Turk. J. Math. Comput. Sci.
16(2)(2024) 471–480
© MatDer
DOI : 10.47000/tjmcs.1507142

On Properties of q-Close-to-Convex Harmonic Functions of Order α

Serkan Çakmak

Department of Management Information Systems, Faculty of Economics, Administrative and Social Science, Istanbul Gelisim University, Istanbul, Türkiye.

Received: 29-06-2024 • Accepted: 07-11-2024

Abstract. In this paper, a novel subclass, denoted as $\mathcal{PH}(q,\alpha)$, is unveiled within the domain of harmonic functions in the open unit disk $\mathbb{E}$. This subclass, comprised of functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}^0$, is characterized by a specific inequality involving the q-derivative operator. Through meticulous analysis, it is demonstrated that functions belonging to $\mathcal{PH}(q,\alpha)$ exhibit remarkable close-to-convexity properties. Furthermore, diverse results such as distortion theorem, coefficient bounds, and a sufficient coefficient condition are yielded by the exploration. Additionally, the closure properties of $\mathcal{PH}(q,\alpha)$ under convolution operations and convex combination are elucidated, underscoring its structural coherence and relevance in the broader context of harmonic mappings.

2020 AMS Classification: 30C45, 30C50, 30C55

Keywords: q-Derivative, harmonic functions, q-close-to-convex functions, coefficient bounds, distortion.

1. Introduction

In the realm of harmonic functions, each function $\mathfrak{f}$ belonging to the class $\mathcal{SH}^0$ can be represented in the form $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where

$$\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} u_s z^s, \quad \mathfrak{v}(z) = \sum_{s=2}^{\infty} v_s z^s. \tag{1.1}$$

Here, both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic functions within the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. When the condition $|\mathfrak{u}'(z)| > |\mathfrak{v}'(z)|$ holds throughout the disk $\mathbb{E}$, it follows that the function $\mathfrak{f}$ is sense-preserving and locally univalent in this domain. Additionally, it is crucial to highlight that if $\mathfrak{v}(z)$ is identically zero, the class $\mathcal{SH}^0$ simplifies to the class $\mathcal{S}$, where $\mathcal{S}$ denotes the class of analytic, univalent, and normalized functions in the unit disk $\mathbb{E}$.

The subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of $\mathcal{S}$ are characterized by their mappings of the unit disk $\mathbb{E}$ onto close-to-convex and starlike domains, respectively. In a similar vein, the subclasses of $\mathcal{SH}^0$ that map the unit disk $\mathbb{E}$ onto corresponding domains are denoted by $\mathcal{SH}^{0,*}$ and $\mathcal{KH}^0$. For a more comprehensive understanding, one can refer to the detailed discussions in [7, 11].

Jackson's q-derivative for a function $\mathfrak{u} \in \mathcal{S}$, where $0 < q < 1$, is defined as follows [10]:

$$D_q\mathfrak{u}(z) = \begin{cases} \frac{\mathfrak{u}(z)-\mathfrak{u}(qz)}{(1-q)z}, & \text{if } z \neq 0, \\ \mathfrak{u}'(z), & \text{if } z = 0. \end{cases}$$

Email address: secakmak@gelisim.edu.tr (S. Çakmak)

It is important to note that if the function $\mathfrak{u}$ is differentiable at z in the classical sense, then the limit as q approaches 1 from the left satisfies $\lim_{q\to 1^-} D_q\mathfrak{u}(z) = \mathfrak{u}'(z)$. In the context of equation (1.1), this q-derivative can be expressed as:

$$D_q\mathfrak{u}(z) = 1 + \sum_{s=2}^{\infty} [s]_q a_s z^{s-1},$$

where the q-number $[s]_q$ is given by $[s]_q = \frac{1-q^s}{1-q}$ for any positive integer s. As q tends to 1 from the left, it is evident that $[s]_q \to s$.

Furthermore, Jackson defined the q-integral as follows [10]:

$$\int_0^z \mathfrak{u}(\zeta)\, d_q\zeta = z(1-q)\sum_{s=0}^{\infty} q^s \mathfrak{u}(zq^s),$$

provided that the series on the right-hand side converges. This definition plays a significant role in the study of q-calculus, particularly in examining the properties of functions within the class $\mathcal{S}$ under q-differentiation and q-integration.

A function $\mathfrak{u} \in \mathcal{S}$ is termed q-starlike in the open unit disk $\mathbb{E}$, and this is denoted by $\mathfrak{u} \in \mathcal{S}^*(q)$, if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zD_q\mathfrak{u}(z)}{\mathfrak{u}(z)}\right\} > \alpha \quad (0 \le \alpha < 1),$$

where D_q represents the Jackson q-derivative. Similarly, a function $\mathfrak{u} \in \mathcal{S}$ is called q-convex in $\mathbb{E}$, denoted by $\mathfrak{u} \in C(q)$, if it fulfills the condition:

$$\operatorname{Re}\left\{\frac{D_q(zD_q\mathfrak{u}(z))}{D_q\mathfrak{u}(z)}\right\} > \alpha \quad (0 \le \alpha < 1).$$

Moreover, if there exists a q-convex function ϕ in $\mathbb{E}$, and if $\mathfrak{u} \in \mathcal{K}(q)$, where $\mathcal{K}(q)$ represents a class of q-close-to-convex functions, such that

$$\operatorname{Re}\left\{\frac{D_q\mathfrak{u}(z)}{D_q\phi(z)}\right\} > \alpha \quad (0 \le \alpha < 1),$$

then, in the particular case where $\phi(z) = z$, it implies that

$$\operatorname{Re}\{D_q\mathfrak{u}(z)\} > \alpha \quad (0 \le \alpha < 1).$$

This definition extends the classical notions of convex and starlike functions to the setting of q-calculus, providing a framework for analyzing these properties in the context of Jackson's q-derivative.

In 2019, Ahuja and Çetinkaya [1] introduced the class of q-harmonic, univalent, and sense-preserving functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, denoted by $\mathcal{SH}_q^0$. A necessary and sufficient condition for a function $\mathfrak{f}$ to belong to the class $\mathcal{SH}_q^0$ is that $\left|\omega_q(z)\right| = \left|\frac{D_q\mathfrak{v}(z)}{D_q\mathfrak{u}(z)}\right| < 1$. Additionally, as $q \to 1^-$, the class $\mathcal{SH}^0$ is recovered. For further details, see [2, 3, 17].

Let us now introduce a novel class of functions utilizing the Jackson q-derivative.

Definition 1.1. The class $\mathcal{PH}(q,\alpha)$ is defined as the set of functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q^0$ that satisfy the inequality:

$$\operatorname{Re}\left\{D_q\mathfrak{u}(z) - \alpha\right\} > \left|D_q\mathfrak{v}(z)\right| \tag{1.2}$$

for $0 \le \alpha < 1$ and $z \in \mathbb{E}$.

If q, 1 is approximated from the left in inequality (1.2), the $P_H^0(\alpha)$ class studied by Li and Ponnusamy [13, 14] is obtained. Also, for $\alpha = 0$, the P_H^0 class studied by Ponnussamy et al. [12] is obtained. For more detailed information on classes of close-to-convex harmonic functions, see references [4–6] .

Definition 1.2. The class $\mathcal{P}(q,\alpha)$ consists of functions $\mathfrak{u} \in \mathcal{S}$ that satisfy the inequality:

$$\operatorname{Re}\left\{D_q\mathfrak{u}(z)\right\} > \alpha$$

for $0 \le \alpha < 1$ and $z \in \mathbb{E}$.

2. Geometric Properties of the Class $\mathcal{PH}(q,\alpha)$.

In this section, we investigate the geometric properties of the class $\mathcal{PH}(q,\alpha)$. We establish key results including coefficient bounds, q-close-to-convexity, and distortion theorem for functions in this class. Specifically, we prove the q-close-to-convexity of harmonic mappings, derive coefficient bounds, and provide sufficient conditions for membership in $\mathcal{PH}(q,\alpha)$.

A mapping $f:\mathbb{E}\to\mathbb{C}$ is termed close-to-convex if it is univalent within the unit disk $\mathbb{E}$ and its image $f(\mathbb{E})$ constitutes a close-to-convex domain. In particular, a function f is referred to as a q-close-to-convex harmonic function if it is close-to-convex and also q-harmonic. One approach to establishing the close-to-convexity of harmonic mappings involves examining their relationship with analytic functions. In this context, Clunie and Sheil-Small provided a useful criterion in their work [7].

Lemma 2.1 ([7]). *Let $\mathfrak{u}$ and $\mathfrak{v}$ be analytic functions in $\mathbb{E}$, and suppose that $|\mathfrak{v}'(0)|<|\mathfrak{u}'(0)|$. If, for each ε with $|\varepsilon|=1$, the function $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ is close-to-convex, then $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is also close-to-convex in $\mathbb{E}$.*

The next result gives a relation between analytic functions of class $\mathcal{P}(q,\alpha)$ and functions of class $\mathcal{PH}(q,\alpha)$.

Theorem 2.2. *The harmonic mapping $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ belongs to the class $\mathcal{PH}(q,\alpha)$ if and only if for each ε with $|\varepsilon|=1$, the function $\mathfrak{F}_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ is in the class $\mathcal{P}(q,\alpha)$.*

Proof. Suppose $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is a member of the class $\mathcal{PH}(q,\alpha)$. For any complex number ε with $|\varepsilon|=1$, let us define $\mathfrak{F}_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$. Then we have:

$$\begin{aligned}\operatorname{Re}\left\{D_q\mathfrak{F}_\varepsilon(z)\right\}&=\operatorname{Re}\left\{D_q(\mathfrak{u}(z)+\varepsilon\mathfrak{v}(z))\right\}\\&=\operatorname{Re}\left\{D_q\mathfrak{u}(z)+\varepsilon D_q\mathfrak{v}(z)\right\}\\&=\operatorname{Re}\left\{D_q\mathfrak{u}(z)\right\}+\operatorname{Re}\left\{\varepsilon D_q\mathfrak{v}(z)\right\}\\&\geq\operatorname{Re}\left\{D_q\mathfrak{u}(z)\right\}-\left|\varepsilon D_q\mathfrak{v}(z)\right|\\&=\operatorname{Re}\left\{D_q\mathfrak{u}(z)\right\}-\left|D_q\mathfrak{v}(z)\right|\\&\geq\alpha.\end{aligned}$$

Therefore, $\mathfrak{F}_\varepsilon$ is in the class $\mathcal{P}(q,\alpha)$. Conversely, assume $\mathfrak{F}_\varepsilon$ is in the class $\mathcal{P}(q,\alpha)$. Then,

$$\operatorname{Re}\left\{D_q\mathfrak{u}(z)\right\}>\operatorname{Re}\left\{-\varepsilon D_q\mathfrak{v}(z)\right\}+\alpha\quad\text{for}\quad z\in\mathbb{E}.$$

By choosing ε appropriately such that $|\varepsilon|=1$, we get,

$$\operatorname{Re}\left\{D_q\mathfrak{u}(z)-\alpha\right\}>\left|D_q\mathfrak{v}(z)\right|\quad\text{for}\quad z\in\mathbb{E}.$$

Hence, $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is in the class $\mathcal{PH}(q,\alpha)$. □

Theorem 2.3. *Any function in the class $\mathcal{PH}(q,\alpha)$ is q-close-to-convex in the domain $\mathbb{E}$.*

Proof. Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ be a function belonging to the class $\mathcal{PH}(q,\alpha)$. By definition, $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is q-harmonic, univalent, and sense-preserving. Since $\mathfrak{f}\in\mathcal{PH}(q,\alpha)$, Theorem 2.2 ensures that $\mathfrak{F}_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ belongs to $\mathcal{P}(q,\alpha)$. This membership implies that $\mathfrak{F}_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ is q-close-to-convex. As q approaches 1 from the left, $\mathfrak{F}_\varepsilon$ becomes close-to-convex in the classical sense. Therefore, by Lemma 2.1, we conclude that $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is close-to-convex. Thus, it follows that $\mathfrak{f}$ is indeed q-close-to-convex in $\mathbb{E}$. □

Theorem 2.4. *Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ belong to the class $\mathcal{PH}(q,\alpha)$. For $s\geq 2$, the following inequality holds:*

$$|v_s|\leq\frac{1-\alpha}{[s]_q}.$$

This bound is sharp, and equality is achieved for the function $\mathfrak{f}(z)=z+\frac{1-\alpha}{[s]_q}\overline{z}^s$.

Proof. Assuming $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{PH}(q, \alpha)$, with the function $\mathfrak{v}(re^{i\theta})$ represented as a series, where $0 \leq \rho < 1$ and $\theta \in \mathbb{R}$, we have the following inequalities:

$$\begin{aligned}\rho^{s-1}[s]_q |v_s| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|D_q\mathfrak{v}(\rho e^{i\theta})\right| d\theta \\ &< \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\{D_q\mathfrak{u}(\rho e^{i\theta}) - \alpha\} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[1 - \alpha + \sum_{s=2}^{\infty}[s]_q u_s \rho^{s-1} e^{i(s-1)\theta}\right] d\theta = 1 - \alpha.\end{aligned}$$

As the parameter ρ approaches 1 from the left side, we achieve the desired bound. □

Theorem 2.5. *Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{PH}(q, \alpha)$. Then, for $s \geq 2$, the following inequalities hold:*

i. $|u_s| + |v_s| \leq \frac{2(1-\alpha)}{[s]_q}$,
ii. $||u_s| - |v_s|| \leq \frac{2(1-\alpha)}{[s]_q}$,
iii. $|u_s| \leq \frac{2(1-\alpha)}{[s]_q}$.

Equality is achieved by the function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2(1-\alpha)}{[s]_q} z^s$.

Proof. (i) Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ belong to the class $\mathcal{PH}(q, \alpha)$. From Theorem 2.2, it follows that $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ belongs to the class $\mathcal{P}(q, \alpha)$ for every ε where $|\varepsilon| = 1$. Consequently, we obtain

$$\mathrm{Re}\{D_q(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z))\} > \alpha$$

for all $z \in \mathbb{E}$. Thus, there exists an analytic function Φ with a positive real part in the unit disk $\mathbb{E}$, which can be expressed in the form $\Phi(z) = 1 + \sum_{s=1}^{\infty} \phi_s z^s$. This implies that

$$D_q(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z)) = \alpha + (1 - \alpha)\Phi(z).$$

By comparing the coefficients of both sides, we derive

$$[s]_q(u_s + \varepsilon v_s) = (1 - \alpha)\phi_{s-1} \quad \text{for } s \geq 2.$$

Given that the real part of $\Phi(z)$ is positive, we have $|\phi_s| \leq 2$ for $s \geq 1$. Furthermore, considering that ε with $|\varepsilon| = 1$ is arbitrary, we can rigorously establish that inequality (i) holds true. The proofs for statements (ii) and (iii) can be derived following similar lines of reasoning and logic. The function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2(1-\alpha)}{[s]_q} z^s$ serves as an example to demonstrate that all the stated inequalities are indeed sharp.

For $s \geq 1$, we obtain $|\phi_s| \leq 2$ since the real component of $\Phi(z)$ is positive. In addition, we can firmly prove that inequality (i) holds since ε with $|\varepsilon| = 1$ is arbitrary. Similar lines of reasoning and logic can be used to derive the proofs for propositions (ii) and (iii). To show that all of the given inequalities are, in fact, sharp, consider the function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2(1-\alpha)}{[s]_q} z^s$. □

We now give a sufficient condition for a function to belong to the class $\mathcal{PH}(q, \alpha)$.

Theorem 2.6. *Suppose $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ belongs to the class $\mathcal{SH}_q^0$, where*

$$\sum_{s=2}^{\infty}[s]_q (|u_s| + |v_s|) \leq 1 - \alpha. \tag{2.1}$$

In that case, $\mathfrak{f}$ belongs to the class $\mathcal{PH}(q, \alpha)$.

Proof. Considering that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is a member of the class $\mathcal{SH}_q^0$. We may write the following set of inequalities using the provided condition (2.1):

$$\mathrm{Re}\{D_q\mathfrak{u}(z) - \alpha\} = \mathrm{Re}\left[1 - \alpha + \sum_{s=2}^{\infty}[s]_q u_s z^{s-1}\right] > 1 - \alpha - \sum_{s=2}^{\infty}[s]_q |u_s| \geq \sum_{s=2}^{\infty}[s]_q |v_s| > \left|\sum_{s=2}^{\infty}[s]_q v_s z^{s-1}\right| = \left|D_q\mathfrak{v}(z)\right|.$$

Consequently, it may be concluded that $\mathfrak{f}$ belongs to the class $\mathcal{PH}(q, \alpha)$. □

Theorem 2.7. *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{PH}(q, \alpha)$. *Then,*

$$|z| + 2\sum_{s=2}^{\infty} \frac{(-1)^{s-1} |z|^s}{[s]_q} \leq |\mathfrak{f}(z)| \leq |z| + 2\sum_{s=2}^{\infty} \frac{|z|^s}{[s]_q}.$$

Equality is achieved by the function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2}{[s]_q} z^s$.

Proof. Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{PH}(q, \alpha)$. Using Theorem 2.2, we have $\mathfrak{F}_\varepsilon = \mathfrak{u} + \varepsilon\, \mathfrak{v} \in \mathcal{P}(q, \alpha)$ for each ε ($|\varepsilon| = 1$). By employing a similar method as in [9], we obtain

$$\frac{1 - (1 - 2\alpha)|z|}{1 + |z|} \leq \left|D_q \mathfrak{F}_\varepsilon(z)\right| \leq \frac{1 + (1 - 2\alpha)|z|}{1 - |z|} \quad (|z| < 1). \tag{2.2}$$

The expression on the right side of inequality (2.2) can be derived using the Taylor series expansion, denoted as

$$\begin{aligned} \left|D_q \mathfrak{F}_\varepsilon(z)\right| &= \left|D_q \mathfrak{u}(z) + \varepsilon D_q \mathfrak{v}(z)\right| \\ &\leq 1 + 2(1 - \alpha) \sum_{s=1}^{\infty} |z|^s. \end{aligned}$$

Similarly, the expression on the left side of inequality (2.2) can be obtained through the Taylor series expansion, denoted as

$$\begin{aligned} \left|D_q \mathfrak{F}_\varepsilon(z)\right| &= \left|D_q \mathfrak{u}(z) + \varepsilon D_q \mathfrak{v}(z)\right| \\ &\leq 1 + 2(1 - \alpha) \sum_{s=1}^{\infty} (-1)^s |z|^s. \end{aligned}$$

Specifically, we obtain the following inequalities:

$$\left|D_q \mathfrak{u}(z)\right| + \left|D_q \mathfrak{v}(z)\right| \leq 1 + 2(1 - \alpha) \sum_{s=1}^{\infty} |z|^s,$$

and

$$\left|D_q \mathfrak{u}(z)\right| - \left|D_q \mathfrak{v}(z)\right| \leq 1 + 2(1 - \alpha) \sum_{s=1}^{\infty} (-1)^s |z|^s.$$

Denote the radial segment from 0 to z by Γ. We get,

$$\begin{aligned} |\mathfrak{f}(z)| &\leq \int_{\Gamma} \left(\left|D_q \mathfrak{u}(\zeta)\right| + \left|D_q \mathfrak{v}(\zeta)\right|\right) \left|d_q \zeta\right| \\ &\leq \int_0^{|z|} \left(1 + 2(1 - \alpha) \sum_{s=1}^{\infty} |\rho|^s\right) d_q \rho \\ &= |z| + 2(1 - \alpha) \sum_{s=1}^{\infty} \frac{|z|^{s+1}}{[s+1]_q} \\ &= |z| + 2(1 - \alpha) \sum_{s=2}^{\infty} \frac{|z|^s}{[s]_q}, \end{aligned}$$

and

$$\begin{aligned}
|\mathfrak{f}(z)| &\leq \int_{\Gamma} \left(\left|D_q\mathfrak{u}(\zeta)\right| - \left|D_q\mathfrak{v}(\zeta)\right|\right)\left|d_q\zeta\right| \\
&\leq \int_0^{|z|} \left(1 + 2(1-\alpha)\sum_{s=1}^{\infty}(-1)^s|\rho|^s\right) d_q\rho \\
&= |z| + 2(1-\alpha)\sum_{s=1}^{\infty}\frac{(-1)^s|z|^{s+1}}{[s+1]_q} \\
&= |z| + 2(1-\alpha)\sum_{s=2}^{\infty}\frac{(-1)^{s-1}|z|^s}{[s]_q}.
\end{aligned}$$

□

3. Exploring Closure Properties of the Class $\mathcal{PH}(q,\alpha)$

In this section, we investigate whether the class $\mathcal{PH}(q,\alpha)$ is closed under convex combinations and convolutions.

Theorem 3.1. *The class $\mathcal{PH}(q,\alpha)$ is closed under convex combinations.*

Proof. Let us consider that $\mathfrak{f}_k = \mathfrak{u}_k + \overline{\mathfrak{v}_k}$ belongs to the class $\mathcal{PH}(q,\alpha)$ for each $k = 1,2,...,n$, and suppose further that the sum $\sum_{k=1}^{n}\varphi_k = 1$ with each coefficient $0 \leq \varphi_k \leq 1$. The convex combination of the functions $\mathfrak{f}_k$ for $k = 1,2,...,n$ can be expressed as:

$$\mathfrak{f}(z) = \sum_{k=1}^{n}\varphi_k\mathfrak{f}_k(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)},$$

where

$$\mathfrak{u}(z) = \sum_{k=1}^{n}\varphi_k\mathfrak{u}_k(z) \text{ and } \mathfrak{v}(z) = \sum_{k=1}^{n}\varphi_k\mathfrak{v}_k(z).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic functions within the open unit disk $\mathbb{E}$, satisfying the initial conditions $\mathfrak{u}(0) = \mathfrak{v}(0) = D_q\mathfrak{u}(0) - 1 = D_q\mathfrak{v}(0) = 0$. Now, we consider the real part of the q-derivative of $\mathfrak{u}$:

$$\begin{aligned}
\operatorname{Re}\{D_q\mathfrak{u}(z) - \alpha\} &= \operatorname{Re}\left[\sum_{k=1}^{n}\varphi_k D_q\mathfrak{u}_k(z) - \alpha\right] \\
&> \sum_{k=1}^{n}\varphi_k\left|D_q\mathfrak{v}_k(z)\right| \\
&\geq \left|D_q\mathfrak{v}(z)\right|,
\end{aligned}$$

showing that $\mathfrak{f}$ belongs to the class $\mathcal{PH}(q,\alpha)$. □

If a sequence $\{a_s\}_{s=0}^{\infty}$ of non-negative real numbers satisfies the following criteria, it is termed a "convex null sequence": as $s \to \infty$, a_s approaches 0, and the inequality

$$a_0 - a_1 \geq a_1 - a_2 \geq a_2 - a_3 \geq \ldots \geq a_{s-1} - a_s \geq \ldots \geq 0$$

holds.

Lemma 3.2 ([8]). *When $\{a_s\}_{s=0}^{\infty}$ is a convex null sequence, then the function*

$$A(z) = \frac{a_0}{2} + \sum_{s=1}^{\infty}a_s z^s$$

is analytic, and the real part of $A(z)$ is positive within the open unit disk $\mathbb{E}$.

The Hadamard product (or convolution) of two harmonic functions is defined similarly to that of analytic functions, with the product applied separately to both the analytic and co-analytic parts of the functions. Let $\mathfrak{f}(z) = \mathfrak{u}_1(z) + \overline{\mathfrak{v}_1(z)}$ and $\mathfrak{k}(z) = \mathfrak{u}_2(z)+\overline{\mathfrak{v}_2(z)}$ be two harmonic functions, where $\mathfrak{u}_1(z) = \sum_{s=0}^{\infty} u_s z^s$ and $\mathfrak{u}_2(z) = \sum_{s=0}^{\infty} c_s z^s$ are analytic functions, and $\mathfrak{v}_1(z) = \sum_{s=0}^{\infty} v_s z^s$ and $\mathfrak{v}_2(z) = \sum_{s=0}^{\infty} d_s z^s$ are co-analytic functions. Then, the Hadamard product of $\mathfrak{f}$ and $\mathfrak{k}$, denoted by $(\mathfrak{f} * \mathfrak{k})(z)$, is defined as

$$(\mathfrak{f} * \mathfrak{k})(z) = (\mathfrak{u}_1 * \mathfrak{u}_2)(z) + \overline{(\mathfrak{v}_1 * \mathfrak{v}_2)(z)},$$

where

$$(\mathfrak{u}_1 * \mathfrak{u}_2)(z) = \sum_{s=0}^{\infty} u_s c_s z^s \quad \text{and} \quad (\mathfrak{v}_1 * \mathfrak{v}_2)(z) = \sum_{s=0}^{\infty} v_s d_s z^s.$$

Thus, the Hadamard product is obtained by multiplying the coefficients of the analytic and co-analytic parts of the harmonic functions separately.

Lemma 3.3 ([16]). *Suppose the function $\Phi(z)$ is analytic within the domain $\mathbb{E}$, satisfying $\Phi(0) = 1$ and $Re\{\Phi(z)\} > 1/2$ throughout $\mathbb{E}$. For any analytic function F defined in $\mathbb{E}$, the function $\Phi * F$ maps to values within the convex hull of the image of $\mathbb{E}$ under F.*

Lemma 3.4. *Let $F \in \mathcal{P}(q, \alpha)$, then $Re\left[\frac{F(z)}{z}\right] > \frac{1}{2}$.*

Proof. Consider F belonging to the class $\mathcal{P}(q, \alpha)$, defined as $F(z) = z + \sum_{s=2}^{\infty} U_s z^s$. Then, the inequality

$$\text{Re}\left[1 + \sum_{s=2}^{\infty} [s]_q U_s z^{s-1}\right] > \alpha \quad (z \in \mathbb{E}),$$

can be equivalently expressed as $\text{Re}\{\Phi(z)\} > \frac{1}{2}$ within the open unit disk $\mathbb{E}$, where

$$\Phi(z) = 1 + \frac{1}{2(1-\alpha)} \sum_{s=2}^{\infty} [s]_q U_s z^{s-1}.$$

Let $\{a_s\}_{s=0}^{\infty}$ be a sequence defined by

$$a_0 = 1 \text{ and } a_{s-1} = \frac{2(1-\alpha)}{[s]_q} \text{ for } s \geq 2.$$

It is clear that the sequence $\{a_s\}_{s=0}^{\infty}$ constitutes a convex null sequence. Utilizing Lemma 3.2, we deduce that the function

$$A(z) = \frac{1}{2} + \sum_{s=2}^{\infty} \frac{2(1-\alpha)}{[s]_q} z^{s-1}$$

is analytic, with $\text{Re}\{A(z)\} > 0$ within $\mathbb{E}$. Expressing

$$\frac{F(z)}{z} = \Phi(z) * \left(1 + \sum_{s=2}^{\infty} \frac{2(1-\alpha)}{[s]_q} z^{s-1}\right),$$

and using Lemma 3.3, we arrive at the conclusion that $\text{Re}\left\{\frac{F(z)}{z}\right\} > \frac{1}{2}$ for $z \in \mathbb{E}$. □

Lemma 3.5. *Let $\mathfrak{u}_m \in \mathcal{P}(q, \alpha)$ for $m = 1, 2$. Then, $\mathfrak{u}_1 * \mathfrak{u}_2 \in \mathcal{P}(q, \alpha)$.*

Proof. Let $\mathfrak{u}_1(z) = z + \sum_{s=2}^{\infty} \mathfrak{U}_s z^s$ and $\mathfrak{u}_2(z) = z + \sum_{s=2}^{\infty} \mathfrak{V}_s z^s$, and

$$U(z) = (\mathfrak{u}_1 * \mathfrak{u}_2)(z) = z + \sum_{s=2}^{\infty} \mathfrak{U}_s \mathfrak{V}_s z^s.$$

Considering

$$\frac{D_q U(z) - \alpha}{1-\alpha} = \frac{D_q \mathfrak{u}_1(z) - \alpha}{1-\alpha} * \frac{\mathfrak{u}_2(z)}{z}. \tag{3.1}$$

Since $\mathfrak{u}_1 \in \mathcal{P}(q,\alpha)$, we get $\mathrm{Re}\left[\frac{D_q\mathfrak{u}_1(z)-\alpha}{1-\alpha}\right] > 0$ $(z \in \mathbb{E})$. Moreover, using Lemma 3.4, $\mathrm{Re}\left[\frac{\mathfrak{u}_2(z)}{z}\right] > \frac{1}{2}$ in $\mathbb{E}$. Now, $\mathrm{Re}\left[D_q U(z)\right] > 0$ in $\mathbb{E}$ results from applying Lemma 3.4 to (3.1). Thus, $U = \mathfrak{u}_1 * \mathfrak{u}_2 \in \mathcal{P}(q,\alpha)$. □

Using Lemma 3.5, we now show that the class $\mathcal{PH}(q)$ is closed under convolutions of its members.

Theorem 3.6. *For $m = 1, 2$, let $\mathfrak{f}_m \in \mathcal{PH}(q,\alpha)$. Then, the convolution $\mathfrak{f}_1 * \mathfrak{f}_2$ also belongs to the class $\mathcal{PH}(q,\alpha)$.*

Proof. Assume that $\mathfrak{f}_m = \mathfrak{u}_m + \overline{\mathfrak{v}_m} \in \mathcal{PH}(q,\alpha)$ for $m = 1, 2$. The convolution $\mathfrak{f}_1 * \mathfrak{f}_2 = \mathfrak{u}_1 * \mathfrak{u}_2 + \overline{\mathfrak{v}_1 * \mathfrak{v}_2}$ is defined as the convolution of the individual components of $\mathfrak{f}_1$ and $\mathfrak{f}_2$. To demonstrate that $\mathfrak{f}_1 * \mathfrak{f}_2$ belongs to the class $\mathcal{PH}(q,\alpha)$, we need to show that the function $F_\varepsilon = \mathfrak{u}_1 * \mathfrak{u}_2 + \varepsilon(\mathfrak{v}_1 * \mathfrak{v}_2)$ belongs to $\mathcal{P}(q,\alpha)$ for every ε with $|\varepsilon| = 1$.

According to Lemma 3.5, the class $\mathcal{P}(q,\alpha)$ is closed under convolutions. For each ε with $|\varepsilon| = 1$, since $\mathfrak{u}_m + \varepsilon\mathfrak{v}_m \in \mathcal{P}(q,\alpha)$ for $m = 1, 2$, we can assert that $\mathcal{P}(q,\alpha)$ includes both U_1 and U_2, where

$$U_1 = (\mathfrak{u}_1 - \mathfrak{v}_1) * (\mathfrak{u}_2 - \varepsilon\mathfrak{v}_2) \quad \text{and} \quad U_2 = (\mathfrak{u}_1 + \mathfrak{v}_1) * (\mathfrak{u}_2 + \varepsilon\mathfrak{v}_2).$$

Given that $\mathcal{P}(q,\alpha)$ is closed under convex combinations, we can form the function

$$U_\varepsilon = \frac{1}{2}(U_1 + U_2) = \mathfrak{u}_1 * \mathfrak{u}_2 + \varepsilon(\mathfrak{v}_1 * \mathfrak{v}_2).$$

This function U_ε also belongs to the class $\mathcal{P}(q,\alpha)$. Therefore, we can conclude that the class $\mathcal{PH}(q,\alpha)$ is closed under convolution operations, meaning that the convolution of any two functions within $\mathcal{PH}(q,\alpha)$ also remains within $\mathcal{PH}(q,\alpha)$. □

4. Examples of Functions in the Class $\mathcal{PH}(q,\alpha)$

Example 4.1. Let $\mathfrak{f}_1(z) = z + \frac{1}{13}z^3 + \frac{3}{13}\overline{z}^3$. According to Theorem 2.5, the function $\mathfrak{f}_1(z)$ belongs to the class $\mathcal{PH}(\frac{1}{3}, \frac{1}{3})$. This function maps the unit disk to a starlike region, thereby making it close-to-convex. Figure 1 illustrates the image of concentric circles inside the unit disk $\mathbb{E}$ under the transformation defined by $\mathfrak{f}_1(z) = z + \frac{1}{13}z^3 + \frac{3}{13}\overline{z}^3$.

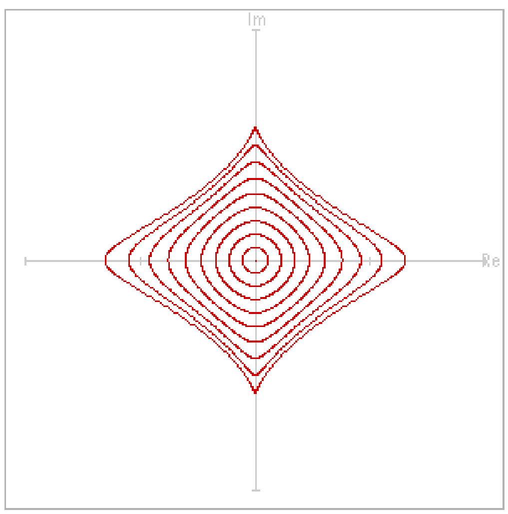

Figure 1. Under the map $\mathfrak{f}_1(z) = z + \frac{1}{13}z^3 + \frac{3}{13}\overline{z}^3$, the image of concentric circles inside the unit disk.

Example 4.2. Let $\mathfrak{f}_2(z) = z + \frac{1}{12}z^2 - \frac{3}{13}\bar{z}^3$. According to Theorem 2.5, the function $\mathfrak{f}_2(z)$ belongs to the class $\mathcal{PH}(\frac{1}{3}, \frac{1}{3})$. This function maps the unit disk to a close-to-convex region. Figure 2 illustrates the image of concentric circles inside the unit disk $\mathbb{E}$ under the transformation defined by $\mathfrak{f}_2(z) = z + \frac{1}{12}z^2 - \frac{3}{13}\bar{z}^3$.

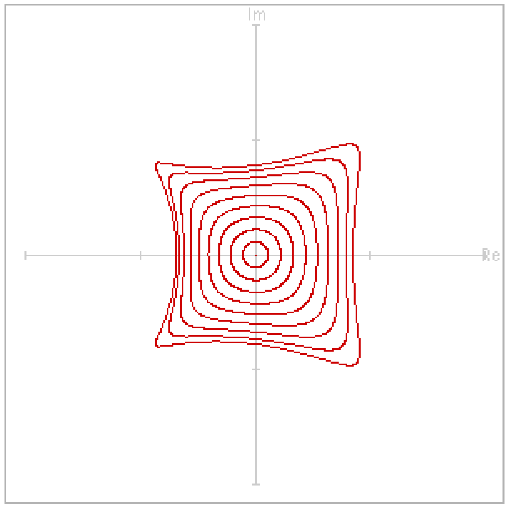

FIGURE 2. Under the map $\mathfrak{f}_2(z) = z + \frac{1}{12}z^2 - \frac{3}{13}\bar{z}^3$, the image of concentric circles inside the unit disk.

Example 4.3. Let $\mathfrak{f}_3(z) = \mathfrak{f}_1(z) * \mathfrak{f}_2(z) = z - \frac{9}{169}\bar{z}^3$, where $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are the functions given in Example 4.1 and Example 4.2. According to Theorem 2.5, the function $\mathfrak{f}_3(z)$ belongs to the class $\mathcal{PH}(\frac{1}{3}, \frac{1}{3})$. This function maps the unit disk to a close-to-convex region. Figure 3 illustrates the image of concentric circles inside the unit disk $\mathbb{E}$ under the transformation defined by $\mathfrak{f}_3(z) = z - \frac{9}{169}\bar{z}^3$.

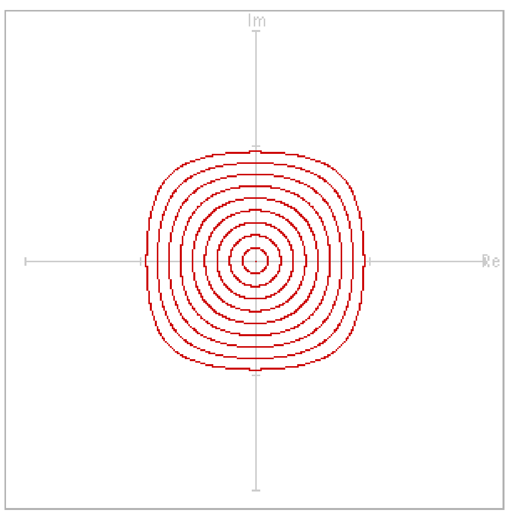

FIGURE 3. Under the map $\mathfrak{f}_3(z) = z - \frac{9}{169}\bar{z}^3$, the image of concentric circles inside the unit disk.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the article.

References

[1] Ahuja, O.P., Çetinkaya, A., *Use of Qquantum calculus approach in mathematical sciences and its role in geometric function theory*, AIP Conference Proceedings, **2095**(2019), 020001-14.

[2] Ahuja, O.P., Çetinkaya, A., *Connecting quantum calculus and harmonic starlike functions*, Filomat, **34**(5)(2020), 1431–1441.

[3] Ahuja, O.P., Çetinkaya, A., Polatoglu, Y., *Harmonic univalent convex functions using a quantum calculus approach*, Acta Universitatis Apulensis, **58**(2019), 67–81.

[4] Çakmak, S., Yaşar, E., Yalçın, S., *New subclass of the class of close-to-convex harmonic mappings defined by a third-order differential inequality*, Hacettepe Journal of Mathematics and Statistics, **51**(1)(2022), 172–186.

[5] Çakmak, S., *Regarding a novel subclass of harmonic multivalent functions defined by higher-order differential inequality*, Iranian Journal of Science, **48**(2024), 1541–1550.

[6] Cakmak, S., Yasar, E., Yalçın Tokgöz, S., *Some basic properties of a subclass of close-to-convex harmonic mappings*, Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics, **15**(2)(2024), 163–173.

[7] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I., **9**(1984), 3–25.

[8] Fejér, L., *Über die Positivität von Summen, die nach trigonometrischen oder Legendreschen Funktionen fortschreiten*, Acta Litt. Ac Sei. Szeged, (1925), 75–86.

[9] Goodman, A.W., Univalent Functions, vol. I, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[10] Jackson, F.H., *On q-functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, **46**(2)(1909), 253–281.

[11] Owa, S., Nunokawa, M., Saitoh, H., Srivastava, H.M., *Close-to-convexity, starlikeness, and convexity of certain analytic functions*, Appl. Math. Lett., **15**(1)(2002), 63–69.

[12] Ponnusamy, S., Yamamoto, H., Yanagihara, H., *Variability regions for certain families of harmonic univalent mappings*, Complex Var. Elliptic Equ., **58**(1)(2013), 23–34.

[13] Li, L., Ponnusamy, S., *Disk of convexity of sections of univalent harmonic functions*, J. Math. Anal. Appl., **408**(2013), 589–596.

[14] Li, L., Ponnusamy, S., *Injectivity of sections of univalent harmonic mappings*, Nonlinear Anal., **89**(2013), 276–283.

[15] Salagean, G.S., *Subclass of univalent functions*, in Complex Analysis-Fifth Romanian Finish Seminar, (1983), 362–372.

[16] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106**(1989), 145–152.

[17] Yalçın, S., Bayram, H., *On harmonic univalent functions involving q-poisson distribution series*, MJPS, **8**(2)(2021).