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Geometry of pointwise hemi-slant warped product submanifolds in para-contact manifolds

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Abstract. In this article, firstly we study pointwise slant, pointwise hemi-slant submanifolds whose ambient spaces are para-cosymplectic manifolds and we prove that there exist pointwise hemi-slant non-trivial warped product submanifolds whose ambient spaces are para- cosymplectic manifolds by giving some examples. We get several theorems and some characterizations. Later, we also obtain some inequalities.

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1. Introduction

Slant submanifold was explained by B.Y. Chen in 1990 and he started the working in pseudo-Riemannian manifolds in 2012 [4]. Then, Almost contact manifold was indicated by I. Sato [10]. S. Zamkovoy researched almost para-contact metric manifolds [12] and An almost para-contact geometry is expressed as $(\mathcal{P}, \xi, \eta)$. Such that, $\mathcal{P}^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on almost para-contact structure. Then, some researchers have been working Riemannian and semi-Riemannian manifolds in last years $[1, 2, 5, 8]$.

Bishop and O'Neill produced notion of warped product manifolds. Warped products are $\mathcal{N}_a$ and $\mathcal{N}_b$ be Riemannian manifolds with $\breve{g}_a$ and $\breve{g}_b$. Then, warped product manifold $\mathcal{N}_x = \mathcal{N}_a \times_k \mathcal{N}_b$ is a product manifold $\mathcal{N}_a \times \mathcal{N}_b$ equipped by $\breve{g}_x = \breve{g}_a + k^2 \breve{g}_b$ and k is a warping function of warped product manifold [3]. Warped products is generally used in differential geometry, theory of general relativity, theory of string, black holes. Warped product pseudo-slant submanifolds whose ambient spaces are Kaehler manifolds were worked by B. Sahin [9]. He proved that the warped product pseudo-slant $N_b^{\perp} \times_k N_a^{\theta}$ submanifold does not exist and he obtained a characterization and an inequality. Later S. Uddin and others worked warped product submanifolds whose ambient spaces are cosymplectic manifolds [11].

This article is organized as follows. In section 2, we introduce pointwise slant submanifolds of para-cosymlectic manifolds. Moreover, we give some definitions, examples and results. In section 3, we introduce proper pointwise hemi-slant submanifolds in para-cosymplectic manifolds and we give theorems, lemmas and examples. In section 4, we define pointwise hemi-slant non-trivial warped product submanifolds in para-cosymlectic manifolds. Also, we give some results and examples. In section 5, we obtain some inequalities.

2. Preliminaries

Let $\bar{\mathcal{N}}_x$ be a $(2\bar{n}+1)$-dimensional almost para-contact metric structure. If it is provided with structure $(\mathcal{P}, \xi, \eta, \breve{g}_1)$, that $\mathcal{P}$ is a tensor field of type $(1, 1)$, η is a one form, ξ is a vector field and $\breve{g}_1$ is to expressed semi-Riemannian metric.

$$\mathcal{P}^2 = \mathcal{I} - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \breve{g}_1(P\mathcal{X}_a, P\mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) + \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b) \tag{1}$$

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These situations require that

$$\mathcal{P}\xi = 0, \quad \eta(\mathcal{P}\mathcal{X}_a) = 0, \quad \eta(\mathcal{X}_a) = \breve{g}_1(\mathcal{X}_a, \xi), \tag{2}$$

$$\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b). \tag{3}$$

An almost para-contact metric manifold is named para-cosymplectic manifold if the following relation is satisfied:

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b = 0, \quad \mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b = \bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \quad \bar{\nabla}_{\mathcal{X}_a}\xi = 0 \tag{4}$$

including any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{N}_x}$.

Let currently, $\mathcal{N}_x$ is a submanifold of $(\mathcal{P}, \xi, \eta, \breve{g}_1)$. The Gauss and Weingarten equations are dedicated by

$$\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b = \nabla_{\mathcal{X}_a}\mathcal{Y}_b + h_1(\mathcal{X}_a, \mathcal{Y}_b), \tag{5}$$

$$\bar{\nabla}_{\mathcal{X}_a}V = -A_V\mathcal{X}_a + \nabla^{\perp}_{\mathcal{X}_a}V, \tag{6}$$

including $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and $V \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$, that h_1 is a second fundamental form of $\mathcal{N}_x$, A_V is the Weingarten endomorphism connected with V and $\nabla^{\perp}$ is the normal connection. A_V and h_1 are related by

$$\breve{g}_1(A_V\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), V), \tag{7}$$

here $\breve{g}_1$ designates the semi-Riemannian metric on $\mathcal{N}_x$ with the one introduced on $\mathcal{N}_x$. For all tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{8}$$

such that $R\mathcal{X}_a$ is the tangential component of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal one. For all normal vector field V,

$$\mathcal{P}V = rV + sV, \tag{9}$$

such that rV and sV are the tangential, normal components of $\mathcal{P}V$, respectively.
From the covariant derivative of the tensor fields R,S,r and s, we get

$$(\nabla_{\mathcal{X}_a}R)Y_b = \nabla_{\mathcal{X}_a}RY_b - R\nabla_{\mathcal{X}_a}Y_b, \tag{10}$$

$$(\nabla_{\mathcal{X}_a}S)Y_b = \nabla^{\perp}_{\mathcal{X}a}S\mathcal{Y}_b - S\nabla_{\mathcal{X}_a}Y_b, \tag{11}$$

$$(\nabla_{\mathcal{X}_a}r)V = \nabla_{\mathcal{X}_a}rV - r\nabla^{\perp}_{\mathcal{X}a}V, \tag{12}$$

$$(\nabla_{\mathcal{X}_a}s)V = \nabla^{\perp}_{\mathcal{X}a}sV - s\nabla^{\perp}_{\mathcal{X}a}V. \tag{13}$$

The mean curvature vector is indicated by

$$H = \frac{1}{n}traceh_1. \tag{14}$$

Definition 1. *We call that a submanifold $\mathcal{N}_x$ of almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is pointwise slant if for all time-like or space-like tangent vector field $\mathcal{X}_a$, the ratio $\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a)/\breve{g}_1(\mathcal{P}\mathcal{X}_a, P\mathcal{X}_a)$ is a function. Moreover, a submanifold $\mathcal{N}_x$ of almost para-contact metric structure $\bar{\mathcal{N}_x}$ is named pointwise slant, if at each point* p $\in \mathcal{N}_x$*, the Wirtinger angle $\theta(X)$ between $\mathcal{P}\mathcal{X}_a$ and $\mathcal{T}_p\mathcal{N}_x$ is dependent of the choice of the non-zero $\mathcal{X}_a \in \mathcal{T}_p\mathcal{N}_x$. In this instance, the Wirtinger angle causes a real-valued function $\theta : \mathcal{T}\mathcal{N}_x - 0 \to \mathcal{R}$ which is named the slant function or Wirtinger function of the pointwise slant submanifold.*

We express that a pointwise slant submanifold whose ambient spaces are almost para-contact manifold is named slant, if its Wirtinger function θ is globally constant. We state that all slant submanifold is a pointwise slant submanifold [9].

If $\mathcal{N}_x$ is a para-complex submanifold, in that case, $\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a$ and the above ratio is equal to 1. Moreover if $\mathcal{N}_x$ is totaly real, then $R = 0$, so $\mathcal{P}\mathcal{X}_a = S\mathcal{X}_a$ and the above ratio equals 0. Hence, both para-complex submanifolds and totally real are the special situations of pointwise slant submanifolds.

Definition 2. *Let $\mathcal{N}_x$ be a proper pointwise slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$. We call that it is of type-1 if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is time-like(spacelike), also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$ and (For type-2) $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.*

Theorem 1. *Let $\mathcal{N}_x$ be a pointwise slant submanifold in almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$. So that, for any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike) and $\mathcal{N}_x$ is the pointwise slant submanifold of type-1-2 necessary and sufficient condition*

$$\textbf{(a)} \quad \mu = R^2 = cosh^2\theta(I - \eta\otimes\xi), \quad \mu \in (1, +\infty) \quad (Type-1), \tag{15}$$

$$\textbf{(b)} \quad \mu = R^2 = cos^2\theta(I - \eta\otimes\xi), \quad \mu \in (0, 1) \quad (Type-2). \tag{16}$$

where θ denotes the slant function of N_x.

Proof. Firstly, if $\mathcal{N}_x$ is a pointwise slant submanifold of type-1 for any spacelike tangent vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike. from the equation of (1), $\mathcal{P}\mathcal{X}_a$ also is. Furthermore, they supply $|R\mathcal{X}_a|/|\mathcal{P}\mathcal{X}_a| > 1$. So, there exists the slant function θ. Because of,

$$\cosh\theta = \frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} = \frac{\sqrt{-\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a))}}{\sqrt{-\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)}} \tag{17}$$

and using (1) and (17) , we have

$$\breve{g}_1(R^2\mathcal{X}_a, \mathcal{X}_a) = \cosh^2\theta(I - \eta\otimes\xi)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a).$$

Thus, we get $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$. So, $\mu = R^2 = \cosh^2\theta(I - \eta\otimes\xi)$.
Also, for any time-like tangent vector field $\mathcal{Z}$, $R\mathcal{Z}$ and $\mathcal{P}\mathcal{Z}$ are spacelike. Therefore, in place of (17), we get

$$\cosh\theta = \frac{|R\mathcal{Z}|}{|\mathcal{P}\mathcal{Z}|} = \frac{\sqrt{\breve{g}_1(R\mathcal{Z}, R\mathcal{Z}))}}{\sqrt{\breve{g}_1(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}$$

Because of $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$, for any spacelike and timelike $\mathcal{X}_a$ it further provides for lightlike vector fields and therefore we get $\mu = R^2 = \cosh^2\theta(I - \eta\otimes\xi)$. Thus, we get (a). In a similar way, we have (b) ☐

Corollary 1. *Let $\mathcal{N}_x$ be a pointwise slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$ with the slant function θ. Later, for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x - <\xi>$, If $\mathcal{N}_x$ is of type-1, type-2, we obtain:*

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cosh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)),\\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= \sinh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)), \end{aligned} \tag{18}$$

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cos^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)),\\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= -\sin^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{19}$$

Corollary 2. *Let $\mathcal{N}_x$ be a pointwise slant submanifold of an almost para-contact metric structure $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$. Later, let $\mathcal{N}_x$ be a pointwise slant submanifold of almost para-contact metric structure $\bar{\mathcal{N}_x}$. Therefore $\mathcal{N}_x$ is a pointwise slant submanifold of (type-1-2) necessary and sufficient condition,*
    * $rS\mathcal{X}_a = -\sinh^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$ *and* $SRX = -sSX$ *(For type-1)*
    * $rS\mathcal{X}_a = \sin^2\theta(\mathcal{X}_a - \eta(\mathcal{X}_a)\xi)$ *and* $SRX = -sSX$ *(For type-2)*
are satisfied for all timelike (spacelike) vector field $\mathcal{X}_a$.

3. Pointwise Hemi-Slant Submanifolds Whose Ambient Spaces are Para-Cosymplectic Manifolds

Definition 3. *A semi-Riemannian submanifold $\mathcal{N}_x$ of almost para-contact manifold $(\bar{\mathcal{N}_x}, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is named to pointwise hemi-slant submanifold if there exist a two orthogonal distributions $D_t^{\perp}$, D_n^{α} with $\mathcal{N}_x$. Such that,*
1) $\mathcal{T}\mathcal{N}_x = \mathcal{D}_t^{\perp} \oplus \mathcal{D}_n^{\alpha} \oplus <\xi>$.
2) *The distribution $\mathcal{D}_t^{\perp}$ is an totally real distribution, $\mathcal{P}\mathcal{D}_t^{\perp} \subset \mathcal{T}^{\perp}\mathcal{N}_x$.*
3) *The distribution $\mathcal{D}_n^{\alpha}$ is a pointwise slant distribution.*
Then, we say θ as function.

Definition 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of an almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. Let $\mathcal{D}_n^\alpha$ be a pointwise slant distribution on $\mathcal{N}_x$. Then, we call that it is of (For type-1) if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike(spacelike), also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$ and (For type-2) $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.*

Theorem 2. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of almost para-contact metric structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. $\mathcal{N}_x$ is the pointwise slant submanifold of type-1-2 necessary and sufficient condition*

$$\textbf{(a)} \quad \mu = R^2 = cosh^2\theta(I - \eta \otimes \xi), \quad \mu \in (1, +\infty), \quad (Type - 1). \tag{20}$$

$$\textbf{(b)} \quad \mu = R^2 = cos^2\theta(I - \eta \otimes \xi), \quad \mu \in (0, 1), \quad (Type - 2). \tag{21}$$

For any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike).

Proof. The proof is proved like the proof of Theorem 1. □

Corollary 3. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold of almost para-contact structure $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. For non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x - < \xi >$, if $\mathcal{D}_n^\alpha$ is of type-1 and type-2, then we obtain (respectively)*

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cosh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= \sinh^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{22}$$

and

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cos^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(X)\eta(Y)), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= -\sin^2\theta(\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) - \eta(\mathcal{X}_a)\eta(\mathcal{Y}_b)). \end{aligned} \tag{23}$$

Lemma 1. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. We get, $A_{\mathcal{P}\mathcal{Z}_a}\mathcal{W}_b = A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a$ is satisfied for any non-null vector fields $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$.*

Proof. For type-*1*-*2* and for $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$, $\mathcal{U}_a \in \Gamma(\mathcal{T}\mathcal{N}_x)$, we write $\mathcal{U}_a = \mathcal{P}_1\mathcal{U}_a + \mathcal{P}_2\mathcal{U}_a + \eta(\mathcal{U}_a)\xi$. Let be $\mathcal{T}N_x = \mathcal{D}_t^\perp \oplus \mathcal{D}_n^\alpha \oplus < \xi >$ and $\mathcal{T}^\perp N_x = \mathcal{P}\mathcal{D}_t^\perp \oplus S\mathcal{D}_n^\alpha \oplus \lambda$
Using (3),(4),(6) and (7), we obtain

$$\begin{aligned} \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, \mathcal{U}_a) &= \breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{U}_a), \mathcal{P}\mathcal{W}_b) \\ &= -\breve{g}_1(-A_{\mathcal{P}\mathcal{Z}_a}\mathcal{U}_a + \nabla^\perp_{\mathcal{U}_a}\mathcal{P}\mathcal{Z}_a, \mathcal{W}_b) \\ &= \breve{g}_1(A_{\mathcal{P}\mathcal{Z}_a}\mathcal{W}_b, \mathcal{U}_a). \end{aligned}$$

□

Lemma 2. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. In this case, the totally real distribution $\mathcal{D}_t^\perp$ is always integrable.*

Proof. For type-1, type-*2* and since $\bar{\mathcal{N}}_x$ is a para-cosymplectic manifold, using equations (1),(3),(4),(5),(6), (8) and from definition of projections for non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{D}_t^\perp$ and $\mathcal{U}_a \in \mathcal{T}N_x$, we write

$$\begin{aligned} \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{P}\mathcal{U}_a) &= -\breve{g}_1(\mathcal{P}[\mathcal{X}_a, \mathcal{Y}_b], \mathcal{U}_a) \\ &= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{U}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P}\mathcal{X}_a, \mathcal{U}_a). \end{aligned}$$

The right hand side of the last equation should be zero. Thus, we derive

$$\begin{aligned} \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{P}\mathcal{U}_a) &= 0, \\ \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], R\mathcal{U}_a) + \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], S\mathcal{U}_a) &= 0, \\ \breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], RP_2\mathcal{U}_a)) &= 0. \end{aligned}$$

From above equation, we have $[\mathcal{X}_a, \mathcal{Y}_b] = 0$. So, $\mathcal{D}_t^\perp$ is integrable. □

Lemma 3. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type-1 and type-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. For $\mathcal{D}_n^\alpha \oplus < \xi >$ to be integrable, necessary and sufficient condition*
1) $\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) = sech^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}))$*(Tip-1)*

2) $\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a, Z) = sec^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, Z), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}))(Tip-2)$
for non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{D}_n^\alpha \oplus <\xi>$ *and* $\mathcal{Z} \in \mathcal{D}_t^\perp$.

Proof. We demonstrate 1) and 2) in a similar method. We will give its proof when $\mathcal{D}_n^\alpha$ is type-1. $\bar{\mathcal{N}}_x$ is a para-cosymplectic manifold, using (1),(2),(3),(4),(5),(6),(7),(8) and Corollary 2, we write

$$\begin{aligned}
\breve{g}_1([\mathcal{X}_a, \mathcal{Y}_b], Z) &= \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b - \bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \eta(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b)\eta(\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}R\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}S\mathcal{Y}_b, \mathcal{Z}) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}rS\mathcal{Y}_b, \mathcal{Z}) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}sS\mathcal{Y}_b, \mathcal{Z}) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}) \\
&= -\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) - sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}) \\
&+ \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

making add subtract $sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})$ above equation, we have

$$\begin{aligned}
cosh^2\theta\breve{g}_1(([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{Z})) &= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) \\
&- cosh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

$$\begin{aligned}
\breve{g}_1(([\mathcal{X}_a, \mathcal{Y}_b], \mathcal{Z})) &= sech^2\theta(\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b) - \breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z})) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z})
\end{aligned}$$

The right hand side of the last equation should be zero, proof is complete. □

Theorem 3. *Let* $\mathcal{N}_x$ *be a pointwise hemi-slant type1-2 submanifold whose ambient space is para-cosymplectic manifold* $(\bar{\mathcal{N}}_x, \mathcal{P}, \xi, \eta, \breve{g}_1)$. *In that case, totally real distribution* $\mathcal{D}_t^\perp$ *describes a totally geodesic foliation, necessary and sufficient condition*

$$\breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}a - A_{SR\mathcal{X}_a}\mathcal{W}_b, \mathcal{Z}_a) = 0 \tag{24}$$

is satisfied for non-null vector fields $\mathcal{Z}_a, \mathcal{W}_b \in \mathcal{D}_t^\perp$, $\mathcal{X}_a \in \mathcal{D}_n^\alpha \oplus <\xi>$.

Proof. For type-*1*, we obtain

$$\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) = \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) - \breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), \mathcal{X}_a).$$

Using (1) and (5), we get

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) + \eta(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b)\eta(\mathcal{X}_a) \\
&= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a).
\end{aligned}$$

Using (6), (8), also from $\mathcal{P}\mathcal{W}$ and $S\mathcal{X}_a$ are orthogonally. We obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, R\mathcal{X}_a) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, S\mathcal{X}_a) \\
&= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, R\mathcal{X}_a) - \breve{g}_1(\nabla^\perp_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, R\mathcal{X}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a, \mathcal{P}\mathcal{W}_b).
\end{aligned}$$

Using (1), (4) and (7). We obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, R\mathcal{X}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a, \mathcal{P}\mathcal{W}_b) \\
&= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}S\mathcal{X}_a, \mathcal{W}_b).
\end{aligned}$$

Using (9) and (Corollary 2 for type-1), we obtain

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) &= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}rS\mathcal{X}_a, \mathcal{W}_b) \\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}sS\mathcal{X}_a, \mathcal{W}_b) \\
&= \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) + sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{X}_a, \mathcal{W}_b) \\
&+ \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}SR\mathcal{X}_a, \mathcal{W}_b).
\end{aligned}$$

Using (5), (6), (7) and because of $\mathcal{W}_b$ and $\mathcal{X}_a$ are orthogonally, we obtain

$$\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a) = \breve{g}_1(h_1(\mathcal{Z}_a, R\mathcal{X}_a), \mathcal{P}\mathcal{W}_b) - sinh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{X}_a)$$

$$\begin{aligned} & - \breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) \\ cosh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a) &= \breve{g}_1(h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b)-\breve{g}_1(h_1(\mathcal{W}_b,\mathcal{Z}_a),SR\mathcal{X}_a) \\ cosh^2\theta\breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a) &= \breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Z}_a)-\breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b,\mathcal{Z}_a). \end{aligned}$$

Thus, the proof is complete. In the same way, we get for type-2 □

Theorem 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant type1-2 submanifold whose ambient space is para-cosymplectic manifold $(\bar{\mathcal{N}}_x,\mathcal{P},\xi,\eta,\breve{g}_1)$. In that case, pointwise slant distribution $\mathcal{D}_n^\alpha\oplus<\xi>$ describes a totally geodesic foliation, necessary and sufficient condition*

$$\breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b-A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}a,\mathcal{Y}_b)=0 \tag{25}$$

is satisfied for non-null vector fields $\mathcal{W}_b\in\mathcal{D}_t^\perp$ and $\mathcal{X}_a,\mathcal{Y}_b\in\mathcal{D}_n^\alpha\oplus<\xi>$.

Proof. For type-1, using (1) and (5), we get

$$\begin{aligned} \breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) \\ &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P}\mathcal{X}_a,\mathcal{P}\mathcal{W}_b). \end{aligned}$$

Using (3), (5),(6), (8) and Corollary 2, we obtain

$$\begin{aligned} \breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(\nabla_{\mathcal{Y}_b}R\mathcal{X}_a,\mathcal{P}\mathcal{W}_b)-\breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b) \\ &+ \breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Y}_b}S\mathcal{X}_a,\mathcal{W}_b) \\ &= -\breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b)+\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}rS\mathcal{X}_a,\mathcal{W}_b) \\ &+ \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}sS\mathcal{X}_a,\mathcal{W}_b) \\ &- \breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b). \\ (1+sinh^2\theta)\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(h_1(\mathcal{Y}_b,R\mathcal{X}_a),\mathcal{P}\mathcal{W}_b)-\breve{g}_1(\bar{\nabla}_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b). \\ (cosh^2\theta)\breve{g}_1(\nabla_{\mathcal{Y}_b}\mathcal{X}_a,\mathcal{W}_b) &= -\breve{g}_1(A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Y}_b)-\breve{g}_1(-A_{SR\mathcal{X}_a}\mathcal{Y}_b,\mathcal{W}_b) \\ &- \breve{g}_1(\nabla^\perp_{\mathcal{Y}_b}SR\mathcal{X}_a,\mathcal{W}_b) \\ &= \breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{W}_b-A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Y}_b). \end{aligned}$$

So, the proof is completed. In the same way, we have for type-2 too. □

Corollary 4. *Let $\mathcal{N}_x$ be a pointwise hemi-slant submanifold type-1,2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. Therefore $\mathcal{N}_x$ is a locally semi-Riemannian product structure, necessary and sufficient condition*

$$A_{\mathcal{P}\mathcal{Y}_b}R\mathcal{X}_a=A_{SR\mathcal{X}_a}\mathcal{Y}_b$$

is satisfied for $\mathcal{X}_a\in\mathcal{D}_n^\alpha\oplus<\xi>$ and $\mathcal{Y}_b\in\mathcal{D}_t^\perp$, that $\mathcal{N}_b^\perp$ is a anti-invariant submanifold and $\mathcal{N}_a^\theta$ is a pointwise slant submanifold of $\bar{\mathcal{N}}_x$.

4. Pointwise Hemi-Slant Non-Trivial Warped Product Submanifolds of Para-Cosymplectic Manifolds

Warped product manifolds were introduced by Bishop and O'Neill [3]. Projections of $\mathcal{N}_a\times\mathcal{N}_b$ are $\beta_1:\mathcal{N}_a\times\mathcal{N}_b\to\mathcal{N}_a$ and $\beta_2:\mathcal{N}_a\times\mathcal{N}_b\to\mathcal{N}_b$. Such that warped product manifold $\mathcal{N}_x=\mathcal{N}_a\times_k\mathcal{N}_b$ is the Riemannian manifold $\mathcal{N}_a\times\mathcal{N}_b=(\mathcal{N}_a\times\mathcal{N}_b,\breve{g})$ with the Riemannian structure. Therefore

$$\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=\breve{g}_1(\beta_{1*}\mathcal{X}_a,\beta_{1*}\mathcal{Y}_b)+(k\circ\beta_1)^2\breve{g}_1(\beta_{2*}\mathcal{X}_a,\beta_{2*}\mathcal{Y}_b)$$

for every vector fields $\mathcal{X}_a,\mathcal{Y}_b\in\Gamma(T\mathcal{N}_x)$, that * indicates the tangent map. The function k is named the warping function of the warped product manifold. Especially, if the warping function is non-constant, the manifold $\mathcal{N}_x$ is named to be non-trivial. $\mathcal{N}_a$ is totally geodesic and $\mathcal{N}_b$ is totally umbilical in $\mathcal{N}_x$.

Lemma 4. *Let $\mathcal{N}_x=\mathcal{N}_a\times_k\mathcal{N}_b$ be a warped product manifold with warping function k, therefore*
1) $\nabla_{\mathcal{X}_a}\mathcal{Y}_b\in\Gamma(\mathcal{T}\mathcal{N}_a)$ *is the lift of* $\nabla_{\mathcal{X}_a}\mathcal{Y}_b$ *on* $\mathcal{N}_a$*;*
2) $\nabla_{\mathcal{X}_a}\mathcal{Z}=\nabla_{\mathcal{Z}}\mathcal{X}_a=(\mathcal{X}_a lnk)\mathcal{Z}$ *;*
3) $\nabla_{\mathcal{Z}}\mathcal{W}=\bar{\nabla}^2_{\mathcal{Z}}\mathcal{W}-(\breve{g}(\mathcal{Z},\mathcal{W})\div k)$ *gradk ;*
are satisfied for non-null vector fields $\mathcal{X}_a,\mathcal{Y}_b\in\mathcal{T}\mathcal{N}_a$ and $\mathcal{Z},\mathcal{W}\in\mathcal{T}\mathcal{N}_b$, where gradk is the gradient of k

introduced as $\breve{g}_a(gradk, \mathcal{X}_a) = \mathcal{X}_a k$ *also* ∇ , $\bar{\nabla}^2$ *define the Levi-Civita connections on* $\mathcal{N}_x$ *and* $\mathcal{N}_b$ *[3]. As a result, we get*

$$||gradk||^2 = \sum_{v=1}^{s}(e_v(k))^2 \tag{26}$$

is satisfied for an orthonormal frame $(e_1, ..., e_s)$ *on* $\mathcal{N}_a$.

Theorem 5. *There does not exist a pointwise hemi-slant non-trivial warped product submanifolds* $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ *whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$ *and* $\xi \in \mathcal{T}\mathcal{N}_b^{\perp}$. *Such that* $\mathcal{N}_b^{\perp}$ *is totally real and* $\mathcal{N}_a^{\theta}$ *is pointwise slant submanifold of* $\bar{\mathcal{N}_x}$.

Proof. The non-existence of warped products pointwise semi-slant submanifolds whose ambient spaces are cosymplectic manifolds had proved by K.S. Park [7]. Similarly, we can demonstrate the non-existence of warped products pointwise hemi-slant submanifolds whose ambient spaces are para-cosymplectic manifolds. □

Let's consider para-cosymplectic structure on $\bar{R}_3^7$:

$$\mathcal{P}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \ \ \mathcal{P}(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \ \ \mathcal{P}(\frac{\partial}{\partial z}) = 0, \ \ \xi = \frac{\partial}{\partial z}, \ \ \eta = dz.$$

Here, η is 1-form, ξ is vector field and $\breve{g}_1 = (+,-,+,-,+,-,+)$. $\breve{g}_1$ is pseudo-Riemannian metric. Also, $(\mathtt{x}_1, y_1, \mathtt{x}_2, y_2, \mathtt{x}_3, y_3, \mathtt{z})$ denotes the cartesian coordinates over $\bar{R}_3^7$. Then $(\bar{R}_3^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$ is a para-cosymplectic manifold.
Let $\mathcal{N}_x$ be a semi-Riemannian submanifold of $\bar{\mathcal{R}}_3^7$ described by $\psi : \mathcal{N}_x \to \bar{\mathcal{R}}_3^7$.

Example 1. *For* $\mathtt{m} + \mathtt{n} > 0$ *and* $m + n \in \mathcal{R}$ *with*

$$\psi(m, \mathtt{n}, \mathtt{c}, t) = (cosh\mathtt{m}, \cosh n, \sinh n, \sinh m, c^3, \alpha, t),$$

$$\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial x_1} + \cosh \mathtt{m}\frac{\partial}{\partial y_2}, \quad \psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial y_1} + \cosh \mathtt{n}\frac{\partial}{\partial x_2},$$

$$\psi_c = +3c^2\frac{\partial}{\partial x_3}, \quad \psi_t = \frac{\partial}{\partial z} = \xi.$$

Then, we get

$$\mathcal{P}\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial y_1} + \cosh \mathtt{m}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial x_1} + \cosh \mathtt{n}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_c = 3c^2\frac{\partial}{\partial y_3}$$

describes a pointwise hemi-slant submanifold $\mathcal{N}_x^4$ *with type-*1 *whose ambient space is para-cosymplectic manifold* $(\bar{\mathcal{R}}_3^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$ *with* $\mu = \mathcal{R}^2 = cosh^2(\mathtt{m}+\mathtt{n})(I - \eta \otimes \xi)$. *Actually* $D_n^{\alpha} = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ *is pointwise slant distribution with hemi-slant function and* $\mathcal{D}_t^{\perp} = span\{\psi_c\}$ *is anti-invariant distribution.*
It is easy to notice that D_n^{α}, $\mathcal{D}_t^{\perp}$ *are integrable. The induced metric tensor* $g_{\mathcal{N}_x}$ *on* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *is given by* $g_{\mathcal{N}_x} = -{d_m}^2 + {d_n}^2 + (9c^4){d_c}^2 + {d_t}^2$.
Thus, $\mathcal{N}_x$ *is a pointwise hemi-slant non-trivial warped product type-1 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{R}}_3^7$ *with warping function* $k = 3c^2$.

Example 2. *For* $\mathtt{m} - \mathtt{n} \in (0, \frac{\pi}{2})$ *with*

$$\psi(\mathtt{m}, \mathtt{n}, c, t) = (\cos m, \cos n, \sin m, \sin n, \sin c, \pi, t),$$

$$\psi_{\mathtt{m}} = -\sin m\frac{\partial}{\partial x_1} + \cos m\frac{\partial}{\partial x_2}, \quad \psi_n = -\sin n\frac{\partial}{\partial y_1} + \cos \mathtt{n}\frac{\partial}{\partial y_2},$$

$$\psi_c = \cos c\frac{\partial}{\partial x_3}, \quad \psi_t = \frac{\partial}{\partial z} = \xi,$$

Then, we get

$$\mathcal{P}\psi_{\mathtt{m}} = -\sin \mathtt{m}\frac{\partial}{\partial y_1} + \cos \mathtt{m}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = -\sin \mathtt{n}\frac{\partial}{\partial x_1} + \cos \mathtt{n}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_c = \cos c\frac{\partial}{\partial y_3}$$

describes a pointwise hemi-slant submanifold with type-2 in $(\bar{\mathcal{R}}_4^7, \mathcal{P}, \xi, \eta, \breve{g}_1)$, *with* $\mu = \mathcal{R}^2 = cos^2(\mathtt{m}-\mathtt{n})(I - \eta \otimes \xi)$. $D_n^{\alpha} = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ *is pointwise slant distribution with hemi-slant function and* $\mathcal{D}_t^{\perp} = span\{\psi_c\}$

is anti-invariant distribution and $\mathcal{P}\psi_c \perp T\mathcal{N}_x = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}, \psi_t\}$.
It is easy to notice that D_n^{α}, $\mathcal{D}_t^{\perp}$ *are integrable. The induced metric tensor* $g_{\mathcal{N}_x}$ *on* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *is given by* $g_{\mathcal{N}_x} = {d_m}^2 - {d_n}^2 + (\cos^2 c){d_c}^2 + {d_t}^2$. *Thus,* $\mathcal{N}_x^4$ *is a pointwise hemi-slant non-trivial warped product type-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{R}}_3^7$ *with warping function* $k = cosc$.

Lemma 5. *Let* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$. *Such that* $\xi \in T\mathcal{N}_b^{\perp}$, *then*
1) $\xi(lnk) = 0$.
2) *For any* $\mathcal{X}_a, \mathcal{Y}_b \in T\mathcal{N}_a^{\theta}$ *and* $\mathcal{Z} \in T\mathcal{N}_b^{\perp}$,

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b). \tag{27}$$

Proof. **1)** For any $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ and $\xi \in T\mathcal{N}_b^{\perp}$, we obtain $\bar{\nabla}_{\mathcal{X}_a}\xi = 0$. Also Using (5),(6) and from Lemma 4 - (2), we obtain $\xi(lnk)\mathcal{X}_a = 0$ which means that $\xi(lnk) = 0$, for any non-zero vector field $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ that proves 1).
2)Using (5),(3), (8),(6), (7), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) &= \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b - \nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{Z}) \\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{Z}) \\
&= -\breve{g}_1(-A_{S\mathcal{Y}_b}\mathcal{X}_a + \nabla^{\perp}_{\mathcal{X}_a}S\mathcal{Y}_b, \mathcal{Z}) \\
&= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b)
\end{aligned}$$

If we relocate $\mathcal{X}_a$ with $R\mathcal{X}_a$ and $\mathcal{Y}_b$ with $R\mathcal{Y}_b$ in (27) , then we get belove results

$$\breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}), S\mathcal{Y}_b), \tag{28}$$

$$\breve{g}_1(h_1(\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b), \tag{29}$$

$$\breve{g}_1(h_1(R\mathcal{X}_a, R\mathcal{Y}_b), \mathcal{P}\mathcal{Z}) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}), SR\mathcal{Y}_b). \tag{30}$$

□

Lemma 6. *Let* $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ *be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold* $\bar{\mathcal{N}_x}$. *Such that* $\xi \in T\mathcal{N}_a^{\theta}$, *then*
1) $\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}, \mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
2) a) *For type-1;*
$\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), SR\mathcal{X}_a) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (\mathcal{X}_a lnk)cosh^2\theta\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
b) *For type-2 ;*
$\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), SR\mathcal{X}_a) = \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) - (\mathcal{X}_a lnk)cos^2\theta\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b)$,
for any $\mathcal{Z}_a, \mathcal{W}_b \in T\mathcal{N}_b^{\perp}$ *and* $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$.

Proof. Using (8), (5) and Lemma 4-(2), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_a), (\mathcal{P}\mathcal{X}_a - R\mathcal{X}_a)) \\
&= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) - \breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}_b, \mathcal{P}\mathcal{X}_a) \\
&= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) - \breve{g}_1(\nabla_{\mathcal{Z}_a}\mathcal{W}, R\mathcal{X}_a).
\end{aligned}$$

By using (4) and from $\mathcal{W}_b$ and $R\mathcal{X}_a$ are orthogonality. Also later using (6), (7) and from Lemma 4-(2), we obtain

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) + \breve{g}_1(\mathcal{W}_b, \nabla_{\mathcal{Z}_a}R\mathcal{X}_a) \\
&= -\breve{g}_1(-A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a, \mathcal{X}_a) + \breve{g}_1(\nabla^{\perp}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b, \mathcal{X}_a) \\
&+ (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b) \\
&= \breve{g}_1((h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b).
\end{aligned}$$

Therefore, Proof 1 is complete. Now, we will demonstrate proof 2(a) for type-1.
If we replace $\mathcal{X}_a$ and $R\mathcal{X}_a$ in the last equation and using (1), we derive

$$\begin{aligned}
\breve{g}_1(h_1(\mathcal{Z}_a, \mathcal{W}_b), S\mathcal{X}_a) &= \breve{g}_1((h_1(\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b) \\
&= \breve{g}_1((h_1(R\mathcal{X}_a, \mathcal{Z}_a), \mathcal{P}\mathcal{W}_b) + R^2(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a, \mathcal{W}_b).
\end{aligned}$$

For type-1, (a);

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) = \breve{g}_1((h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b)+cosh^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).$$

For type-2, (b);

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),SR\mathcal{X}_a) = \breve{g}_1((h_1(R\mathcal{X}_a,\mathcal{Z}_a),\mathcal{P}\mathcal{W}_b)+cos^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).$$

□

Theorem 6. *Let $\mathcal{N}_x=\mathcal{N}_a^\theta\times_k\mathcal{N}_b^\perp$ be a pointwise hemi-slant non-trivial warped product type1-2 submanifold whose ambient space is para-cosymplectic manifold $\bar{\mathcal{N}}_x$. Then $\mathcal{N}_x$ is locally a mixed geodesic warped product pointwise submanifold $\mathcal{N}_a^\theta\times_k\mathcal{N}_b^\perp$ necessary and sufficient condition*

$$A_{\mathcal{P}\mathcal{Z}_a}\mathcal{X}_a=0,\quad A_{S\mathcal{X}_a}\mathcal{Z}_a=R\mathcal{X}_a(\varphi)\mathcal{Z}_a, A_{SR\mathcal{X}_a}\mathcal{Z}_a=cosh^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_a \quad (Type1), \tag{31}$$

$$A_{\mathcal{P}\mathcal{Z}_a}\mathcal{X}_a=0,\quad A_{S\mathcal{X}_a}\mathcal{Z}_a=R\mathcal{X}_a(\varphi)\mathcal{Z}_a A_{SR\mathcal{X}_a}\mathcal{Z}_a=cos^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_a \quad (Tip-2) \tag{32}$$

are satisfied for any $\mathcal{X}_a\in D_n^\alpha\oplus<\xi>$ and $\mathcal{Z}_a\in D_t^\perp$, that φ is a function on $\mathcal{N}_x$ and $\mathcal{W}_b(\varphi)=0$ is satisfied for any $\mathcal{W}_b\in D_t^\perp$.

Proof. Using advantage of Lemmas 4 and 5, we demonstrate that $\mathcal{N}_x$ is a mixed geodesic warped product pointwise submanifold. Let $\mathcal{N}_x$ be a hemi-slant submanifold with the slant distribution $\mathcal{D}_n^\alpha\oplus<\xi>$ and the anti-invariant distribution $\mathcal{D}_t^\perp$ with the cases shown in (31) and (32). Also using these conditions and Theorem 4, the distribution $\mathcal{D}_n^\alpha\oplus<\xi>$ describes a totally geodesic foliation and utilizing Lemma 2, $\mathcal{D}_t^\perp$ is integrable, imagine $h^\perp$ be the second fundamental form of the leaf $\mathcal{N}_b^\perp$ of $\mathcal{D}_t^\perp$ in $\mathcal{N}_x$, Also for any $\mathcal{X}_a\in\mathcal{D}_n^\alpha\oplus<\xi>$ and $\mathcal{W}_b,\mathcal{Z}_a\in\mathcal{D}_t^\perp$.
Utilizing (5), (1), (3), (4) and (8), we have

$$\begin{aligned}\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= \breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a)\\ &= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,\mathcal{P}\mathcal{X}_a)+\eta(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b)\eta(\mathcal{P}\mathcal{X}_a)\\ &= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,R\mathcal{X}_a)-\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_a}\mathcal{P}\mathcal{W}_b,S\mathcal{X}_a).\end{aligned}$$

Utilizing (6) and therefore $\mathcal{P}\mathcal{W}_b$ and $S\mathcal{X}_a$ are orthogonality, we obtain

$$\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = -\breve{g}_1((A_{\mathcal{P}\mathcal{W}_b}\mathcal{Z}_a,R\mathcal{X}_a)+\breve{g}_1(\mathcal{P}\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}S\mathcal{X}_a).$$

Utilizing (1),(3),(4) and (9), we get

$$\begin{aligned}\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= -\breve{g}_1((A_{\mathcal{P}\mathcal{W}_b}R\mathcal{X}_a,\mathcal{Z}_a)-\breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}rS\mathcal{X}_a)\\ &- \breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}sS\mathcal{X}_a).\end{aligned}$$

Utilizing first condition of (31) and Corollary 2, we derive

$$\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = \breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}sinh^2\theta\mathcal{X}_a)+\breve{g}_1(\mathcal{W}_b,\bar{\nabla}_{\mathcal{Z}_a}SR\mathcal{X}_a).$$

Therefore, orthogonality of $\mathcal{W}_b$ with $\mathcal{X}_a$, using (5),(6) and (31), we derive

$$\begin{aligned}\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= -sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_a}\mathcal{W}_b,\mathcal{X}_a)\\ &+ \breve{g}_1(\mathcal{W}_b,(-A_{SR\mathcal{X}_a}\mathcal{Z}_a+\nabla^\perp_{\mathcal{Z}_a}SR\mathcal{X}_a))\\ &= -sinh^2\theta\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a)-\breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{Z}_a,\mathcal{W}_b),\\ -cosh^2\theta\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) &= \breve{g}_1(A_{SR\mathcal{X}_a}\mathcal{Z}_a,\mathcal{W}_b)\\ &= cosh^2\theta\mathcal{X}_a(\varphi)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b).\end{aligned}$$

From the describtion of gradient, we obtain

$$\breve{g}_1(h^\perp(\mathcal{Z}_a,\mathcal{W}_b),\mathcal{X}_a) = -\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b)\breve{g}_1(grad\varphi,\mathcal{X}_a).$$

So that, $h^\perp(\mathcal{Z}_a,\mathcal{W}_b)=-\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b)\breve{g}_1 grad\varphi$ for vectors $\mathcal{Z}_a,\mathcal{W}_b\in D_t^\perp$. $H=-grad\varphi$ and $\mathcal{N}_b^\perp$ is totally umbilical in $\mathcal{N}_x$
Now, we explain $grad\varphi$ is parallel suitable to the normal connection $\mathcal{D}_t^\perp$ of $\mathcal{N}_b^\perp$ in $\mathcal{N}_x$. For $\mathcal{X}_a\in\mathcal{D}_n^\alpha\oplus<\xi>$ and $\mathcal{W}_b\in D_t^\perp$, we derive

$$\begin{aligned}\breve{g}_1(D_{\mathcal{W}_b}grad\varphi,\mathcal{X}_a) &= \breve{g}_1(\nabla_{\mathcal{W}_b}grad\varphi,\mathcal{X}_a)\\ &= \mathcal{W}_b\breve{g}_1(grad\varphi,\mathcal{X}_a)-\breve{g}_1(grad\varphi,\nabla_{\mathcal{W}_b}\mathcal{X}_a)\\ &= \mathcal{W}_b(\mathcal{X}_a(\varphi))-\breve{g}_1(grad\varphi,[\mathcal{W}_b,\mathcal{X}_a])-\breve{g}_1(grad\varphi,\nabla_{\mathcal{X}_a}\mathcal{W}_b)\end{aligned}$$

$$= \mathcal{X}_a(\mathcal{W}_b\varphi) + \breve{g}_1(\nabla_{\mathcal{X}_a} grad\varphi, \mathcal{W}_b) = 0.$$

So, $\mathcal{W}_b\varphi = 0$ is satisfied for every $\mathcal{W}_b \in \mathcal{D}_t^\perp$ also $\nabla_{\mathcal{X}_a} grad\varphi \in \mathcal{D}_n^\alpha \oplus <\xi>$ therefore $\mathcal{D}_n^\alpha \oplus <\xi>$ is totally geodesic. We understand that mean curvature of $\mathcal{N}_b^\perp$ is parallel. So that, the leaves of $\mathcal{D}_t^\perp$ are totally umbilical with parallel mean curvature $H = -grad\varphi$. So, $\mathcal{N}_b^\perp$ is called the extrinsic sphere in $\mathcal{N}_x$. By considering Hiepko ([6]), we attain that $\mathcal{N}_x$ is a warped product pointwise submanifold and the proof is completed for type-*1*.
In a similarly way, for type-2 is also proved. □

5. An Optimal Inequality

Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a s-dimensional pointwise hemi-slant non-trivial warped product submanifold whose ambient space is $(2m+1)$-dimensional para-cosymplectic manifold $\bar{\mathcal{N}_x}$. Such that, $\mathcal{N}_b^\perp$ is dimension d_1 and $\mathcal{N}_a^\theta$ is dimension $d_2 = 2p+1$ so ξ is tangent to $\mathcal{N}_a^\theta$. We take tangent spaces of $\mathcal{N}_b^\perp$ and $\mathcal{N}_a^\theta$ by $\mathcal{D}_t^\perp$ and $\mathcal{D}_n^\alpha \oplus <\xi>$. We create orthonormal frames according to type-1 and type-2. Firstly for type-1, the orthonormal frames of $\mathcal{D}_t^\perp$ and $\mathcal{D}_n^\alpha \oplus <\xi>$, respectively;
$\{\mathtt{E}_1, \mathtt{E}_2, .., \mathtt{E}_{d_1}\}$ and $\{\mathtt{E}_{d_1+1} = \mathtt{E}_1^*, ..., \mathtt{E}_{d_1+\mathtt{p}} = \mathtt{E}_\mathtt{p}^*, \mathtt{E}_{d_1+\mathtt{p}+1} = \mathtt{E}_{\mathtt{p}+1}^* = sech\theta R\mathtt{E}_1^*, ...$
$, \mathtt{E}_{d_1+2\mathtt{p}} = \mathtt{E}_{2\mathtt{p}}^* = sech\theta R\mathtt{E}_\mathtt{p}^*, \mathtt{E}_{d_1+2\mathtt{p}+1} = \mathtt{E}_{2\mathtt{p}+1}^* = \xi\}$ that θ is nonconstant.
At the moment, we will give orthonormal frames of the normal subbundles of $\mathcal{P}D_t^\perp$, SD_n^α and λ. This frames respectively are
$\{\mathtt{E}_{s+1} = \bar{\mathtt{E}}_1 = \mathcal{P}\mathtt{E}_1, \mathtt{E}_{s+2} = \bar{\mathtt{E}}_2 = \mathcal{P}\mathtt{E}_2, ..., \mathtt{E}_{s+d_1} = \bar{\mathtt{E}}_{d_1} = \mathcal{P}\mathtt{E}_{d_1}\}$,
$\{\mathtt{E}_{s+d_1+1} = \bar{\mathtt{E}}_{d_1+1} = csch\theta S\mathtt{E}_1^*, \mathtt{E}_{s+d_1+2} = \bar{\mathtt{E}}_{d_1+2} = csch\theta S\mathtt{E}_2^*, ..., \mathtt{E}_{s+d_1+p} = \bar{\mathtt{E}}_{d_1+p} = csch\theta S\mathtt{E}_p^*, \mathtt{E}_{s+d_1+p+1} = \bar{\mathtt{E}}_{d_1+p+1} = csch\theta sech\theta SR\mathtt{E}_1^*, ..., \mathtt{E}_{s+d_1+p+p} = \bar{\mathtt{E}}_{d_1+p+p} = csch\theta sech\theta SR\mathtt{E}_p^*\}$ and
$\{\mathtt{E}_{2s} = \bar{\mathtt{E}}_s, ..., \mathtt{E}_{2m+1} = \bar{\mathtt{E}}_{2(m-s+1)}\}$. where θ is the slant function.
Lets assume that
* on $\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim(\mathcal{D}_t^\perp)$; also, supposed that $\breve{g}_1(\mathtt{E}_v, \mathtt{E}_v) = 1$.
* on $\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p+1}$, where $2p+1 = dim(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.
* on $\mathcal{P}\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim\mathcal{P}(\mathcal{D}_t^\perp)$ also $\breve{g}_1(\mathcal{P}\mathtt{E}_v, \mathcal{P}\mathtt{E}_v) = -1$.
* on $\mathcal{S}\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p+1}$, where $2p+1 = dim\mathcal{S}(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.

Theorem 7. *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a s-dimensional mixed geodesic warped product pointwise hemi-slant of type-1 submanifold whose ambient space is $(2m+1)$- dimensional para-cosymplectic manifold $\bar{\mathcal{N}_x}$. So that $\mathcal{N}_a^\theta$ is a proper pointwise slant submanifold of dimension $2p+1$ and $\mathcal{N}_b^\perp$ is a totally real submanifold of dimension d_1 of $\bar{\mathcal{N}_x}$. So that $\mathcal{N}_b^\perp$ is spacelike. Then*
1) *The squared norm of the second fundamental form of $\mathcal{N}_x$ supplies*

$$||h_1||^2 \leq d_1 coth^2\theta ||gradlnk||^2, \tag{33}$$

where $grad(lnk)$ is the gradient of lnk.

2) *If the equality sign of (33) holds the same way, then $\mathcal{N}_a^\theta$ is totally geodesic and $\mathcal{N}_b^\perp$ is totally umbilical in $\bar{\mathcal{N}_x}$.*

Proof. From description $||h_1||^2 = ||h_1(\mathcal{D}_m, \mathcal{D}_m)||^2 + 2||h_1(\mathcal{D}_m, \mathcal{D}_t^\perp)||^2 + ||h_1(\mathcal{D}_t^\perp, \mathcal{D}_t^\perp)||^2$, that $\mathcal{D}_m = \mathcal{D}_n^\alpha \oplus <\xi>$. Because of $\mathcal{N}_x$ is mixed geodesic, the middle term of the right-hand side should be zero. In that case, we obtain

$$||h_1||^2 = \sum_{r=s+1}^{2m+1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \mathtt{E}_r)^2 + \sum_{r=s+1}^{2m+1} \sum_{l,b=1}^{d_1} \breve{g}_1(h_1(\mathtt{E}_l^*, \mathtt{E}_b^*), \mathtt{E}_r)^2$$

This equation can be seperated for the $\mathcal{P}D_t^\perp$, SD_n^α and λ components as follows

$$\begin{aligned} ||h_1||^2 = & \sum_{r=1}^{d_1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \bar{\mathtt{E}}_r)^2 \\ & + \sum_{r=d_1+1}^{2p+d_1} \sum_{v,w=1}^{2p+1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \bar{\mathtt{E}}_r)^2 \end{aligned}$$

$$+\sum_{r=s}^{2(m-s+1)}\sum_{v,w=1}^{2p+1}\breve{g}_1(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=1}^{d_1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=d_1+1}^{2p+1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$$
$$+\sum_{r=s}^{2(m-s+1)}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2 \quad (34)$$

Utilizing(27) and (30), the first term of right-hand side in the last equation vanishes same way and we should leave all the terms except the fifth term in the last equation, then we have
$||h_1||^2 \le \sum_{r=d_1+1}^{2p+d_1}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),\bar{\mathtt{E}}_r)^2$
Using the frame of $S\mathcal{D}_n^\alpha$, we get,

$$\begin{aligned}||h_1||^2 \quad &\le \quad \sum_{w=1}^{p}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),csch\theta S\mathtt{E}^*_w)^2\\ &+ \quad \sum_{w=1}^{p}\sum_{l,b=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_l,\mathtt{E}_b),csch\theta sech\theta SR\mathtt{E}^*_w)^2\end{aligned}$$

Utilizing Lemma 5 and Lemma 6-1, we obtain

$$\begin{aligned}||h_1||^2 \quad &\le \quad csch^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(R\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}_l,\mathtt{E}_b)^2\\ &+ \quad coth^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}_l,\mathtt{E}_b)^2\\ &= \quad (csch^2\theta\sum_{w=1}^{p}(R\mathtt{E}^*_w lnk)^2+coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2\\ &= \quad d_1(csch^2\theta\sum_{w=1}^{p}(R\mathtt{E}^*_w lnk)^2+coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2)\\ &= \quad d_1(csch^2\theta\sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_w,Rlnk)^2+coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2)\end{aligned}$$

By using (26), the above equation will be simlified as

$$\begin{aligned}||h_1||^2 \quad &\le \quad d_1[csch^2\theta(||Rgradlnk||^2-\sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_{p+w},Rgradlnk)^2)\\ &+ \quad coth^2\theta\sum_{w=1}^{p}\sum_{l,b=1}^{d_1}(\mathtt{E}^*_w lnk)^2\breve{g}_1(\mathtt{E}^*_w lnk)^2]\\ &, \quad (for \quad Rgradlnk\in D_m \quad and \quad R\xi=0 \quad)\\ &= \quad d_1[csch^2\theta(||Rgradlnk||^2-cosh^2\theta\sum_{w=1}^{p}\breve{g}_1(\mathtt{E}^*_w,gradlnk)^2)\\ &+ \quad coth^2\theta\sum_{w=1}^{p}(\mathtt{E}^*_w lnk)^2]\end{aligned}$$

$$= \quad d_1[coth^2\theta||Rgradlnk||^2 - coth^2\theta\sum_{w=1}^{p}(\mathtt{E}_w^* lnk)^2 + coth^2\theta\sum_{w=1}^{p}(\mathtt{E}_w^* lnk)^2]$$

Last equation specifies in (33) and from the leaving terms in (34), we have the following connections from the second and the third leaving terms of (34).

$\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m)S\mathcal{D}_n^{\alpha}) = 0$, $\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m),\lambda) = 0$ that intend

$$h_1(\mathcal{D}_m,\mathcal{D}_m)\bot S\mathcal{D}_n^{\alpha}, \quad h_1(\mathcal{D}_m,\mathcal{D}_m)\bot\lambda \Rightarrow h_1(\mathcal{D}_m,\mathcal{D}_m) \in \mathcal{P}\mathcal{D}_t^{\bot} \tag{35}$$

Because of a mixed geodesic warped product pointwise submanifold and from Theorem 5 , we derive $\breve{g}_1(h_1(\mathcal{D}_m,\mathcal{D}_m),\mathcal{P}\mathcal{D}_t^{\bot}) = 0$. Such that

$$h_1(\mathcal{D}_m,\mathcal{D}_m)\bot\mathcal{P}\mathcal{D}_t^{\bot} \tag{36}$$

When we take into account (35) and (36), understand that $h_1(\mathcal{D}_m,\mathcal{D}_m) = 0$ using this connection with the fact that $\mathcal{N}_a^{\theta}$ is totaly geodesic in $\mathcal{N}_x$ ([3]).

From the leaving fourth and the sixth terms of (34) on the right side, we determine that $\breve{g}_1(h_1(\mathcal{D}_t^{\bot},\mathcal{D}_t^{\bot}),\mathcal{P}\mathcal{D}_t^{\bot}) = 0$, $\breve{g}_1(h_1(\mathcal{D}_t^{\bot},\mathcal{D}_t^{\bot}),\lambda) = 0$, we get

$$h_1(\mathcal{D}_t^{\bot},\mathcal{D}_t^{\bot})\bot\mathcal{P}\mathcal{D}_t^{\bot}, \quad h_1(\mathcal{D}_t^{\bot},\mathcal{D}_t^{\bot})\bot\lambda \Rightarrow h_1(\mathcal{D}_t^{\bot},\mathcal{D}_t^{\bot}) \in S\mathcal{D}_t^{\bot} \tag{37}$$

For a mixed geodesic, from Lemma 5(1), we derive

$$\breve{g}_1(h_1(\mathcal{Z}_a,\mathcal{W}_b),S\mathcal{X}_a) = (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_a,\mathcal{W}_b) \tag{38}$$

for any $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ and $\mathcal{Z}_a,\mathcal{W}_b \in T\mathcal{N}_b^{\bot}$.

Therefore, by the connections (37), (38) and substantially $\mathcal{N}_b^{\bot}$ is totally umbilical in $\mathcal{N}_x$ [3], we obtain that $\mathcal{N}_b^{\bot}$ is totally umbilical in $\bar{\mathcal{N}}_x$. □

Remark 1. *If $\mathcal{N}_b^{\bot}$ manifold of Theorem 7 is totally umbilical and timelike, equation (33) should be modified by*

$$||h_1||^2 \geq d_1 coth^2\theta||gradlnk||^2, \tag{39}$$

where $grad(lnk)$ is the gradient of lnk.

Theorem 8. *Let $\mathcal{N}_x = \mathcal{N}_a^{\theta}\times_k\mathcal{N}_b^{\bot}$ be a s-dimensional mixed geodesic warped product pointwise hemi-slant submanifold whose ambient space is $(2\mathtt{m}+1)$- dimensional para-cosymplectic manifold $\bar{\mathcal{N}}_x$. So that $\mathcal{N}_a^{\theta}$ is a pointwise slant submanifold and $\mathcal{N}_b^{\bot}$ is a totally real submanifold of dimension d_1 of $\bar{\mathcal{N}}_x$. Hence, $\mathcal{N}_b^{\bot}$ is spacelike and timelike. Then, (for type-2)*

$$||h_1||^2 \leq d_1cot^2\theta||gradlnk||^2(respectively, ||h_1||^2 \geq d_1cot^2\theta||gradlnk||^2), \tag{40}$$

where $grad(lnk)$ is the gradient of lnk.

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