA) 2022 and Trends in International

Mathematics and Science Study (TIMSS) 2022 both investigated students' mathematical reasoning skills when they made the framework for the mathematical literacy assessment. The PISA (2022) Turkey Report indicates that Türkiye's reasoning performance is below the OECD average. Consequently, there is a must to enhance reasoning abilities in our country. National and international curricula emphasize the necessity of creating conducive circumstances for the cultivation of mathematical reasoning skills. Indicators essential for students to develop reasoning skills include 'defending the validity and truth of inferences,' 'formulating logical generalizations and inferences,' and 'articulating and applying mathematical patterns and relationships when analyzing a mathematical context' (MoNE, 2013).

The National Council of Teachers of Mathematics (NCTM) (2000) asserts that cultivating students' thinking relies on certain assumptions and principles and that students should be motivated to defend and formulate assumptions. Much research has indicated that justification and generalization are essential across all grade levels and are pivotal to the learning process (Blanton & Kaput 2003; Carraher et al., 2006; Ellis 2007; Lannin 2005) as quoted in Melhuish et al. (2020). By analyzing students' generalization of mathematical concepts, educators can discern the extent of their conceptual comprehension. For secondary school students, comprehending generalization is crucial for enhancing conceptual knowledge (Angraini, 2023). Justification is essential for students to comprehend significant mathematical structures, concepts, and procedures in the classroom (Thanheiser et al., 2021). Staples et al. (2012) underscored the significance of centering justification as a pedagogical activity and asserted that it should be incorporated into the K-12 curriculum. Furthermore, it is asserted that reasoning and proof should be included in all educational processes beginning from early life (Harel & Sowder, 2007; NCTM, 2000).

An examination of the literature reveals that studies are exploring mathematical reasoning skills from diverse perspectives (Bragg & Herbert, 2018; Çiftci, 2015; Çoban, 2010; Francisco & Maher, 2011; Herbert, 2014; Herbert & Bragg, 2021; Herbert & Williams, 2023; Herbert et al., 2022; Lannin, 2005; Loong et al., 2018; Marasabessy, 2021; Mata-Pereira & Ponte, 2017; Öz, 2017; Vale et al., 2017b). Jeannotte and Kieran (2017) noted that the process aspect of mathematical reasoning is inadequately addressed in the literature (Ellis, 2007; Herbert et al., 2022; Jeannotte & Kieran, 2017; Lannin et al., 2011; Lin & Tsai, 2016; Loong et al., 2018; Mason, 1982; Pedemonte, 2007; Peker, 2020; Stylianides, 2007, 2008; Widjaja & Vale, 2021). Emphasizing comparison, generalization, and justification activities, which signify reasoning skills, is crucial in designing learning environments to foster these skills and enhance teachers' knowledge. This study aimed to elucidate the current state of middle school students' comparison, generalization, and justification thinking processes during a mathematical task. Analyses of the students' reasoning and problem-solving approaches are useful in elucidating the nature of reasoning and the dynamics of the processes involved (Serrazina et al., 2024). Selecting suitable tasks or problems that will elucidate the thinking processes under examination is crucial. Vale et al. (2017a) asserted, "What else could it be?" activities such as "Which one does not belong?" can offer thinking possibilities across many mathematical ideas and different primary school levels (Small, 2011). The task “What else might it be?” was employed in our study to elucidate the thinking processes of middle school students transitioning from elementary school. The research is crucial for assessing the comparison, generalization, and justification processes of secondary school students in our country. The study's conclusions are significant since they offer insights for teachers and academics. The study aimed to address the subsequent research questions:

- How are the comparing processes of secondary school students?
- How are middle school students' generalizing processes?
- How are the justifying processes of middle school students?

Method

Research Design

As the study's objective is to describe the reasoning processes of sixth-grade students by implementing a reasoning activity that involves contrasting, generalizing, and justifying processes within the scope of the research, a case study, which is a qualitative research approach, was adopted. A case study is a research method favored for addressing "how" and "why" questions, particularly in contexts where researchers lack control over events or phenomena (Yin, 2009).

The study involved 27 sixth-grade students from two public schools who voluntarily engaged in a mathematical reasoning workshop organized by the researchers. The schools situated in the Eastern Anatolia Region of Türkiye exhibit similarities in socio-economic terms. The sample included 22 female students and 5 male students. The selection of 6th-grade students was based on the expectation that they would possess foundational skills related to numerical phenomena.

Data Collection Tools

The study utilized Small's (2011) "What else could it be?" activity. The Turkish version of the activity in the data collection tool received support from a language expert. Furthermore, the perspectives of three researchers with expertise in mathematics education were obtained regarding the implementation of the activity in the study. This activity enables students to compare, generalize, and justify, as the numerical set in the question stem encompasses multiple relationships and shared characteristics. The activity comprises three open-ended sub-questions. The initial inquiry of the activity is, "These numbers (30, 12, 18) belong together or not because…”. The subsequent question is, "Other numbers that belong with this group are…”. The third question is “How do you know that all these numbers belong and fit with your reason? Use words numbers or drawings to explain”.

Data Collection Process

The data were collected during the mathematical reasoning workshop. The participants were informed in advance of the workshop's date, time, and location. The university provided transport support for students to attend the workshop. The workshop took place in a meeting room that accommodated students comfortably, facilitating ease of writing. Students engaged in the workshop alongside their teachers. The comfort of students in the environment is crucial for their active participation in the researchers' directives. The participants were required to provide written responses independently. The preference for written data collection was due to the presence of students from two different schools in the workshop, as it was anticipated that they might feel uncomfortable expressing themselves verbally. During the workshop, an overview was provided for the initial 15 minutes to enhance awareness of mathematical reasoning skills. Subsequently, the participants were allotted 20 minutes to engage in the "What else could it be" activity. The participants’ responses were collected and retained for analysis after the session.

Data Analysis

Content analysis was implemented to analyze the research data. The data were analyzed through continuous comparison and grouping of relevant information. The researchers analyzed the data to generate codes, categories, and themes. The researchers collaboratively finalized the analysis process and achieved consensus on all code categories and themes. The researchers drew inspiration for naming certain codes, categories, and themes from the works of Ellis (2007), Vale et al. (2017b), and Widjaja et al. (2021). Two weeks post-analysis, the data underwent re-evaluation, leading to the finalization of the analysis.

Role of the Researcher

Before the implementation process, researchers invited teachers and students to participate in the workshop voluntarily. At the outset of the workshop, the researchers provided an overview of mathematical reasoning skills. The nature of mathematical reasoning skills necessitates the frequent use of "why" and "if..." questions. Consequently, an effort was made to highlight how the "if..." structure underpins these reasoning skills. Subsequently, the researchers provided the data collection instrument to the students and remained available in the environment to address any potential enquiries from the students. In addressing potential student enquiries, efforts were made to avoid directing the students. The researchers conducted data arrangement, preparation for analysis, and the analysis process following data collection.

Credibility, Transferability, Consistency, and Verifiability of the Research

The research's credibility was established through a comprehensive presentation of the methodologies employed, management of researcher biases (by engaging with an unfamiliar group), and data analysis conducted by two expert mathematics educators. The research stages were described in detail to enhance the transferability of the study. Using direct quotes and comparing the data to one another helped to verify consistency. The confirmability of the study was established through a detailed explanation of the analysis method, comprehensive descriptions of the participants, data collection tools, data storage procedures, and the researcher's role.

Ethical Issues

The students, accompanied by their teachers, participated in the workshop, with verbal consent obtained for their involvement in the activities. In the data collection phase, participants were instructed to articulate their ideas without restriction. In the research, the actual names of the students who participated in the workshop were not utilized; instead, the student names were assigned codes such as S1 and S2. The data were transferred directly and unaltered. Ethics committee approval was secured (Atatürk University Ethics Committee 05.07.2023/7).

Findings

The analysis of data concerning the comparison, generalization, and justification processes—indicators of students' mathematical reasoning skills—was elucidated through tables, followed by examples that illustrate the diversity of analyses presented.

Findings Related to Students' Comparing Process

The comparative situations of the students were derived from analyzing their responses to the question**,** "These numbers (30, 12, 18) belong together or not because**…"** in the data collection instrument. The responses to this question should demonstrate recognition of features such as magnitude, order, place value, multiples, factors, and the classification of numbers as odd or even. Table 1 displays the codes, categories, and themes obtained from the evaluation of the students' responses.

Table 1 Codes, Categories, and Themes for the "Comparing" Process

Themes	Categories	Codes	Participants
Recognizing the relationship	Addition/ subtraction relationship	Addition relationship	S6, S7, S8, S9, S10, S11, S15, S23, S24, S25
		Subtraction relationship	S1, S6, S8, S9, S17, S22, S23, S24
	Pattern finding	Pattern finding	S7, S10, S13, S15, S19
		Inability to find a pattern	S14
Recognizing the common features	Multiples/ factors	Having factors of 2,3,6	S1, S2, S4, S8, S10, S13, S18, S20, S21, S22, S23, S24, S25
		Being a multiple of 6	S7, S12, S15, S19, S26
		Having factors of 2,3	S3, S7, S16, S27
		Having factors of 3,6	S9
		Having a factor 2	S5
	Even/ odd number	Being an even number	S1, S2, S4, S8, S11, S12, S17, S18, S19, S20, S21, S22, S23, S24, S25, S26, S27
		The units digits are even and the tens digits are odd	S17
	Digit value	Being two digits	S19, S20, S17
		Increasing the number of ones digits	S11

Analysis of Table 1 reveals that all students identified at least one relationship or common feature. The predominant relations identified were the "addition/subtraction" relation within the theme of recognizing relations and the "multiples/factors" common feature in the theme of recognizing common features. Analysis of the codes reveals that the group elements exhibit common characteristics: they are even, consist of multiples of 2, 3, and 6, and demonstrate an "addition/subtraction" relationship among them. Some students identified multiple relationships or common features, resulting in a total frequency that exceeded the number of participants. Figure 1 presents selected excerpts from the student responses.

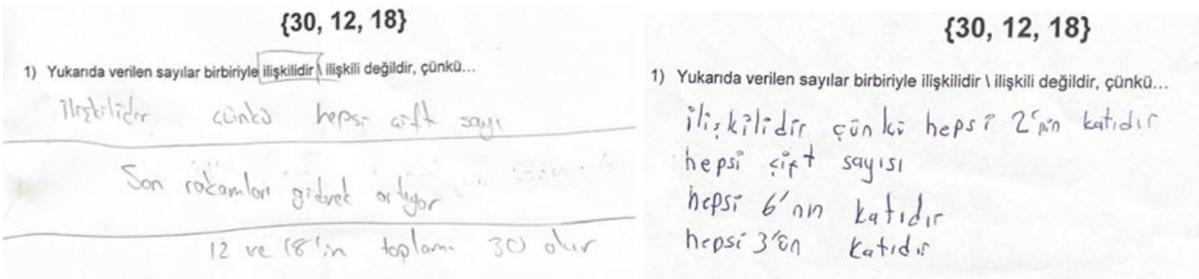

Figure 1 Respectively S11 and S2's Answers About Comparing

Figure 1 illustrates S11's assertion regarding the additive relationship among the group elements, alongside S2's claim that the group elements possess factors of 2, 3, and 6. The relationship indicating that the sum of the numbers 12 and 18 in the group equals 30 was identified by numerous students through comparison. Furthermore, as indicated in S2's response, numerous students identified a shared characteristic through comparison, noting that all numbers within the group are multiples of 2, 3, and 6, and are classified as even numbers.

Findings Related to Students' Generalizing Process

The students' generalizing capacity was assessed through an analysis of their responses to the question, " Other numbers that belong with this group are…", within the data collection tool. The provided answers (30, 12, 18) indicate the necessity to extend the number set by incorporating additional values. All answers identifying the relationships or common features established by the student for the number group (30, 12, 18) were categorised under the theme "Expanding the group for the determined relationship/common feature." Answers that failed to consider all relationships or common features identified by students in the generalisation step were categorised under the theme "Inability to expand the group for the determined relationship/common feature." Table 2 presents the codes, categories, and themes derived from the analysis of student responses.

Table 2 Codes, Categories, and Themes for the "Generalizing" Process

Themes	Categories	Codes	Participants
Expanding the group for the determined relationship/common feature	Expanding the group through a single relationship/common feature	Having a factor of 2	S5
		Addition relationship	S6
	Expanding the group over multiple relationships/common features	Being even and having a factor of 2	S21
		Being even and having multipliers 2,3	S27
		Being a multiple of 6 and being an even number	S26
Inability to expand the group for the determined relationship/common feature	Forming a group that provides some of the determined relationships/common features	Having factors of 2,3,6 and being an even number	S1, S2, S8
		Addition relationship	S9
		Pattern finding	S10
		Being an even number	S17, S18, S20
		Even number and two-digits	S19
		Addition relation and having factors of 2, 3, 6	S23, S24
	Creating separate groups that fulfill some of the determined relationships/common features	Having a factor of 2 or 3	S3, S4
		Being an even number	S4
	Creating separate groups that fulfill each of the determined relationship/common features	Addition/subtraction relationship	S7, S11, S14, S15, S22, S25
		Being an even number	S11, S12, S22, S25
		Increasing the number of ones digits	S11
		Being a multiple of 6	S7, S12, S15
		Having factors of 2, 3, 6	S7, S13, S25
		Pattern finding	S7, S13, S14, S15
		Having a factor of 2, 3 or 6	S16, S22

Analysis of Table 2 reveals that only five students expanded the group based on the relationships or common features identified during the comparison process. The other students were unable to identify the relationships or common features collectively and attempted to expand the group based on one or a few of these features. The students experienced challenges in expanding the number group by integrating all related features identified during the comparison process. However, they did not encounter difficulties in adding new numbers associated with a specific feature. During the analysis, it was observed that students who recognized both "addition relationship" and "pattern finding" relations, or the combination of "addition relationship," "pattern finding," and "being two digits" relations/common features simultaneously in the comparing phase, were unable to generalize all of these relations/common features. A student who extends the group based on the addition relation is unlikely to extend the group according to the -18+6 pattern rule. A student applying the -18+6 pattern rule to expand the group cannot achieve a superset that meets the criterion of being two-digit numbers. For this reason, students are expected to choose one of these properties and make a generalization accordingly. Figures 2 and 3 present excerpts from the students' responses.

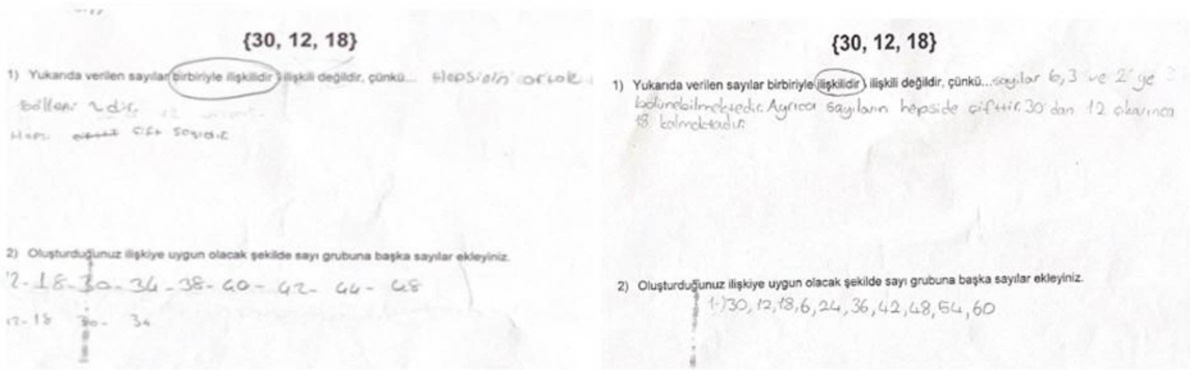

Figure 2 Respectively S21 and S1's Answers About The Generalizing Process

Figure 2 illustrates that S21 successfully generalized the number group. S21 identified the shared characteristics of "being an even number" and "having a factor of 2," subsequently incorporating additional numbers into the group that exhibited both traits, thereby broadening the numerical set. S1, conversely, was unable to generalize the numerical group. S1 identified the shared characteristics of "having factors of 2, 3, 6" and "being an even number," as well as the "subtraction relationship" during comparisons. However, in the generalizing phase, S1 focused solely on the common features of "having factors of 2, 3, 6" and "being an even number," neglecting the "subtraction relationship." Consequently, the relations and common properties identified by S1 for the specified number group do not precisely align with those of the supergroup he attempted to establish, thus he cannot be regarded as having generalized the set.

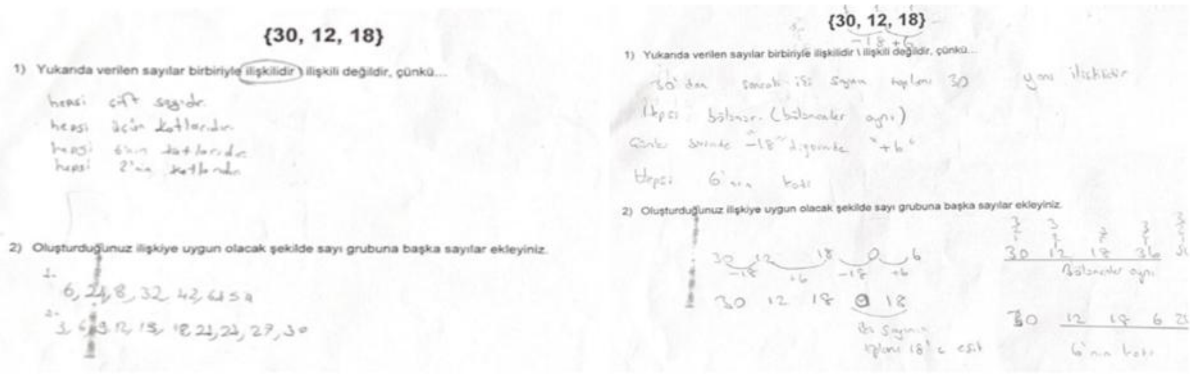

Figure 3 Respectively S4 and S7's Answers About the Generalizing Process

Figure 3 illustrates that S4 was unable to generalize the number group. S4 identified the shared characteristics of "being an even number" and "having factors of 2, 3, and 6" during comparison, subsequently creating a distinct group by incorporating numbers that exhibited some of the common features he discerned while generalizing. Although S4 identified "being a multiple of 6" as a common feature during comparisons, he failed to incorporate this feature in his generalizations. S7 was unable to generalize the number group. S7 observed the connections between "pattern finding" and "addition relationship," as well as the shared characteristics of "having factors of 2, 3" and "being a multiple of 6." While making comparisons, S7 attempted to expand the group individually for each scenario, neglecting to consider all relationships and common features collectively. Upon careful analysis, it is evident that the "addition relationship" and "pattern finding" relations cannot be simultaneously established. Consequently, the student should either refrain from writing one of the relations or indicate that the two relations cannot be simultaneously achieved.

Findings Related to Students' Justifying Process

The students' capacity for justification was assessed through an analysis of their responses to the prompt, “How do you know that all these numbers belong and fit with your reason? Use words numbers or drawings to explain”. Students are required to justify the expansion process undertaken based on the relationships and common features identified. Table 3 presents the codes, categories, and themes derived from the analysis of student responses.

Table 3 Codes, Categories, and Themes for the "Justification" Process

Themes	Categories	Codes	Participants
Inability to justify	No answer/irrelevant response	No answer/irrelevant response	S2,S16
	Writing the specified relationship/common feature verbatim	Writing the specified relationship/common feature verbatim	S1, S3, S4, S6, S7, S8, S9, S11, S12, S13, S14, S17, S18, S19, S20, S22, S23, S24, S25, S26
To be able to justify	To be able to make partial justification for the determined relationship/common feature	Pattern finding	S10
		Being a multiple of 6	S15
		Having factors of 2, 3	S27
	To be able to make a justification for the determined relationship/ common feature	Being an even number and having factors 2, 3, 6	S21
		Having a factor of 2	S5

Analysis of Table 3 reveals that the majority of students were unable to justify. The students believed they justified by reiterating the relationship and common features identified. This situation may stem from the students' insufficient experience in justification. Participants' responses that the specified relationship or common feature characteristic verbatim without any justification, as well as those that provided entirely irrelevant answers, were categorized under the theme of "Inability to justify." A subset of students conducted the verification process for a single relationship or common feature. Only two students extended the group by utilizing the initially identified relationship/common feature and successfully justified this expansion. Participants' responses that justified were categorized under the theme "To be able to justify." The number of individuals capable of justifying is notably low. Figures 4 and 5 present excerpts from student responses concerning the justification process.

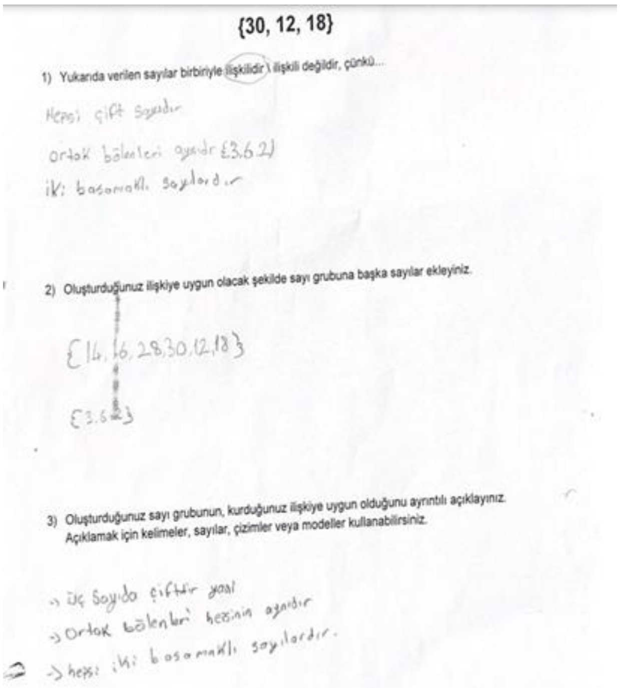

Figure 4 S20's Answer about Justifying Process

As seen in S20, there were many students who tended to write the determined relationship exactly. As can be seen in Figure 4, S20 formed a number group without using all of the relations/common features he identified in the comparing process. In the justification step, he wrote the relationship/common features he found without providing any justification.

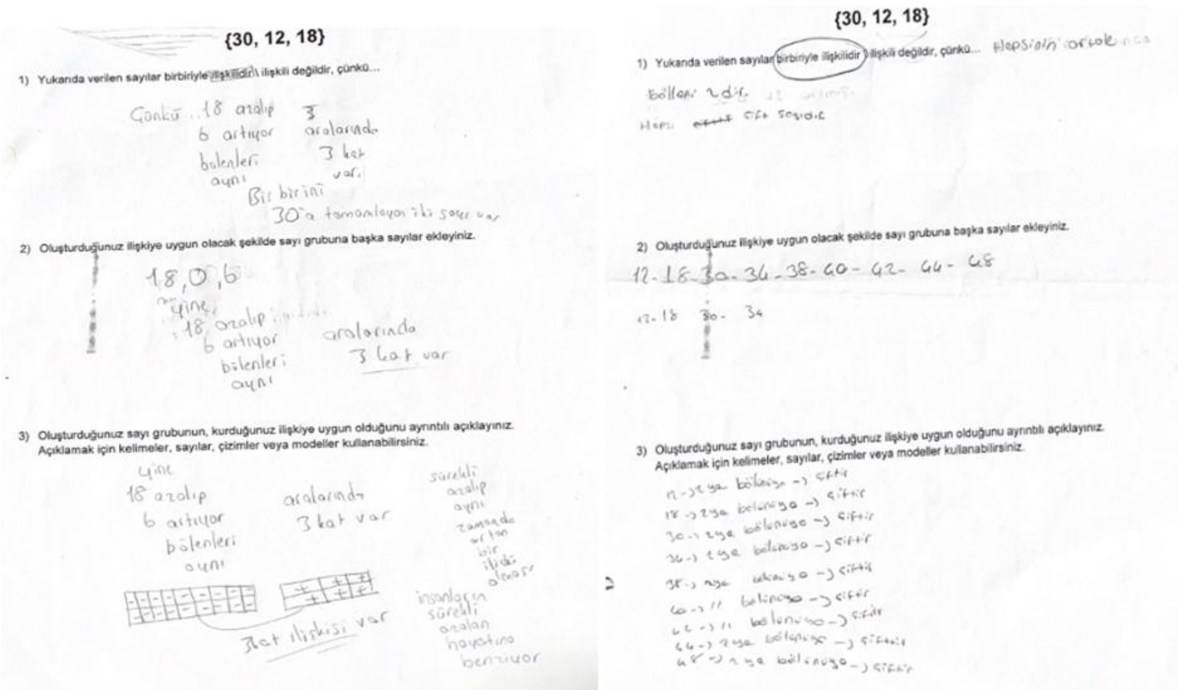

Figure 5 Respectively S10 and S21's Answers about the Justifying Process

The answer of S10, which was evaluated in the category of "to be able to make partial justification for the determined relationship/common feature", is presented in Figure 5. As can be seen in the figure, S10 did not justify the relationship/common feature that he stated as "numbers complement each other by 30" among the relationship/common features he found, while he justified the other relationship/common features. The answer of S21 in the category of "to be able to make justification for the determined relationship / common feature" is given in Figure 5. S21 expanded the number group for the relationship/common feature he identified and was able to justify all the relationship/common features he found.

Discussion

Our findings are essential for enhancing students' higher-level reasoning by identifying the common characteristics of this number group, extending the group to a broader set, and providing justification for their assumptions when faced with a number group. Upon examination of the students' responses, it was observed that they demonstrated at least one reasoning action in the processes of comparison, generalization, and justification. Upon comprehensive consideration, it was determined that the number of students who completed these three processes in a connected manner was relatively low. The research allowed students to articulate their thoughts freely through open-ended questions, mitigating grade-related anxiety. Vale et al. (2017a) underscore the fact that open-ended tasks that necessitate making assumptions about a shared characteristic offer an opportunity for reasoning actions.

The students demonstrated the ability to make comparisons to identify the relationships and common features among the numbers in the specified group. The students identified multiple relationships and common features within the given number group through their comparisons. The predominant relationship identified within the theme of "recognizing the relationship" is the "addition/subtraction" relationship. The investigation of the number group (30, 12, 18) primarily concentrated on the operations of addition and subtraction concerning establishing a relationship among the numbers. Additionally, some students sought patterns in their attempts to establish relationships between numbers. It is important to note that the number of students (7) attempting to identify a pattern was limited. The recognition of the "addition/subtraction" relationship by the majority (18) may be attributed to its minimal requirement for advanced reasoning in identifying common features among numbers. Jeannotte and Kieran (2017) asserted that identifying a pattern extends beyond mere comparison, as the process primarily highlights only similarities and differences. As an example of higher-level reasoning in the process of comparing and contrasting, Ellis (2007) and Vale et al. (2017b) also mentioned recognizing relationships that call for deeper thought rather than identifying relationships that are readily recognized. Our study indicates that students demonstrated a greater awareness of the addition/subtraction relation compared to the pattern-finding relation. Within the theme of "recognizing the common feature," it was noted that students identified the common features of "multiples/multipliers" and "even/odd numbers" more frequently than other common features. Students identified that the common divisors of the numbers in the group (30, 12, 18) are 2, 3, and 6. Although a minority of students identified only one or two common divisors, the majority successfully recognized all three common divisors. Furthermore, a notable characteristic that was often highlighted was the property of being an even number. Students recognized that every number in the group is an even number. Alongside these common properties, another category that some students emphasized is "digit value." According to Vale et al. (2017a), when students compare, they employ numerical information such as factors, multiples, place value, counting patterns, and number order. The potential to identify a relationship or common feature may be associated with students' prior knowledge of numerical phenomena. Consequently, students' capacity for comparison is linked to gaps in their prior knowledge.

The majority of students were unable to generalize the provided number group. It was observed that, during comparisons, the group could not be expanded by incorporating additional numbers related to the identified relationships or common features. Similarly, Rodrigues et al. (2021) highlighted that instructors and pre-service teachers struggle with scenarios requiring the generalization process, and Ersoy et al. (2017) also highlighted that secondary school students struggle with it. To achieve comprehensive generalization, it is essential to consider all identified relationships and common features holistically. Generalization necessitates that students recognize similarities and fundamental principles across diverse examples or contexts (Angraini, 2023). Malara (2012) describes the generalization process as a sequential cognitive activity that entails the analysis of specific instances and shared characteristics, subsequently applying these insights to all established common features. The groups formed by students who cannot generalize do not qualify as a superset when considering the established relationships or common features. The students demonstrated the ability to generalize across one or more relations or common features. It remains unclear whether students generalized over a single relation or common feature due to an inability to identify additional relations or because it was more straightforward to generalize in this manner. Students may encounter difficulties in the process of generalization, even when they recognize relationships or common features during comparison. This may lead them to concentrate on familiar relationships or features that they can easily manipulate.

The analysis of the justifying process revealed that students could not typically provide justifications. Only two students successfully generalized and justified the relationship or common feature identified within the number group during the comparison process. The findings align with previous research indicating a low incidence of students providing complete and convincing justifications (Özmusul & Bindak, 2022) and a general unfamiliarity among students with the practice of justifying their solutions (Reyes-Hernandez & Mooney, 2021). Furthermore, many studies indicate that students struggle with generalization and justification (Chazan, 1993; English & Warren, 1995; Knuth et al., 2002). When the answers of the students who could not justify are analyzed, it is noteworthy that the students tended to write the same relationship/common features they found. Students experienced challenges in generalizing the relationship or common feature identified within the number group to a broader context. Additionally, they struggled to establish and interpret the causal connections between the conjectures they developed. Understanding the rationale and methodology behind a task is crucial for the development of mathematical thinking in students, rather than merely executing tasks mechanically (Dikkartın Övez & İnce, 2024). Lins (2001) asserts that justifications offer insight into the overarching concept of generalization and its characteristics. Generalizing and justifying are closely related concepts (Ellis 2007; Kirwan 2015; Lannin 2005). Furthermore, justifying serves as a mechanism that aids students in uncovering the applications of various elements in mathematics, thereby enhancing their comprehension of mathematical concepts (Hanna, 2000). Students' inability to justify may be linked to their exposure to justifying activities within the learning environment and their underlying conceptual understanding. Assigning students tasks that facilitate the establishment of mathematical relationships, encourage discussion of their reasoning, and require justification within the learning environment allows for opportunities to justify and generalize (Staples & Newton, 2016; Stein et al., 2008). Bozkurt et al. (2017) indicated that secondary school teachers predominantly employed questions that elicited short answers in the classroom, while questions necessitating long answers and deeper comprehension, such as those involving justification and criticism/interpretation, were utilized to a lesser degree. Furthermore, a lack of emphasis on the rationale behind any procedure or phenomenon presented to students will diminish their engagement in hypothesizing, justifying, and generalizing activities (Mukuka et al., 2023). Jackson and Stenger (2024) emphasized that generalization should be conveyed through lessons designed to communicate overarching statements regarding the subjects studied in the classroom environment. It is essential for teachers to priorities comparing, generalizing, and justifying within their classrooms and to incorporate these elements into their teaching methodologies. Teachers are essential in facilitating students' engagement in mathematical reasoning (Ellis et al., 2019). Widjaja et al. (2021) highlighted the necessity of offering primary school students opportunities to engage in reasoning processes that involve comparing and contrasting, verifying conjectures, and generalizing. This study examined the process aspect of mathematical reasoning and concluded that the majority of middle school students were unable to complete the comparing, generalizing, and justifying processes holistically. When these processes were analyzed individually, the majority of students demonstrated the ability to make comparisons based on more superficial reasoning compared to other processes. Nonetheless, the proportion of students who engaged in the generalizing and justification processes necessitating higher-level reasoning significantly declined. However, what is expected here, is necessary to generalize the common features or relationships identified, to broaden the dataset, and to provide justification for this scenario.

The study's findings and results led to recommendations for researchers, educators, and program developers.

The process dimension of mathematical reasoning skills is gaining significance. Future studies should incorporate a broader range of research participants and methodologies. Given the extended duration necessary for students to develop a skill, further longitudinal studies are essential. Given the variability in students' modes of expression, it is beneficial to explore alternative assessment methods beyond solely evaluating their reasoning through written responses.

A conducive classroom climate is essential for fostering the development of mathematical reasoning skills in educational settings. Creating a supportive and non-judgmental environment is essential for facilitating student expression of thoughts. To address deficiencies in students' mathematical reasoning skills, it is essential to provide additional activities that facilitate reasoning opportunities and allow for individual reasoning development.

The acquisition of the mathematical reasoning skills, often implicitly integrated into curricula, can be emphasized more prominently. Teachers' professional development activities, as implementers of the curriculum, can be enhanced through more effective methods and instructional strategies for teaching this skill to students, facilitated by both in-service training and curricular improvements.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest

The authors have no relevant financial or non-financial interests to disclose.

Funding

The authors did not receive support from any organization for the submitted work.

Credit author statement

All authors contributed to the study's conception and design. Material preparation, data collection and analysis were performed by [Öz, T.] and [Çiftçi, Z.]. The first draft of the manuscript was written by [Öz, T.] and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Research involving Human Participants and/or Animals

The participants were verbally informed.

Study-specific approval by the appropriate ethics committee for research involving humans and/or animals (The research study that underpins this publication was provided by Ataturk University, Registration number 05.07.2023/7).

Matematiksel Akıl Yürütme Etkinliği: Karşılaştır, Genelle, Gerekçele

Özet:

Matematiksel akıl yürütme becerisinin önemi hem ulusal hem de uluslararası müfredatlarda sıklıkla vurgulamaktadır. Özellikle son yıllarda, matematiksel akıl yürütmenin süreç yönü, karşılaştırma, genelleme ve gerekçelendirme açısından ele alınmıştır. Bu süreç becerilerine odaklanmak, matematiksel akıl yürütme becerilerinin geliştirilmesi için öğrenme ortamlarının hazırlanmasında ve öğretmenlerin farkındalığının artırılmasında oldukça önemlidir. Bu çalışma ile ortaokul öğrencilerinin akıl yürütme etkinliği kapsamında karşılaştırma, genelleme ve gerekçelendirme durumları incelenmiştir. Araştırmanın katılımcılarını matematiksel akıl yürütme atölyesine katılan 6. sınıf seviyesindeki 27 öğrenci oluşturmaktadır. Araştırma verileri öğrencilerin cevaplandırdığı 3 açık uçlu alt problemden oluşan akıl yürütme etkinliği ile toplanmıştır. Veriler içerik analizi ile analiz edilmiştir. Araştırmanın sonuçlarına göre, ortaokul öğrencilerinin karşılaştırma yapabildiklerini fakat genelleme ve gerekçelendirme basamaklarında problem yaşadıklarını göstermektedir. Bir bütün olarak düşünüldüğünde ise bu üç süreci de bağlantılı bir şekilde tamamlayan öğrenci sayısının çok az olduğu sonucuna varılmıştır.

Anahtar kelimeler: Matematiksel akıl yürütme becerisi, karşılaştırma, genelleme, gerekçelendirme.

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