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Research Article

The Jinc- function: a note on the relevant generalizations and applications

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. Jinc and sinc-functions are well known special functions with important applications in Spectral theory, Fourier Optics and diffraction problems from circular apertures. The first are less widely known than the latter and should be more properly framed within the context of special functions. In this article, we present a unified point of view to the relevant generalizations, propose generalized forms and touch on application perspectives.

Keywords: Jinc-function, Fourier-optics, Quantum-diffraction.

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1. INTRODUCTION

The Jinc-function naturally emerges in problems such as the diffraction of spherical/plane waves converging on circular apertures [2, 5].
Its definition involves the use of the cylindrical Bessel of first order, according to the identity [1]:

$$\begin{aligned} &Jinc(x) = \frac{J_1(x)}{\left(\dfrac{x}{2}\right)} \\ &J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{\Gamma(r+\alpha+1)\, r!} \equiv \alpha\, th - \text{order first kind cylindrical bessel.} \end{aligned} \tag{1.1}$$

A current definition is $Jinc(x) = J_1(x)/x$, but the adoption of the form in equation 1.1 is more convenient for the generalizations considered in this article. In the case of the Fraunhofer diffraction by circular apertures, the scattered Airy pattern distribution exhibits a Jinc-like behavior, with respect to the radial coordinate, namely [9, 3]:

$$F(x,y) = Jinc\left(\sqrt{x^2+y^2}\right). \tag{1.2}$$

In order to fix the terms of the forthcoming discussion, we start from equation (1.1) rewritten as:

$$Jinc(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\dfrac{x}{2}\right)^{2r}}{(1+r)!r!}. \tag{1.3}$$

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The umbral image [15] of (1.3) is a simple Gaussian and can be written as:

$$Jinc(x) = \hat{c}e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0. \tag{1.4}$$

The operator $\hat{c}$ is the umbral operator, satisfying the property:

$$\hat{c}^\alpha \hat{c}^\beta = \hat{c}^{\alpha+\beta} \tag{1.5}$$

and characterized by the following action on the umbral vacuum φ_0:

$$\hat{c}^\alpha \varphi_0 = \frac{1}{\Gamma(\alpha+1)}. \tag{1.6}$$

Since we have reduced the Jinc-function to a Gaussian, namely an elementary transcendent function, we can evaluate the relevant integral using straightforward methods. Therefore, writing

$$\int_{-\infty}^{\infty} Jinc(x)dx = \hat{c}\int_{-\infty}^{+\infty} e^{-\hat{c}\left(\frac{x}{2}\right)^2} dx\varphi_0 \tag{1.7}$$

and, treating the umbral operator $\hat{c}$ as an ordinary constant, we find:

$$\hat{c}\int_{-\infty}^{+\infty} e^{-\hat{c}\left(\frac{x}{2}\right)^2} dx\varphi_0 = 2\hat{c}\sqrt{\frac{\pi}{\hat{c}}}\varphi_0 = 2\sqrt{\pi}\hat{c}^{\frac{1}{2}}\varphi_0. \tag{1.8}$$

The use of equations (1.5) and (1.5) yields:

$$2\sqrt{\pi}\hat{c}^{\frac{1}{2}}\varphi_0 = 2\sqrt{\pi}\frac{1}{\Gamma\left(\frac{3}{2}\right)} \tag{1.9}$$

and, therefore, we eventually find:

$$\int_{-\infty}^{\infty} Jinc(x)\,dx = 4. \tag{1.10}$$

The function in equation (1.2) realizes what is known as ***sombrero*** ($somb(x)$) function and, from the mathematical point of view, is the Fourier transform of the 2-d circle function [10, 6].
Although this aspect of the problem will be touched on in the final section, here we like to mention another important property.
Thus, we consider the integral:

$$L(x) = \int_{-\infty}^{+\infty} Jinc\left(\sqrt{x^2+y^2}\right)\,dy. \tag{1.11}$$

The use of the previously outlined umbral technique yields:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} Jinc\left(\sqrt{x^2+y^2}\right)\,dy = \hat{c}e^{-\hat{c}\left(\frac{x}{2}\right)^2}\int_{-\infty}^{+\infty} e^{-\hat{c}\left(\frac{y}{2}\right)^2} dy\varphi_0\\
&=2\sqrt{\pi}\,\hat{c}^{\frac{1}{2}}e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 = 2\sqrt{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{\hat{c}^{r+\frac{1}{2}}\left(\frac{x}{2}\right)^{2r}}{r!}\varphi_0\\
&=2\sqrt{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{2r}}{r!\Gamma\left(r+\frac{3}{2}\right)}.
\end{aligned} \tag{1.12}$$

In conclusion, if we apply the definition of Bessel function in equation (1.1), we end up with:

$$L(x) = 2\sqrt{\pi}\frac{J_{\frac{1}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{1}{2}}}. \tag{1.13}$$

For future convenience we also note that the use of the series definition of the Struve function [7, 11, 12],

$$\mathcal{H}_\alpha(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\left(r+\frac{3}{2}\right)\Gamma\left(r+\alpha+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2r+\alpha+1} \tag{1.14}$$

allows to write:

$$\int_{-\infty}^{+\infty} Jinc\left(\sqrt{x^2+y^2}\right)\, dy = \sqrt{\frac{8\pi}{x}}\mathcal{H}_{-\frac{1}{2}}(x). \tag{1.15}$$

This introduction has been addressed to assess the formal environment in which we develop the forthcoming discussion. The paper consists of two further sections, in which we propose a generalization of the Jinc and their link with previously known families of special functions and the concluding section is dedicated to comments on the relevant applications.

2. Generalized Forms of Jinc's

The genesis of the function (1.1) is associated with the Fourier transform of the $circ(x)$ function, which is defined as:

$$circ(r) = \begin{matrix} 1, r<1 \\ \frac{1}{2}, r=0 \\ 0, r>0. \end{matrix} \tag{2.16}$$

The relevant Fourier transform is:

$$\begin{aligned} \hat{F}\left(circ(r)\right) &= somb(\rho) \\ somb(\rho) &= 2\pi\, Jinc\left(2\pi\rho\right). \end{aligned} \tag{2.17}$$

The analogy with the $\operatorname{sinc}(x) = \sin(x)/x$ has a twofold origin. It is, indeed, associated with the Fourier transform of the step function, but it is also corroborated by the analogy between the first two cylindrical Bessel ($J_{0,1}(x)$) with the circular functions $(\cos(x), \sin(x))$. Postponing a deeper discussion on this point to the final section, here we note that the $jinc$ function is well behaved at the origin:

$$\lim_{x\to 0} Jinc(x) = 1 \tag{2.18}$$

If we introduce the n-th order Jinc as (with n any real):

$$Jinc_n(x) = \frac{J_n(x)}{\left(\frac{x}{2}\right)^n}, \tag{2.19}$$

we find that they are well behaved at the origin ($Jinc_n(0) = 1/\Gamma(n+1)$) and that:

$$\begin{aligned} L(x) &= 2\sqrt{\pi} Jinc_{\frac{1}{2}}(x) \\ Jinc(x) &= Jinc_1(x). \end{aligned} \tag{2.20}$$

The point we like to rise is that the function defined in eq. (2.19) are by no means new. They are indeed expressible in terms of the so called Tricomi fuctions, namely:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,\Gamma(n+r+1)} \tag{2.21}$$

so that:

$$Jinc_n(x) = C_n\left(-\left(\frac{x}{2}\right)^2\right). \tag{2.22}$$

Most of the properties of the Tricomi functions can be extended to the Jinc functions as it will be shown below. The 0-th order Tricomi is known as the Laguerre exponential and is an eigenfunctions of the Laguerre derivative [14], namely:

$$[\partial_x x \partial_x]\, C_0(\lambda x) = -\lambda C_0(\lambda x) \tag{2.23}$$

which, translated to the $Jinc_0(x)$, yields:

$$\frac{1}{x}\partial_x x \partial_x Jinc_0(\lambda x) = -\lambda^2 Jinc_0(\lambda x) \tag{2.24}$$

while for the associated differential equation we find:

$$\begin{aligned} &xz'' + z' + xz(x) = 0 \\ &z = Jinc_0(x). \end{aligned} \tag{2.25}$$

The n-th order Tricomi are eigenfunctions of the operator $-[\partial_x x \partial_x + n\partial_x]$, therefore the equations for the $Jinc_n(x)$ writes:

$$\begin{aligned} &xz'' + (1+n)z' + xz(x) = 0 \\ &z = Jinc_n(x). \end{aligned} \tag{2.26}$$

The previous equations exhibit a Bessel-like form and will be further discussed in the forthcoming section.
As already noted the umbral methods offer an efficient tool to work out the relevant properties. Accordingly, it is straightforward to get:

$$\begin{aligned} &Jinc_n(x) = \hat{c}^n e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 \\ &\int_{-\infty}^{\infty} Jinc_n(x)dx = \hat{c}^n \int_{-\infty}^{+\infty} e^{-\hat{c}\left(\frac{x}{2}\right)^2} dx\varphi_0 = 2\sqrt{\pi}\hat{c}^{n-\frac{1}{2}}\varphi_0 = 2\frac{\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}. \end{aligned} \tag{2.27}$$

Furthermore, the use of the following properties of the Gaussian under successive derivatives

$$\begin{aligned} &\partial_x^n e^{-ax^2} = (-1)^n H_n(2ax, -a)e^{-ax^2} \\ &H_n(x,y) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^{n-2r} y^r}{(n-2r)!r!} \end{aligned} \tag{2.28}$$

can be exploited to work out, in finite terms, the following higher order derivatives:

$$\partial_x^m Jinc_n(x) = (-1)^m \hat{c}^n H_m\left(\frac{\hat{c}}{2}x, -\frac{\hat{c}}{4}\right) e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0. \tag{2.29}$$

The use of the second of equation (2.28) and of the first of (2.27) yields the desired result:

$$\partial_x^m Jinc_n(x) = \left(-\frac{1}{2}\right)^m \sum_{r=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(-1)^r m! x^{n-r}}{(m-2r)!r!} Jinc_{m+n-r}(x). \tag{2.30}$$

It is evident that the use of the mathematical means, we have outlined, is fairly useful to explore the properties of a Bessel-like family of functions, which have been only tangentially studied within the context of the relevant applications.
In the recent past [2], the possibility of introducing more general forms of Jinc functions has been discussed within the context of studies regarding their application to focusing and diffraction of circular apertures.
The functions introduced in [2], in terms of $Jinc_n(x)$ reads:

$$\Phi inc_n(x) = -\frac{1}{2}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!} Jinc_{r+1}(x) \tag{2.31}$$

In umbral terms equation (2.31) we can cast eq. (2.31) as a combination of Gaussians, namely:

$$\Phi inc_n(x) = -\frac{\hat{c}}{2}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}\hat{c}^r e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 \tag{2.32}$$

and, according to equation (2.27), we obtain:

$$\int_{-\infty}^{+\infty} \Phi inc_n(x)\, dx = -\sqrt{\pi}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}\frac{1}{\Gamma\left(r+\frac{3}{2}\right)}. \tag{2.33}$$

Furthermore, keeping the derivative of the Φinc as expressed in equation (2.32), we obtain:

$$\begin{aligned}\partial_x \Phi inc_n(x) &= -\frac{x}{4}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}\hat{c}^{r+2} e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 \\ &= \frac{x}{2(n+1)}\left[\Phi inc_{n+1}(x) + \frac{1}{2} Jinc_1(x)\right]\end{aligned} \tag{2.34}$$

the n-th derivative can be computed in finite form, using the procedure leading to equation (2.30).
Before closing this section, we discuss the evaluation of the generating function of the Φinc, in particular we consider the derivation of the following infinite sum:

$$G_\Phi(x;\xi) = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}\Phi inc_n(x). \tag{2.35}$$

In order to write equation (2.35) in finite form, we note that, on account of the fact that $r! = \int_0^\infty e^{-t}t^r dt$, it is possible to set:

$$\begin{aligned}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}a^r &= \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}\int_0^\infty e^{-t}(ta)^r\, dt \\ &= \int_0^\infty e^{-t}(1-ta)^n\, dt\end{aligned} \tag{2.36}$$

applying this identity to the umbral form in equation (2.32), we find:

$$-\frac{\hat{c}}{2}\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}\hat{c}^r e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 = -\frac{\hat{c}}{2}\int_0^\infty (1-\hat{c}t)^n e^{-t} dt e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 \tag{2.37}$$

and, accordingly:

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\,\Phi inc_n(x) = -\frac{\hat{c}}{2}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\int_0^{\infty}(1-\hat{c}t)^n e^{-t}dt e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0 = -\frac{\hat{c}}{2}e^{\xi}\int_0^{\infty}e^{-t}e^{-\hat{c}\left(t\xi+\left(\frac{x}{2}\right)^2\right)}dt\varphi_0. \tag{2.38}$$

Therefore, in conclusion, we find:

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\,\Phi inc_n(x) = -\frac{\hat{c}}{2}e^{\xi}\int_0^{\infty}e^{-t}e^{-\hat{c}\left(t\xi+\left(\frac{x}{2}\right)^2\right)}dt\varphi_0 = -\frac{e^{\xi}}{2}\int_0^{\infty}e^{-t}Jinc_1\left(2\sqrt{t\xi+\left(\frac{x}{2}\right)^2}\right)dt. \tag{2.39}$$

The same technique eventually yields:

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\,Jinc_n(x) = Jinc_1\left(2\sqrt{t\xi+\left(\frac{x}{2}\right)^2}\right). \tag{2.40}$$

In the forthcoming section we discuss, along with the relevant use in applications, the impact $Jinc_n(x)$ in the study of further Bessel-like functions.

3. JINC FUNCTIONS , CIRCULAR FUNCTIONS, PRODUCTS OF JINC FUNCTIONS AND DOUBLE VACUA UMBRAL IMAGES

The use of the Umbral/Algebraic methods, touched in the previous sections, has allowed a fairly significant level of freedom in defining a class of functions, "interpolating" between Bessel and circular functions [15]. A step further in this direction allows the possibility of defining the cosine function in terms of its Gaussian image [15]:

$$\cos(x) = e^{-{}_{(2,1)}\hat{d}\,x^2}\gamma_0 \qquad {}_{(\alpha,\beta)}\hat{d}^k\gamma_0 = \frac{\Gamma(k+1)}{\Gamma(\alpha\,k+\beta)}. \tag{3.41}$$

The sine function can accordingly be written as:

$$\sin(x) = 2x_{(2,1)}\hat{d}e^{-{}_{2,1}\hat{d}\,x^2}\gamma_0 \tag{3.42}$$

thus, getting the following Gaussian image for the sinc:

$$\mathrm{sinc}(x) = 2_{(2,1)}\hat{d}e^{-{}_{2,1}\hat{d}\,x^2}\gamma_0. \tag{3.43}$$

The use of the paradigm underlying the derivation of the integrals discussed in the previous section, yields the well- known result:

$$\int_{-\infty}^{+\infty}\mathrm{sinc}(x)\,dx = 2_{(2,1)}\hat{d}^{\frac{1}{2}}\gamma_0 = 2\sqrt{\pi}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} = \pi \tag{3.44}$$

which leads to further mathematical speculations, associated with the derivation of integrals like

$$\begin{aligned}\int_{-\infty}^{+\infty}\operatorname{sinc}(\alpha\,x^2+\beta\,x)\,dx &= 2_{(2,1)}\hat{d}\int_{-\infty}^{+\infty}e^{-{}_{(2,1)}\hat{d}\,(\alpha\,x^2+\beta\,x)}dx\gamma_0\\ &= 2\sqrt{\frac{\pi}{\alpha}}{}_{(2,1)}\hat{d}^{\frac{1}{2}}e^{{}_{(2,1)}\hat{d}\frac{\beta^2}{4\alpha}}\gamma_0\end{aligned}\tag{3.45}$$

which has been obtained by writing in umbral form the elementary Gaussian identity:

$$\int_{-\infty}^{+\infty}e^{-(\alpha\,x^2+\beta\,x)}\,dx=\sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}}.\tag{3.46}$$

The point to be clarified is whether the last term in equation (3.45) has any meaning from the mathematical point of view. The use of the properties of the umbral operator$_{2,1}\hat{d}$ yields (the correctness of the result has been checked numerically):

$$\begin{aligned}&\int_{-\infty}^{+\infty}\operatorname{sinc}(\sqrt{\alpha\,x^2+\beta\,x})\,dx\\ =&2\sqrt{\frac{\pi}{\alpha}}\sum_{r=0}^{\infty}\frac{\Gamma\left(r+\frac{3}{2}\right)}{r!\Gamma(2(r+1))}\left(\frac{\beta}{2\sqrt{\alpha}}\right)^{2r}\\ =&\frac{\pi}{\sqrt{\alpha}}I_0\left(\frac{\beta}{2\sqrt{\alpha}}\right)\end{aligned}\tag{3.47}$$

providing a further example of integral representation of O-th order cylindrical Bessel, whose implications will be discussed in a forthcoming dedicated article.
Regarding however the case involving the Jinc functions we obtain:

$$\int_{-\infty}^{+\infty}Jinc_n(2\sqrt{\alpha\,x^2+\beta\,x})\,dx=\sqrt{\frac{\pi}{\alpha}}Jinc_{n-1/2}\left(\frac{\beta^2}{4\alpha}\right).\tag{3.48}$$

We have so far used the Gaussian as umbral image of Bessel-like functions. It is however well known that any image can be used to get an umbral representation of a given function and the choice is matter of convenience and one can choose the most suitable for the solution of a given problem. In the following we apply this statement to the derivation of integrals involving squares of Jinc functions.
To this aim we remind that the squares of Bessel functions can be written as [4]:

$$[J_n(x)]^2=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{2r+2n}\Gamma\left[2r+1\right]}{r!^2\Gamma(r+n+1)^2}.\tag{3.49}$$

Therefore, defining

$$JJinc_n(x)=\left[\frac{J_n(x)}{\left(\frac{x}{2}\right)^n}\right]^2\tag{3.50}$$

we find:

$$JJinc_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{2r}\Gamma\left[2r+1\right]}{r!^2\Gamma(r+n+1)^2}.\tag{3.51}$$

The function $J_0(x)$ can usefully be exploited as umbral image $JJinc_n(x)$ and, indeed, we write:

$$\begin{aligned} JJinc_n(x) &= \hat{c}^n J_0\left(\sqrt{\hat{b}}x\right)\varphi_0 \\ \hat{b}^r\varphi_0 &= \frac{\Gamma(2\,r+1)}{\Gamma(r+1)}. \end{aligned} \tag{3.52}$$

Furthermore, by keeping into account that

$$\int_{-\infty}^{+\infty} J_0\left(\sqrt{b}x\right)\,dx = \frac{2}{\sqrt{b}} \tag{3.53}$$

we find:

$$\int_{-\infty}^{+\infty} JJinc_n\,(x)\,\,dx = 2\,\hat{b}^{n-\frac{1}{2}}\varphi_0 = 2\frac{\Gamma\left[2\,\left(n-\frac{1}{2}\right)+1\right]}{\Gamma\left(\left(n-\frac{1}{2}\right)+1\right)^2} = 2\frac{\Gamma\,(2n)}{\Gamma\left(n+\frac{1}{2}\right)^2}. \tag{3.54}$$

Let us now remind that:

$$\left[J_n(x)\right]^2 = \sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{2r+2n}\Gamma\left[2r+1\right]}{r!^2\Gamma(r+n+1)^2}. \tag{3.55}$$

A final point we touch here is the study of the following generalized Jinc function :

$$JJinc_{\mu.\nu}(x) = \frac{J_\nu(x)J_\mu(x)}{\left(\frac{x}{2}\right)^{\mu+\nu}} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\nu+\mu+2k+1)}{k!\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\mu+\nu+k+1)}\left(\frac{x}{2}\right)^{2k} \tag{3.56}$$

which can be reduced to a easily manageable image by the use of the following two vacuum operators

$$\begin{aligned} \hat{b}_1^\nu &= e^{\nu\frac{\partial}{\partial z_1}} \\ \hat{b}_2^\mu &= e^{\mu\frac{\partial}{\partial z_2}} \end{aligned} \tag{3.57}$$

acting on the vacuum

$$\varphi_0 = \frac{\Gamma(z_1+z_2+1)}{\Gamma(z_1+1)\Gamma(z_2+1)} \tag{3.58}$$

in such a way that:

$$\hat{b}_1^\nu\hat{b}_2^\mu\varphi_0|_{z_1=z_2=0} = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}. \tag{3.59}$$

If we use now the image

$$Jinc_{\mu+\nu}(x) = \frac{J_{\nu+\mu}(x)}{\left(\frac{x}{2}\right)^{\mu+\nu}}, \tag{3.60}$$

we can end up with

$$JJinc_{\mu,\nu}(x) = \hat{b}_1^\nu\hat{b}_2^\mu Jinc_{\mu+\nu}(\sqrt{\hat{b}_1\hat{b}_2}x)\varphi_0 \tag{3.61}$$

and get the relevant infinite integral as reported below:

$$\begin{aligned}
&\hat{b}_1^{\nu}\hat{b}_2^{\mu}\int_{-\infty}^{+\infty} Jinc_{\mu+\nu}(\sqrt{\hat{b}_1\hat{b}_2}x)dx\,\varphi_0 \\
=&2\sqrt{\pi}\frac{\hat{b}_1^{\nu}\hat{b}_2^{\mu}}{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\hat{b}_1^{-\frac{1}{2}}\hat{b}_2^{-\frac{1}{2}}\varphi_0 \\
=&2\sqrt{\pi}\frac{\Gamma(\mu+\nu)}{\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}.
\end{aligned} \tag{3.62}$$

We have so far treated the Jinc functions and the associated generalizations using a fairly abstract point of view, in the forthcoming part we will discuss physical applications where this type of functions are used. In particular we discuss the case of the Fraunhofer diffraction in two-iris wave-particle duality experiments, where integrals Involving functions of the type $Jinc_{\mu+\nu}(x)$ play a crucial role.

4. Relevant applications of Jinc functions in Physics

Jinc functions play a fundamental role in several contexts of physics, due to the fact that the Fourier transform of a uniform distribution over a finite circular area of radius r_0 coincides with the area of the circle πr_0^2 multiplied by a Jinc function:

$$\int_0^{2\pi} d\varphi \int_0^{\infty} dr r circ\left(\frac{r}{r_0}\right) e^{-ikr\cos(\phi-\varphi)} = 2\pi\int_0^{r_0} dr r J_0(kr) = \pi r_0^2 Jinc(kr_0). \tag{4.63}$$

In the following we are going to list a few topical applications of the above result.

4.1. **High-power lasers.** Chirped Pulse Amplification (CPA) is the most common technique used to amplify laser beams to unprecedented levels of peak power, up to the PetaWatt level. CPA is based on the principle that a short seed pulse is amplified while stretched, to avoid damages in the active media. A laser beam that enters the amplifier with a gaussian profile (or any other bell-shape distribution), is firstly amplified on-axis, where there is a larger amount of seeding energy. Once the saturation level is reached by the photon energy amplified on-axis, the tails of the profile start accumulating energy more efficiently than the central part of the beam. For high-gain, multi-pass amplifiers, the final effect on the seed beam, at the end of the amplification process, is that of a super-gaussian distribution, which can be assimilated to a circular uniform distribution of radius r_0. The flat-profile beam is then recompressed and focused for laser-matter interactions at high-intensity. Considering an aperture of the focusing element that is much larger than the beam diameter, the laser field at the focal plane is evaluated via the Kirchoff's integral [8]:

$$E(r,z=f) = \frac{-ike^{ik\left(f-\frac{r^2}{2f}\right)}}{f}\int_0^{\infty} dr' r' E_0(r') J_0\left(\frac{kr'r}{f}\right) = \frac{-ike^{ik\left(f-\frac{r^2}{2f}\right)}}{f}\pi r_0^2 Jinc\left(\frac{kr_0 r}{f}\right), \tag{4.64}$$

where f is the focal length and $k = 2\pi/\lambda$, with λ the radiation wavelength. The flat-profile of the laser beam impinging on the focusing optics has been defined as $E_0(r) = E_0 circ(r/r_0)$. Thus, the radiation intensity in the focal point is:

$$I = \frac{c\varepsilon_0}{2}\left|E(r,f)\right|^2 = \frac{c\varepsilon_0\pi^2k^2r_0^4}{2f^2}Jinc^2\left(\frac{kr_0r}{f}\right), \tag{4.65}$$

where c is the speed of light in vacuum and ε_0 is the vacuum dielectric constant. A criterion to evaluate the beam radius R at the focal plane, is to impose that the argument of the Jinc function is equal to the first zero of the Bessel J_1, defined as $j_1 \simeq 3.832$:

$$\frac{kr_0R}{f} = j_1 \to R = \frac{j_1 f}{kr_0} = \frac{j_1\lambda f}{2\pi r_0}. \tag{4.66}$$

Typical values are $\lambda = 1\ \mu m$, $f = 1\ m$, $r_0 = 0.1\ m$. This means that the focal spot radius is $R \simeq 6\ \mu m << r_0$. The evaluation of the total laser energy passes through an integral of the type:

$$\begin{aligned} &\int_0^\infty drrJinc^2\left(\frac{kr_0r}{f}\right) = \int_0^\infty drrJinc^2(\kappa r) \\ =&\hat{c}_1\hat{c}_2\int_0^\infty drre^{-\hat{c}_1\left(\frac{\kappa r}{2}\right)^2}e^{-\hat{c}_2\left(\frac{\kappa r}{2}\right)^2}\varphi_0^{(1)}\varphi_0^{(2)} = \frac{2\hat{c}_1\hat{c}_2\varphi_0^{(1)}\varphi_0^{(2)}}{\kappa^2(\hat{c}_1+\hat{c}_2)} = \frac{2}{\kappa^2} = \frac{\lambda^2f^2}{2\pi^2r_0^2} = \frac{2R^2}{j_1^2}, \end{aligned} \tag{4.67}$$

where we have defined the action of the entangled umbral operators $\hat{c}_1$ and $\hat{c}_2$ on the double vacuum $\varphi_0^{(1)}\varphi_0^{(2)}$ as:

$$\left(\frac{\hat{c}_1\hat{c}_2}{\hat{c}_1+\hat{c}_2}\right)^\alpha\varphi_0^{(1)}\varphi_0^{(2)} \equiv \frac{1}{\Gamma(\alpha+1)}. \tag{4.68}$$

4.2. **Diffraction limits in optics, acoustics and telecommunications.** The Jinc function is also known as Airy pattern. The first zero of the Bessel J_1 determines the limits of the so-called Airy disk. The airy patterns were discovered in the context of astronomical observation, where it was observed that the size of stars appeared different depending on the instrumentation. The aperture of the lenses of telescopes limits the resolution of these instruments. This can be seen using Eq. 4.66. Let's assume here a lens of radius r_0, used to focus a uniform wavefront of light of larger size compared to the optical aperture. If the distance between two far objects is d, then $\theta = d/f$ will be, approximately, the angular separation from the observer's point of view, placed at the focal plane. The minimum resolvable distance is then [8]:

$$\frac{d}{f} \simeq \theta \simeq 1.22\frac{\lambda}{2r_0}. \tag{4.69}$$

The same limits and description apply to focusing elements used in acoustics and to antennas exploited in telecommunications.

4.3. **Fraunhofer diffraction in two-iris wave-particle duality experiments.** Fraunhofer diffraction affects any plane wavefront passing through a circular aperture, determining a a Jinc function pattern (see Eq. 4.64). This description also underlies the two-iris experiment for the verification of the wave-particle duality. Considering a quantum particle that can pass a wall only through two different circular holes, with centers in $x = \pm a$, both of radius r_0, the wave function ψ of the particle evaluated at a certain distance R from the holes is given in the Born approximation by [8]:

(4.70)

$$\psi = \psi_1 + \psi_2, \psi_{1,2}(r,t) \simeq e^{i\frac{pz}{\hbar}} + f_{1,2}(\theta_x,\theta_y)\frac{e^{i\frac{pr}{\hbar}}}{r}$$

$$f_{1,2}(\theta_x,\theta_y) = -\frac{mV_0L}{\hbar^2}\int_0^{r_0}drrJ_0(\tilde{q}_{1,2}r) = -\frac{mV_0Lr_0^2}{2\hbar^2}Jinc(\tilde{q}_{1,2}r_0), \quad \tilde{q}_{1,2} \simeq \frac{p}{\hbar}\sqrt{\left(\theta_x \mp \frac{a}{R}\right)^2 + \theta_y^2},$$

where V_0 and L are empirical constants, m is the mass of the particle, $\hbar$ is the Planck constant,

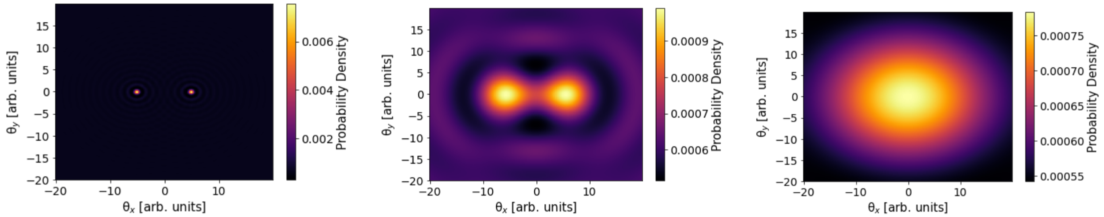

FIGURE 1. Probability density $|\psi_1+\psi_2|^2$ for the two-iris experiment. From left to right the momentum of the particle decreases.

$\theta_{x,y}$ are the observation angles in the plane of the two holes and in the perpendicular one, respectively, and p is the momentum of the particle. The partial wave functions $\psi_{1,2}$, represent the passage of the particle through one or the other iris, respectively. Fig. 1 shows the outcome of the two-iris experiment, based on the Eq. 4.70. From left to right the momentum of the particle decreases, reaching values comparable to $\hbar/r_0$, leading to diffraction phenomena. From the leftest situation, showing very localized particle states, mirroring the two entrance irises, one moves to the rightest situation where the interference is such to create a maximum probability at the center. The normalization of the wavefunction is achieved via integrals of the type:

$$\begin{aligned}&\int d\theta_x \int d\theta_y Jinc(\tilde{q}_1 r_0) Jinc(\tilde{q}_2 r_0)\\ &=\hat{c}_1\hat{c}_2\int d\theta_x\int d\theta_y e^{-\hat{c}_1\left(\frac{\tilde{q}_1 r_0}{2}\right)^2} e^{-\hat{c}_2\left(\frac{\tilde{q}_2 r_0}{2}\right)^2}\varphi_0^{(1)}\varphi_0^{(2)} = \frac{4\pi\hbar^2}{p^2 r_0^2} Jinc\left(\frac{2par_0}{\hbar R}\right)\\ &\int d\theta_x\int d\theta_y Jinc^2(\tilde{q}_{1,2} r_0)\\ &=\hat{c}_1\hat{c}_2\int d\theta_x\int d\theta_y e^{-\hat{c}_1\left(\frac{\tilde{q}_{1,2} r_0}{2}\right)^2} e^{-\hat{c}_2\left(\frac{\tilde{q}_{1,2} r_0}{2}\right)^2}\varphi_0^{(1)}\varphi_0^{(2)} = 4\pi\left(\frac{\hat{c}_1\hat{c}_2}{\hat{c}_1+\hat{c}_2}\right)\varphi_0^{(1)}\varphi_0^{(2)} = 4\pi.\end{aligned} \tag{4.71}$$

It is worth noting that integrals in Eqs. 4.71 would, normally, be very difficult to solve without the umbral approach.

5. DISCUSSION

The article has provided a comprehensive treatment of Jinc functions and of the relevant applications. The mathematical tools we have employed to study the relevant properties has been the use of the indicial umbral calculus [15]. Such a choice has been dictated by mere convenience and other means can be usefully exploited. To give an example we consider the use of the generating function method.
To this aim we remind the Jacobi-Anger generating function [4]

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \tag{5.72}$$

which, once applied to the Jinc, yields:

$$\sum_{n=-\infty}^{+\infty} t^n Jinc_n(x) = e^{t-\frac{x^2}{4t}}. \tag{5.73}$$

We use this last identity to evaluate the integral:

$$Iinc_n(\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha\, x^2} Jinc_n(2\sqrt{\beta\, x})dx. \tag{5.74}$$

Multiplying both sides of (5.74) by t^n and summing on the index n we find:

$$\sum_{n=-\infty}^{+\infty} t^n Iinc_n(\alpha.\beta) = e^t \int_{-\infty}^{+\infty} e^{-\alpha\, x^2 - \frac{\beta\, x}{t}} dx = \sqrt{\frac{\pi}{\alpha}} e^{t + \frac{\beta^2}{4\alpha\, t^2}}. \tag{5.75}$$

In order to get the explicit expression of the integral in (5.74) we note that:

$$\begin{aligned} e^{t+\frac{x}{t^2}} &= \sum_{n=-\infty}^{+\infty} t^n (x^n C_n^{(2)}(x)) \\ C_n^{(1.2)}(x) &= \sum_{s=0}^{\infty} \frac{x^{2s}}{(n+2s)!s!}, \end{aligned} \tag{5.76}$$

where $C_n^{(2)}(x)$ is an n-th order Bessel-Wright function [13]. Therefore expanding the rhs of (5.75) and equating the t-like power coefficients, we end up with:

$$Iinc_n(\alpha, \beta) = \sqrt{\frac{\pi}{\alpha}} C_n^{(2)} \left(\frac{\beta^2}{4\alpha} \right). \tag{5.77}$$

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