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Research Article

Categorical isomorphisms for Hopf braces

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Abstract

In this paper, we introduce the category of brace triples in a braided monoidal setting and prove that it is isomorphic to the category of s-Hopf braces, which are a generalization of cocommutative Hopf braces. After that, we obtain a categorical isomorphism between the category of finite cocommutative Hopf braces and a certain subcategory of the category of cocommutative post-Hopf algebras, which supposes an expansion to the braided monoidal setting of the equivalence obtained for the category of vector spaces over a field $\mathbb{K}$ by Y. Li, Y. Sheng and R. Tang.

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1. Introduction

Hopf braces were born in [2] as the quantum version of skew braces, introduced by L. Guarnieri and L. Vendramin in [9]. The importance of these objects is fundamentally due to the fact that they provide solutions of the Quantum Yang-Baxter equation, that is a relevant subject in mathematical physics (see [19] and [4]). Despite the simplicity of its formulation, finding solutions of the Yang-Baxter equation is not an easy task. In fact, the problem of classifying all the solutions of the equation is still open and different approaches have been proposed since the end of the last century. One of them was proposed by Drinfel'd in [5], that consists of studying non-degenerate set-theoretical solutions. Research into this kind of solutions with the involutive property was what gave rise to the concept of brace introduced by Rump in [17] for which skew braces are a generalization. So, a skew brace consists of two different group structures, $(G,.)$ and $(G,\star)$, satisfying the following compatibility condition

$$g\star(h.t)=(g\star h).g^{-1}.(g\star t) \tag{1.1}$$

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for all $g, h, t \in G$, where g^{-1} denotes the inverse of g with respect to the group structure $(G, .)$. These structures are useful to find non-degenerate solutions of the Yang-Baxter equation not neccesarily involutive. The linearization of skew braces gives rise to the notion of Hopf brace defined by I. Angiono, C. Galindo and L. Vendramin in [2]: If (H, ϵ, Δ) is a coalgebra, a Hopf brace structure over H consists of two different Hopf algebra structures,

$$H_1 = (H, 1, \cdot, \epsilon, \Delta, \lambda), \quad H_2 = (H, 1_\circ, \circ, \epsilon, \Delta, S),$$

where λ and S denote the antipodes, satisfying the following compatibility condition

$$g \circ (h \cdot t) = (g_1 \circ h) \cdot \lambda(g_2) \cdot (g_3 \circ t) \quad \forall g, h, t \in H$$

which generalizes (1.1). Moreover, as was pointed in [2, Corollary 2.4], cocommutative Hopf braces give rise to solutions of Yang-Baxter equation too.

On the other hand, without going into detail, it is not irrelevant to highlight the relationship between Hopf braces and invertible 1-cocycles, which are nothing more than coalgebra isomorphisms between two different Hopf algebras, $\pi\colon H \to B$, such that B is a H-module algebra. In [2, Theorem 1.12] it is proved that the category of Hopf braces with H_1 fixed is equivalent to the category of invertible 1-cocycles $\pi\colon H_1 \to B$. Moreover, González Rodríguez and Rodríguez Raposo proved in [8] that this result remains valid in the case that H_1 is not fixed (see also [6]).

Therefore, motivated by the fact that cocommutative Hopf braces induce solutions of the Quantum Yang-Baxter equation, in this paper we study another objects that characterise the structure of Hopf braces in the cocommutative setting. So, given C a braided monoidal category, in Section 3 we introduce the category of brace triples (see Definition 3.1) and the category of s-Hopf braces (see Definition 3.6). Note that s-Hopf braces generalize cocommutative Hopf braces because both categories are the same under cocommutativity assumption (see Remark 3.9). After that, a functor from brace triples to s-Hopf braces is constructed explicitly (see Theorem 3.10), and another from s-Hopf braces to brace triples (see Theorem 3.17), ending the section with the main Theorem 3.18 where we prove that the previous correspondence gives rise to a categorical isomorphism. As a consequence (see Corollary 3.19), we obtain a categorical isomorphism between cocommutative Hopf braces and cocommutative brace triples, that is to say, a cocommutative Hopf brace is no more than a cocommutative Hopf algebra H together with a pair of morphisms, $T_H\colon H \to H$ and $\gamma_H\colon H \otimes H \to H$, satisfying some compatibility conditions between them and the Hopf algebra structure over H.

In Section 4, we introduce the notion of post-Hopf algebra in a braided monoidal category C (see Definition 4.1), which generalizes the one introduced by Y. Li, Y. Sheng and R. Tang in [14] for a category of vector spaces over a field $\mathbb{K}$ (see also [3]). After proving some interesting properties of these objects that can be deduced from the definition, we obtain a functor from finite brace triples to post-Hopf algebras (see Theorem 4.6). At this point, hypothesis of cocommutativity acquires significant importance and it is essential, together with technical condition (4.13), to prove the existence of a functor from the category of cocommutative post-Hopf algebras that satisfy (4.13) to the category of finite cocommutative Hopf braces (see Theorem 4.15). Therefore, Theorem 4.16 is the main result of this section, where we prove that this correspondence induces a categorical isomorphism between $\mathsf{cocPost\text{-}Hopf}^\star$, the category of cocommutative post-Hopf algebras satisfying (4.13), and finite cocommutative brace triples. So, as a consequence, $\mathsf{cocPost\text{-}Hopf}^\star$, finite cocommutative brace triples and finite cocommutative Hopf braces are isomorphic, which suppose a generalization of [14, Theorem 2.13] to the braided monoidal setting.

In the following diagram it is possible to consult a summary of the categorical relationships that can be seen along this paper. Detailed notation information will be introduced throughout the paper.

$$
\begin{tikzcd}
\mathsf{BT}^{\mathsf{f}} \arrow[r, hook] \arrow[d, "P"] & \mathsf{BT} \arrow[r, shift left, "F"] \arrow[r, phantom, "\simeq"] & \mathsf{sHBr} \arrow[l, shift left, "G"] \\
\mathsf{Post\text{-}Hopf} & \mathsf{cocBT} \arrow[u, hook] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\simeq"] & \mathsf{cocHBr} \arrow[u, hook] \arrow[l, shift left, "G'"] \\
\mathsf{cocPost\text{-}Hopf}^{\star} \arrow[u, hook] \arrow[r, shift right, "G''\circ Q"'] \arrow[r, phantom, "\simeq"] \arrow[rr, bend right, "Q"'] & \mathsf{cocBT}^{\mathsf{f}} \arrow[u, hook] \arrow[l, shift right, "P'"'] \arrow[r, shift left, "F''"] \arrow[r, phantom, "\simeq"] & \mathsf{cocHBr}^{\mathsf{f}} \arrow[u, hook] \arrow[l, shift left, "G''"]
\end{tikzcd}
$$

Figure 1. Categorical relationships between HBr, BT and $\mathsf{Post\text{-}Hopf}$.

2. Preliminaries

Throughout this paper we are going to denote by C a strict braided monoidal category with tensor product $\otimes$, unit object K and braiding c.

As can be found in [15], a monoidal category is a category C together with a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, called tensor product, an object K of C, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in C, called associativity, right unit and left unit constraints, respectively, which satisfy the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$
$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where for each object X in C, id_X denotes the identity morphism of X. A monoidal category is called strict if the previous constraints are identities. It is an important result (see for example [13]) that every non-strict monoidal category is monoidal equivalent to a strict one, so the strict character can be assumed without loss of generality. Then, results proved in a strict setting hold for every non-strict monoidal category that include, between others, the category $\mathbb{K}$-Vect of vector spaces over a field $\mathbb{K}$, the category R-Mod of left modules over a commutative ring R or the category of sets, Set. For simplicity of notation, given objects M, N, P in C and a morphism $f : M \to N$, in most cases we will write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

A braiding for a strict monoidal category C is a natural family of isomorphisms

$$c_{M,N} : M \otimes N \to N \otimes M$$

subject to the conditions

$$c_{M,N\otimes P} = (N \otimes c_{M,P}) \circ (c_{M,N} \otimes P), \;\; c_{M\otimes N,P} = (c_{M,P} \otimes N) \circ (M \otimes c_{N,P}).$$

A strict braided monoidal category C is a strict monoidal category with a braiding. These categories were introduced by Joyal and Street in [11] (see also [12]) motivated by the theory of braids and links in topology. Note that, as a consequence of the definition, the equalities $c_{M,K} = c_{K,M} = id_M$ hold, for all object M of C. Moreover, if C is braided with braiding c, then C is also braided with braiding c^{-1}. We will denote by $\overline{\mathsf{C}}$ the category C with braiding c^{-1}.

If the braiding satisfies that $c_{N,M} \circ c_{M,N} = id_{M\otimes N}$, for all M, N in C, we will say that C is symmetric. In this case, we call the braiding c a symmetry for the category C.

In the following definitions we sum up some basic notions in the braided monoidal setting.

Definition 2.1. An algebra in C is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in C and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \to B$ in C is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$.

If A, B are algebras in C, the tensor product $A \otimes B$ is also an algebra in C where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

Definition 2.2. A coalgebra in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in C and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in C such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, a morphism $f : D \to E$ in C is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$.

Given D, E coalgebras in C, the tensor product $D \otimes E$ is a coalgebra in C where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Definition 2.3. Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra and $A = (A, \eta_A, \mu_A)$ an algebra in C. By $\mathrm{Hom}(D, A)$ we denote the set of morphisms $f : D \to A$ in C. With the convolution operation $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$, $\mathrm{Hom}(D, A)$ is an algebra where the unit element is $\eta_A \circ \varepsilon_D = \varepsilon_D \otimes \eta_A$.

Definition 2.4. Let A be an algebra. The pair (M, φ_M) is a left A-module if M is an object in C and $\varphi_M : A \otimes M \to M$ is a morphism in C satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two left A-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$.

The composition of morphisms of left A-modules is a morphism of left A-modules. Then left A-modules form a category that we will denote by ${}_{\mathsf{A}}\mathsf{Mod}$.

Definition 2.5. We say that X is a bialgebra in C if (X, η_X, μ_X) is an algebra, $(X, \varepsilon_X, \delta_X)$ is a coalgebra, and ε_X and δ_X are algebra morphisms (equivalently, η_X and μ_X are coalgebra morphisms). Moreover, if there exists a morphism $\lambda_X : X \to X$ in C, called the antipode of X, satisfying that λ_X is the inverse of id_X in $\mathrm{Hom}(X, X)$, i.e.,

$$id_X * \lambda_X = \eta_X \circ \varepsilon_X = \lambda_X * id_X, \tag{2.1}$$

we say that X is a Hopf algebra. A morphism of Hopf algebras is an algebra-coalgebra morphism. Note that, if $f : X \to Y$ is a Hopf algebra morphism the following equality holds:

$$\lambda_Y \circ f = f \circ \lambda_X. \tag{2.2}$$

With the composition of morphisms in C we can define a category whose objects are Hopf algebras and whose morphisms are morphisms of Hopf algebras. We denote this category by Hopf.

Note that if $X = (X, \eta_X, \mu_X, \varepsilon_X, \delta_X, \lambda_X)$ is a Hopf algebra in C such that its antipode, λ_X, is an isomorphism, then $X^{cop} = (X, \eta_X, \mu_X, \varepsilon_X, c^{-1}_{X,X} \circ \delta_X, \lambda^{-1}_X)$ is a Hopf algebra in $\overline{\mathsf{C}}$ (see [16]).

A Hopf algebra is commutative if $\mu_X \circ c_{X,X} = \mu_X$ and cocommutative if $c_{X,X} \circ \delta_X = \delta_X$. It is easy to see that in both cases $\lambda_X \circ \lambda_X = id_X$.

If X is a Hopf algebra, relevant properties of its antipode, λ_X, are the following: It is antimultiplicative and anticomultiplicative

$$\lambda_X \circ \mu_X = \mu_X \circ (\lambda_X \otimes \lambda_X) \circ c_{X,X}, \quad \delta_X \circ \lambda_X = c_{X,X} \circ (\lambda_X \otimes \lambda_X) \circ \delta_X, \tag{2.3}$$

and leaves the unit and counit invariant, i.e.,

$$\lambda_X \circ \eta_X = \eta_X, \ \ \varepsilon_X \circ \lambda_X = \varepsilon_X. \tag{2.4}$$

So, it is a direct consequence of these identities that, if X is commutative, then λ_X is an algebra morphism and, if X is cocommutative, then λ_X is a coalgebra morphism.

In the following definition we recall the notion of left module (co)algebra.

Definition 2.6. Let X be a Hopf algebra. An algebra A is said to be a left X-module algebra if (A, φ_A) is a left X-module and η_A, μ_A are morphisms of left X-modules, i.e.,

$$\varphi_A \circ (X \otimes \eta_A) = \varepsilon_X \otimes \eta_A, \;\; \varphi_A \circ (X \otimes \mu_A) = \mu_A \circ \varphi_{A\otimes A}, \tag{2.5}$$

where $\varphi_{A\otimes A} = (\varphi_A \otimes \varphi_A) \circ (X \otimes c_{X,A} \otimes A) \circ (\delta_X \otimes A \otimes A)$ is the left action on $A \otimes A$.

Definition 2.7. Let X be a Hopf algebra. A coalgebra D is said to be a left X-module coalgebra if (D, φ_D) is a left X-module and ε_D, δ_D are morphisms of left X-modules, in other words, the following equalities hold:

$$\varepsilon_D \circ \varphi_D = \varepsilon_H \otimes \varepsilon_D, \;\; \delta_D \circ \varphi_D = \varphi_{D\otimes D} \circ (H \otimes \delta_D). \tag{2.6}$$

Equivalently, (D, φ_D) is a left X-module coalgebra if and only if φ_D is a coalgebra morphism.

The following result will be interesting along this paper.

Theorem 2.8. *Let $X = (X, \eta_X, \mu_X, \varepsilon_X, \delta_X, \lambda_X)$ and $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be Hopf algebras in C such that there exists a morphism $\varphi_H \colon X \otimes H \to H$ satisfying the following conditions:*

(i) $\varphi_H \circ (X \otimes \mu_H) = \mu_H \circ (\varphi_H \otimes \varphi_H) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes H \otimes H)$,

(ii) *φ_H is a coalgebra morphism.*

Then, $\varphi_H \circ (X \otimes \eta_H) = \varepsilon_X \otimes \eta_H$ holds.

Proof. The equality follows by:

$\varphi_H \circ (X \otimes \eta_H)$

$=(\varphi_H \otimes (\varepsilon_H \circ \varphi_H)) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by (ii), naturality of c and (co)unit properties)

$=\mu_H \circ (\varphi_H \otimes (\eta_H \circ \varepsilon_H \circ \varphi_H)) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by unit property)

$=\mu_H \circ (\varphi_H \otimes ((id_H * \lambda_H) \circ \varphi_H)) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by (2.1))

$=\mu_H \circ (\varphi_H \otimes (\mu_H \circ (H \otimes \lambda_H) \circ (\varphi_H \otimes \varphi_H) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes \delta_H))) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by (ii))

$=\mu_H \circ ((\mu_H \circ (\varphi_H \otimes \varphi_H) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes H \otimes H)) \otimes (\lambda_H \circ \varphi_H)) \circ (X \otimes ((H \otimes c_{X,H} \otimes H) \circ (c_{X,H} \otimes \delta_H))) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by naturality of c, coassociativity of δ_X and associativity of μ_H)

$=\mu_H \circ ((\varphi_H \circ (X \otimes \mu_H)) \otimes (\lambda_H \circ \varphi_H)) \circ (X \otimes ((H \otimes c_{X,H} \otimes H) \circ (c_{X,H} \otimes \delta_H))) \circ (\delta_X \otimes \eta_H \otimes \eta_H)$ (by (i))

$=\mu_H \circ (H \otimes \lambda_H) \circ (\varphi_H \otimes \varphi_H) \circ (X \otimes c_{X,H} \otimes H) \circ (\delta_X \otimes (\delta_H \circ \eta_H))$ (by naturality of c and unit property)

$=(id_H * \lambda_H) \circ \varphi_H \circ (X \otimes \eta_H)$ (by (ii))

$=\eta_H \circ \varepsilon_H \circ \varphi_H \circ (X \otimes \eta_H)$ (by (2.1))

$=\varepsilon_X \otimes \eta_H$ (by (ii) and (co)unit properties). □

Corollary 2.9. *Let $X = (X, \eta_X, \mu_X, \varepsilon_X, \delta_X, \lambda_X)$ and $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be Hopf algebras in C. If (H, φ_H) is a left X-module coalgebra and μ_H is a morphism of left X-modules, then (H, φ_H) is a left X-module algebra.*

In the braided setting the definition of Hopf brace is the following:

Definition 2.10. Let $H = (H, \varepsilon_H, \delta_H)$ be a coalgebra in C. Let's assume that there are two algebra structures (H, η_H^1, μ_H^1), (H, η_H^2, μ_H^2) defined on H and suppose that there exist two endomorphism of H denoted by λ_H^1 and λ_H^2. We will say that

$$(H, \eta_H^1, \mu_H^1, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^1, \lambda_H^2)$$

is a Hopf brace in C if:

(i) $H_1 = (H, \eta_H^1, \mu_H^1, \varepsilon_H, \delta_H, \lambda_H^1)$ is a Hopf algebra in C.
(ii) $H_2 = (H, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^2)$ is a Hopf algebra in C.
(iii) The following equality holds:

$$\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H),$$

where

$$\Gamma_{H_1} := \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H).$$

Following [7], a Hopf brace will be denoted by $\mathbb{H} = (H_1, H_2)$ or in a simpler way by $\mathbb{H}$.

Definition 2.11. If $\mathbb{H}$ is a Hopf brace in C, we will say that $\mathbb{H}$ is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$, i.e., if H_1 and H_2 are cocommutative Hopf algebras in C.

Note that by [18, Corollary 5], if H is a cocommutative Hopf algebra in the braided monoidal category C, the identity

$$c_{H,H} \circ c_{H,H} = id_{H \otimes H} \tag{2.7}$$

holds.

Definition 2.12. Given two Hopf braces $\mathbb{H}$ and $\mathbb{B}$ in C, a morphism f in C between the two underlying objects is called a morphism of Hopf braces if both $f : H_1 \to B_1$ and $f : H_2 \to B_2$ are Hopf algebra morphisms.

Hopf braces together with morphisms of Hopf braces form a category which we denote by HBr. Moreover, cocommutative Hopf braces constitute a full subcategory of HBr which we will denote by cocHBr.

Let $\mathbb{H}$ be a Hopf brace in C. Then

$$\eta_H^1 = \eta_H^2, \tag{2.8}$$

holds and, by [2, Lemma 1.7], in this braided setting the equality

$$\Gamma_{H_1} \circ (H \otimes \lambda_H^1) = \mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \tag{2.9}$$

also holds. Moreover, in our braided context [2, Lemma 1.8] and [2, Remark 1.9] hold and then we have that the algebra (H, η_H^1, μ_H^1) is a left H_2-module algebra with action Γ_{H_1} and μ_H^2 admits the following expression:

$$\mu_H^2 = \mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (\delta_H \otimes H). \tag{2.10}$$

In addition, by [2, Lemma 2.2], Γ_{H_1} is a coalgebra morphism when $\mathbb{H}$ is cocommutative.

Lemma 2.13. *Let $\mathbb{H}$ be a Hopf brace in C. The equality*

$$\Gamma_{H_1} \circ (H \otimes \lambda_H^2) \circ \delta_H = \lambda_H^1 \tag{2.11}$$

holds.

Proof. The equality follows by:

$$\begin{aligned}
&\Gamma_{H_1} \circ (H \otimes \lambda_H^2) \circ \delta_H \\
=&\mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes \lambda_H^2) \circ \delta_H \text{ (by definition of } \Gamma_{H_1}) \\
=&\mu_H^1 \circ (\lambda_H^1 \otimes (id_H * \lambda_H^2)) \circ \delta_H \text{ (by coassociativity of } \delta_H) \\
=&\lambda_H^1 \text{ (by (2.1) and (co)unit properties).}
\end{aligned}$$

□

To conclude this introductory section we will remember the notion of finite object in C, since they are going to be of special interest throughout Section 4.

Definition 2.14. An object P in C is finite if there exists an object P^*, called the dual of P, and a C-adjunction $P \otimes - \dashv P^* \otimes -$ between the tensor functors.

We will denote by a_P and b_P the unit and the counit of the previous C-adjunction, respectively. Finite objects in C constitute a full subcategory of C that we will denote by C^{f}. Note that, for every finite object P in C, we have a natural algebra structure in C over the tensor object $P^* \otimes P$ as we can see in the following lemma, whose proof is straightforward.

Lemma 2.15. *Let P be a finite object in C, then $P^* \otimes P$ is an algebra in C with product and unit given by*

$$\mu_{P^*\otimes P} := P^* \otimes b_P(K) \otimes P$$

and

$$\eta_{P^*\otimes P} := a_P(K),$$

respectively.

Moreover, it is going to be useful the following lemma.

Lemma 2.16. *If P is a finite object in C, then*

$$(c_{P,P^*} \otimes P) \circ (P \otimes a_P(K)) = (P^* \otimes c^{-1}_{P,P}) \circ (a_P(K) \otimes P). \tag{2.12}$$

Proof. The equality (2.12) follows by:

$$\begin{aligned} &(c_{P,P^*} \otimes P) \circ (P \otimes a_P(K)) \\ =&(P^* \otimes (c^{-1}_{P,P} \circ c_{P,P})) \circ (c_{P,P^*} \otimes P) \circ (P \otimes a_P(K)) \text{ (by the isomorphism condition for } c_{P,P}) \\ =&(P^* \otimes c^{-1}_{P,P}) \circ (a_P(K) \otimes P) \text{ (by naturality of } c \text{ and } c_{P,K} = id_K). \end{aligned}$$

□

3. Brace triples and Hopf braces

The aim of this part is to prove that we can characterise Hopf braces in C via another structures. This new structures will be known as brace triples.

Definition 3.1. Consider $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ a Hopf algebra in C with λ_H an isomorphism and let $\gamma_H : H \otimes H \to H$ and $T_H : H \to H$ be morphisms in C. We will say that (H, γ_H, T_H) is a brace triple if the following conditions hold:

- (i) $(\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = (\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)$.
- (ii) γ_H is a coalgebra morphism, i.e.:
  - (ii.1) $\delta_H \circ \gamma_H = (\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,
  - (ii.2) $\varepsilon_H \circ \gamma_H = \varepsilon_H \otimes \varepsilon_H$.
- (iii) $\gamma_H \circ (H \otimes \mu_H) = \mu_H \circ (\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)$.
- (iv) $\gamma_H \circ (H \otimes \gamma_H) = \gamma_H \circ ((\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)) \otimes H)$.
- (v) $\gamma_H \circ (\eta_H \otimes H) = id_H$.
- (vi) T_H is an isomorphism in C such that the following equalities are verified:
  - (vi.1) $\delta_H \circ T_H = c_{H,H} \circ (T_H \otimes T_H) \circ \delta_H$.
  - (vi.2) $\varepsilon_H \circ T_H = \varepsilon_H$
  - (vi.3) $\mu_H \circ (H \otimes \gamma_H) \circ ((\delta_H \circ T_H) \otimes H) = \mu_H \circ (H \otimes \gamma_H) \circ (((T_H \otimes T_H) \circ \delta_H) \otimes H)$.
  - (vi.4) $\gamma_H \circ (H \otimes T_H) \circ \delta_H = \lambda_H$.
  - (vi.5) $\gamma_H \circ (T_H \otimes H) \circ \delta_H = \lambda^{-1}_H \circ T_H$.

Remark 3.2. Given a brace triple (H, γ_H, T_H), conditions (ii) and (iii) of Definition 3.1 imply that

$$\gamma_H \circ (H \otimes \eta_H) = \varepsilon_H \otimes \eta_H \tag{3.1}$$

holds by Theorem 2.8.

Remark 3.3. Let (H, γ_H, T_H) be a brace triple. Note that condition (vi.5) of Definition 3.1 is equivalent to

$$\gamma_H \circ (H \otimes T_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H = \lambda_H^{-1}. \tag{3.2}$$

In fact, on the one side, suppose that (vi.5) of Definition 3.1 holds. Then, we have that:

$$\begin{aligned}
&\lambda_H^{-1}\\
=&\lambda_H^{-1} \circ T_H \circ T_H^{-1} \text{ (by the condition of isomorphism for } T_H)\\
=&\gamma_H \circ (T_H \otimes H) \circ \delta_H \circ T_H^{-1} \text{ (by (vi.5) of Definition 3.1)}\\
=&\gamma_H \circ (T_H \otimes H) \circ (T_H^{-1} \otimes T_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \text{ (by (vi.1) of Definition 3.1 and the}\\
&\text{isomorphism condition for } T_H \text{ and } c_{H,H})\\
=&\gamma_H \circ (H \otimes T_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \text{ (by the condition of isomorphism for } T_H).
\end{aligned}$$

On the other side, suppose now that (3.2) holds. Then,

$$\begin{aligned}
&\lambda_H^{-1} \circ T_H\\
=&\gamma_H \circ (H \otimes T_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \circ T_H \text{ (by (3.2))}\\
=&\gamma_H \circ (H \otimes T_H^{-1}) \circ c_{H,H}^{-1} \circ c_{H,H} \circ (T_H \otimes T_H) \circ \delta_H \text{ (by (vi.1) of Definition 3.1)}\\
=&\gamma_H \circ (T_H \otimes H) \circ \delta_H \text{ (by the isomorphism condition for } c_{H,H} \text{ and } T_H).
\end{aligned}$$

Definition 3.4. Let (H, γ_H, T_H) and (B, γ_B, T_B) be brace triples and $f\colon H \to B$ a morphism in C. We will say that f is a morphism of brace triples if f is a Hopf algebra morphism and

$$f \circ \gamma_H = \gamma_B \circ (f \otimes f) \tag{3.3}$$

holds.

Brace triples and their morphisms form a category which we will denote by BT.

Remark 3.5. Suppose that (H, γ_H, T_H) is a brace triple with H cocommutative. Under this condition, note that (i) of Definition 3.1 always holds and take also into account that (vi.1) becomes $\delta_H \circ T_H = (T_H \otimes T_H) \circ \delta_H$. This implies that (vi.3) always holds in the cocommutative setting. Moreover, $\lambda_H^{-1} = \lambda_H$ (due to $\lambda_H \circ \lambda_H = id_H$), so this implies that (vi.5) becomes $\gamma_H \circ (T_H \otimes H) \circ \delta_H = \lambda_H \circ T_H$. Therefore, as a consequence of (vi.4) and (vi.5), we obtain that

$$\gamma_H \circ (H \otimes T_H) \circ \delta_H = \lambda_H = \gamma_H \circ (T_H \otimes H) \circ \delta_H \circ T_H^{-1}.$$

Cocommutative brace triples constitute a full subcategory of BT which we will denote by cocBT.

Definition 3.6. Let $\mathbb{H}$ be a Hopf brace in C. We will say that $\mathbb{H}$ is an s-Hopf brace if the following conditions hold:

(i) $(\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = (\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)$.

(ii) λ_H^1 and λ_H^2 are isomorphisms in C such that the following conditions hold:

- (ii.1) $\mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ ((\delta_H \circ \lambda_H^2) \otimes H) = \mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (((\lambda_H^2 \otimes \lambda_H^2) \circ \delta_H) \otimes H)$.
- (ii.2) $\Gamma_{H_1} \circ (\lambda_H^2 \otimes H) \circ \delta_H = (\lambda_H^1)^{-1} \circ \lambda_H^2$.

With the obvious morphisms, s-Hopf braces constitute a full subcategory of HBr, and we will denote it by sHBr.

Remark 3.7. Let's assume that C is symmetric. Under this assumption, condition (i) of Definition 3.6 means that (H_1, Γ_{H_1}) is in the cocommutativity class of H_2 following the notion introduced in [1, Definition 2.1 and Definition 2.2].

Remark 3.8. Note that, for an s-Hopf brace $\mathbb{H}$, condition (ii.2) of Definition 3.6 is equivalent to

$$\Gamma_{H_1} \circ (H \otimes (\lambda_H^2)^{-1}) \circ c_{H,H}^{-1} \circ \delta_H = (\lambda_H^1)^{-1}. \tag{3.4}$$

Indeed, suppose that (ii.2) of Definition 3.6 holds, then:

$$\begin{aligned}
&(\lambda_H^1)^{-1}\\
=&(\lambda_H^1)^{-1} \circ \lambda_H^2 \circ (\lambda_H^2)^{-1} \text{ (by the condition of isomorphism for } \lambda_H^2)\\
=&\Gamma_{H_1} \circ (\lambda_H^2 \otimes H) \circ \delta_H \circ (\lambda_H^2)^{-1} \text{ (by (ii.2) of Definition 3.6)}\\
=&\Gamma_{H_1} \circ (\lambda_H^2 \otimes H) \circ ((\lambda_H^2)^{-1} \otimes (\lambda_H^2)^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \text{ (by (2.3) and the isomorphism condition}\\
&\text{for } \lambda_H^2 \text{ and } c_{H,H})\\
=&\Gamma_{H_1} \circ (H \otimes (\lambda_H^2)^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \text{ (by the condition of isomorphism for } \lambda_H^2).
\end{aligned}$$

On the other hand, we have that:

$$\begin{aligned}
&(\lambda_H^1)^{-1} \circ \lambda_H^2\\
=&\Gamma_{H_1} \circ (H \otimes (\lambda_H^2)^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \circ \lambda_H^2 \text{ (by (3.4))}\\
=&\Gamma_{H_1} \circ (H \otimes (\lambda_H^2)^{-1}) \circ c_{H,H}^{-1} \circ c_{H,H} \circ (\lambda_H^2 \otimes \lambda_H^2) \circ \delta_H \text{ (by (2.3))}\\
=&\Gamma_{H_1} \circ (\lambda_H^2 \otimes H) \circ \delta_H \text{ (by the isomorphism condition for } \lambda_H^2 \text{ and } c_{H,H}).
\end{aligned}$$

Remark 3.9. Consider $\mathbb{H}$ a cocommutative s-Hopf brace. Under cocommutativity assumption, note that (i) of Definition 3.6 always holds. In addition, $\delta_H \circ \lambda_H^2 = (\lambda_H^2 \otimes \lambda_H^2) \circ \delta_H$, so (ii.1) always holds too. Moreover, under cocommutativity conditions, λ_H^k is an involution for all $k = 1, 2$. Therefore, condition (ii.2) of Definition 3.6 is satisfied. Indeed:

$$\begin{aligned}
&\Gamma_{H_1} \circ (\lambda_H^2 \otimes H) \circ \delta_H\\
=&\mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ ((\delta_H \circ \lambda_H^2) \otimes H) \circ \delta_H \text{ (by definition of } \Gamma_{H_1})\\
=&\mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (((\lambda_H^2 \otimes \lambda_H^2) \circ \delta_H) \otimes H) \circ \delta_H \text{ (by (2.3) and cocommutativity of } \delta_H)\\
=&\mu_H^1 \circ ((\lambda_H^1 \circ \lambda_H^2) \otimes (\lambda_H^2 * id_H)) \circ \delta_H \text{ (by coassociativity of } \delta_H)\\
=&\mu_H^1 \circ ((\lambda_H^1 \circ \lambda_H^2) \otimes (\eta_H \circ \varepsilon_H)) \circ \delta_H \text{ (by (2.1))}\\
=&\lambda_H^1 \circ \lambda_H^2 \text{ (by (co)unit property).}
\end{aligned}$$

So, under cocommutativity supposition, every Hopf brace is an s-Hopf brace, that is to say, $\mathsf{sHBr} = \mathsf{cocHBr}$.

In this first result, we will prove that every brace triple induces an s-Hopf brace in C.

Theorem 3.10. *Let (H, γ_H, T_H) be a brace triple in C. Then $\mathbb{H}_{\mathsf{BT}} = (H, H_{\mathsf{BT}})$ is an s-Hopf brace in C being H_{BT} the Hopf algebra structure defined by $H_{\mathsf{BT}} = (H, \eta_H, \mu_H^{\mathsf{BT}}, \varepsilon_H, \delta_H, T_H)$, where $\mu_H^{\mathsf{BT}} := \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)$.*

Proof. At first we will prove that H_{BT} is a Hopf algebra in C. Note that we already know that $(H, \varepsilon_H, \delta_H)$ is a coalgebra in C and that η_H is a coalgebra morphism. We begin by proving the unit property for μ_H^{BT}. Indeed, on the one side,

$$\begin{aligned}
&\mu_H^{\mathsf{BT}} \circ (\eta_H \otimes H)\\
=&\mu_H \circ (H \otimes \gamma_H) \circ ((\delta_H \circ \eta_H) \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}})\\
=&\mu_H \circ (\eta_H \otimes (\gamma_H \circ (\eta_H \otimes H))) \text{ (by the condition of coalgebra morphism for } \eta_H)\\
=&\mu_H \circ (\eta_H \otimes H) \text{ (by (v) of Definition 3.1)}\\
=&id_H \text{ (by unit property),}
\end{aligned}$$

and, on the other side,

$$
\begin{aligned}
&\mu_H^{\mathsf{BT}} \circ (H \otimes \eta_H) \\
=&\mu_H \circ (H \otimes (\gamma_H \circ (H \otimes \eta_H))) \circ \delta_H \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=&\mu_H \circ (H \otimes \varepsilon_H \otimes H) \circ (\delta_H \otimes \eta_H) \text{ (by (3.1))} \\
=&id_H \text{ (by (co)unit property).}
\end{aligned}
$$

The associativity of μ_H^{BT} follows by

$$
\begin{aligned}
&\mu_H^{\mathsf{BT}} \circ (\mu_H^{\mathsf{BT}} \otimes H) \\
=&\mu_H \circ (H \otimes \gamma_H) \circ ((\delta_H \circ \mu_H) \otimes H) \circ (H \otimes \gamma_H \otimes H) \circ (\delta_H \otimes H \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=&\mu_H \circ (H \otimes \gamma_H) \circ (((\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (\delta_H \circ \gamma_H))) \otimes H) \circ (\delta_H \otimes H \\
&\otimes H) \text{ (by the condition of coalgebra morphism for } \mu_H) \\
=&\mu_H \circ (H \otimes \gamma_H) \circ (((\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H)) \otimes H) \circ (\delta_H \otimes ((\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \\
&\otimes H) \circ (\delta_H \otimes \delta_H))) \otimes H) \circ (\delta_H \otimes H \otimes H) \text{ (by (ii.1) of Definition 3.1)} \\
=&\mu_H \circ (\mu_H \otimes (\gamma_H \circ (\mu_H \otimes H))) \circ (H \otimes ((\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)) \otimes \gamma_H \\
&\otimes H) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ (((H \otimes \delta_H) \circ \delta_H) \otimes \delta_H \otimes H) \text{ (by naturality of } c \text{ and} \\
&\text{coassociativity of } \delta_H) \\
=&\mu_H \circ (\mu_H \otimes (\gamma_H \circ (\mu_H \otimes H))) \circ (H \otimes ((\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes \gamma_H \otimes H) \\
&\circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ (((H \otimes \delta_H) \circ \delta_H) \otimes \delta_H \otimes H) \text{ (by (i) of Definition 3.1)} \\
=&\mu_H \circ ((\mu_H \circ (H \otimes \gamma_H)) \otimes (\gamma_H \circ ((\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)) \otimes H))) \circ (H \otimes ((H \otimes c_{H,H}) \\
&\circ (\delta_H \otimes H)) \otimes H \otimes H) \circ (\delta_H \otimes \delta_H \otimes H) \text{ (by naturality of } c \text{ and coassociativity of } \delta_H) \\
=&\mu_H \circ (H \otimes (\mu_H \circ (\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H))) \circ (H \otimes H \otimes H \otimes \gamma_H) \\
&\circ (\delta_H \otimes \delta_H \otimes H) \text{ (by (iv) of Definition 3.1 and associativity of } \mu_H) \\
=&\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes (\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H))) \text{ (by (iii) of Definition 3.1)} \\
=&\mu_H^{\mathsf{BT}} \circ (H \otimes \mu_H^{\mathsf{BT}}) \text{ (by definition of } \mu_H^{\mathsf{BT}}).
\end{aligned}
$$

Also, μ_H^{BT} is a coalgebra morphism. On the one hand, by the condition of coalgebra morphism for μ_H, (ii.2) of Definition 3.1 and the counit property, it is straightforward to compute that $\varepsilon_H \circ \mu_H^{\mathsf{BT}} = \varepsilon_H \otimes \varepsilon_H$ and, on the other hand,

$$
\begin{aligned}
&\delta_H \circ \mu_H^{\mathsf{BT}} \\
=&\delta_H \circ \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=&(\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (\delta_H \circ \gamma_H)) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra} \\
&\text{morphism for } \mu_H) \\
=&(\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes ((\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H))) \\
&\circ (\delta_H \otimes H) \text{ (by (ii.1) of Definition 3.1)} \\
=&(\mu_H \otimes \mu_H) \circ (H \otimes ((\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)) \otimes \gamma_H) \circ (H \otimes H \otimes c_{H,H} \\
&\otimes H) \circ (((H \otimes \delta_H) \circ \delta_H) \otimes \delta_H) \text{ (by naturality of } c \text{ and coassociativity of } \delta_H) \\
=&(\mu_H \otimes \mu_H) \circ (H \otimes ((\gamma_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes \gamma_H) \circ (H \otimes H \otimes c_{H,H} \otimes H) \\
&\circ (((H \otimes \delta_H) \circ \delta_H) \otimes \delta_H) \text{ (by (i) of Definition 3.1)} \\
=&((\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)) \otimes (\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H))) \circ (H \otimes c_{H,H} \otimes H) \\
&\circ (\delta_H \otimes \delta_H) \text{ (by coassociativity of } \delta_H \text{ and naturality of } c) \\
=&(\mu_H^{\mathsf{BT}} \otimes \mu_H^{\mathsf{BT}}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by definition of } \mu_H^{\mathsf{BT}}).
\end{aligned}
$$

So, H_{BT} is a bialgebra in C. From now on, we will denote by $*_{\mathsf{BT}}$ the convolution in $\mathrm{Hom}(H, H_{\mathsf{BT}})$.

The conditions for T_H to be the antipode for H_{BT} follows from the following facts. First note that

$$
\begin{aligned}
& id_H *_{\mathsf{BT}} T_H \\
=& \mu_H^{\mathsf{BT}} \circ (H \otimes T_H) \circ \delta_H \text{ (by definition of } *_{\mathsf{BT}}) \\
=& \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes T_H) \circ \delta_H \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=& \mu_H \circ (H \otimes (\gamma_H \circ (H \otimes T_H) \circ \delta_H)) \circ \delta_H \text{ (by coassociativity of } \delta_H) \\
=& id_H * \lambda_H \text{ (by (vi.4) of Definition 3.1)} \\
=& \varepsilon_H \otimes \eta_H \text{ (by (2.1))}.
\end{aligned}
$$

On the other side,

$$
\begin{aligned}
& T_H *_{\mathsf{BT}} id_H \\
=& \mu_H^{\mathsf{BT}} \circ (T_H \otimes H) \circ \delta_H \text{ (by definition of } *_{\mathsf{BT}}) \\
=& \mu_H \circ (H \otimes \gamma_H) \circ ((\delta_H \circ T_H) \otimes H) \circ \delta_H \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=& \mu_H \circ (H \otimes \gamma_H) \circ (((T_H \otimes T_H) \circ \delta_H) \otimes H) \circ \delta_H \text{ (by (vi.3) of Definition 3.1)} \\
=& \mu_H \circ (T_H \otimes (\gamma_H \circ (T_H \otimes H) \circ \delta_H)) \circ \delta_H \text{ (by coassociativity of } \delta_H) \\
=& \mu_H \circ (T_H \otimes (\lambda_H^{-1} \circ T_H)) \circ \delta_H \text{ (by (vi.5) of Definition 3.1)} \\
=& \mu_H \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H \circ T_H \text{ (by (vi.1) of Definition 3.1 and the condition of} \\
& \text{isomorphism for } c_{H,H}) \\
=& \eta_H \circ \varepsilon_H \circ T_H \text{ (by (2.1) for } H^{cop}) \\
=& \varepsilon_H \otimes \eta_H \text{ (by (vi.2) of Definition 3.1)}
\end{aligned}
$$

Therefore, H_{BT} is a Hopf algebra in C.

To conclude the proof we have to show that (iii) of Definition 2.10 holds. Note that

$$\Gamma_H^{\mathsf{BT}} = \gamma_H \tag{3.5}$$

holds. Indeed,

$$
\begin{aligned}
& \Gamma_H^{\mathsf{BT}} \\
=& \mu_H \circ (\lambda_H \otimes \mu_H^{\mathsf{BT}}) \circ (\delta_H \otimes H) \text{ (by definition of } \Gamma_H^{\mathsf{BT}}) \\
=& \mu_H \circ (\lambda_H \otimes (\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H))) \circ (\delta_H \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=& \mu_H \circ ((\lambda_H * id_H) \otimes \gamma_H) \circ (\delta_H \otimes H) \text{ (by associativity of } \mu_H \text{ and coassociativity of } \delta_H) \\
=& \gamma_H \text{ (by (2.1) and (co)unit property)}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& \mu_H \circ (\mu_H^{\mathsf{BT}} \otimes \Gamma_H^{\mathsf{BT}}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H) \\
=& \mu_H \circ ((\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)) \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H) \\
& \text{(by definition of } \mu_H^{\mathsf{BT}} \text{ and (3.5))} \\
=& \mu_H \circ (H \otimes (\mu_H \circ (\gamma_H \otimes \gamma_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H))) \circ (\delta_H \otimes H \otimes H) \\
& \text{(by associativity of } \mu_H \text{ and coassociativity of } \delta_H) \\
=& \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes \mu_H) \text{ (by (iii) of Definition 3.1)} \\
=& \mu_H^{\mathsf{BT}} \circ (H \otimes \mu_H) \text{ (by definition of } \mu_H^{\mathsf{BT}}).
\end{aligned}
$$

Finally, by (3.5) and thanks to axioms (i), (vi), (vi.3) and (vi.5) of Definition 3.1, conditions (i), (ii), (ii.1) and (ii.2) of Definition 3.6 are obvious. □

Remark 3.11. When H is a cocommutative Hopf algebra, we recover [10, Remark 4.5].

Corollary 3.12. *Let (H, γ_H, T_H) be a brace triple in C. Then, (H, γ_H) is a left H_{BT}-module algebra.*

Proof. Thanks to the fact that $\mathbb{H}_{\mathsf{BT}} = (H, H_{\mathsf{BT}})$ is a Hopf brace, we know that $(H, \Gamma_H^{\mathsf{BT}})$ is a left H_{BT}-module algebra. Due to being $\Gamma_H^{\mathsf{BT}} = \gamma_H$, as we have proved in the previous result, we conclude that (H, γ_H) is a left H_{BT}-module algebra. □

Remark 3.13. Let's assume that C is symmetric. Under this assumption and thanks to the previous corollary, axiom (i) of Definition 3.1 means that (H, γ_H) is in the cocommutativity class of H_{BT}.

Corollary 3.14. *Let* (H, γ_H, T_H) *be a cocommutative brace triple, then*

$$T_H \circ T_H = id_H. \tag{3.6}$$

Therefore, conditions (vi.4) *and* (vi.5) *of Definition 3.1 are equivalent in the cocommutative setting.*

Proof. As was proved in Theorem 3.10, T_H is the antipode for the Hopf algebra H_{BT}. Then, if H is cocommutative, H_{BT} is cocommutative too and, as a consequence, (3.6) holds. □

Corollary 3.15. *If* $f\colon (H, \gamma_H, T_H) \to (B, \gamma_B, T_B)$ *is a morphism of brace triples in* C, *then*

$$f \circ T_H = T_B \circ f.$$

Proof. It is enough to see that $f\colon H_{\mathsf{BT}} \to B_{\mathsf{BT}}$ is a Hopf algebra morphism. Due to the fact that H_{BT} and B_{BT} are Hopf algebras in C with the same underlying coalgebra structure and the same unit morphisms as H and B, respectively, it is enough to prove that f is compatible with the products μ_H^{BT} and μ_B^{BT}. Indeed,

$$\begin{aligned}
& f \circ \mu_H^{\mathsf{BT}} \\
=& f \circ \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}}) \\
=& \mu_B \circ (f \otimes f) \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H) \text{ (by the condition of algebra morphism for } f\colon H \to B) \\
=& \mu_B \circ (B \otimes \gamma_B) \circ (((f \otimes f) \circ \delta_H) \otimes f) \text{ (by (3.3))} \\
=& \mu_B \circ (B \otimes \gamma_B) \circ (\delta_B \otimes B) \circ (f \otimes f) \text{ (by the condition of coalgebra morphism for } f) \\
=& \mu_B^{\mathsf{BT}} \circ (f \otimes f) \text{ (by definition of } \mu_B^{\mathsf{BT}}).
\end{aligned}$$

So, due to being $f\colon H_{\mathsf{BT}} \to B_{\mathsf{BT}}$ a Hopf algebra morphism in C, we can apply (2.2) what concludes the proof. □

Theorem 3.10 implies that there exist a functor $F\colon \mathsf{BT} \longrightarrow \mathsf{sHBr}$ defined on objects by $F((H, \gamma_H, T_H)) = \mathbb{H}_{\mathsf{BT}}$ and on morphisms by the identity. To see that F is well-defined on morphisms, we have to prove that if f is a morphism in BT, then f is a morphism in HBr. To verify this fact, it is enough to compute that $f \circ \mu_H^{\mathsf{BT}} = \mu_B^{\mathsf{BT}} \circ (f \otimes f)$, what we have just seen in the proof of Corollary 3.15.

Moreover, we can also construct a brace triple from every s-Hopf brace. First of all, we are going to prove the following lemma.

Lemma 3.16. *Let* $\mathbb{H}$ *be a Hopf brace in* C. *If* Γ_{H_1} *satisfies condition* (i) *of Definition 3.6, then* Γ_{H_1} *is a coalgebra morphism.*

Proof. On the one hand, it is straightforward to see that $\varepsilon_H \circ \Gamma_{H_1} = \varepsilon_H \otimes \varepsilon_H$. Let's see that $\delta_H \circ \Gamma_{H_1} = (\Gamma_{H_1} \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$. Indeed:

$$\begin{aligned}
&\delta_H \circ \Gamma_{H_1}\\
=&\delta_H \circ \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H) \text{ (by definition of } \Gamma_{H_1})\\
=&(\mu_H^1 \otimes \mu_H^1) \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \lambda_H^1) \otimes (\delta_H \circ \mu_H^2)) \circ (\delta_H \otimes H) \text{ (by the condition of}\\
&\text{coalgebra morphism for } \mu_H^1)\\
=&(\mu_H^1 \otimes \mu_H^1) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ (\lambda_H^1 \otimes \lambda_H^1) \circ \delta_H) \otimes ((\mu_H^2 \otimes \mu_H^2) \circ (H \otimes c_{H,H}\\
&\otimes H) \circ (\delta_H \otimes \delta_H))) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra morphism for } \mu_H^2 \text{ and (2.3))}\\
=&(H \otimes (\mu_H^1 \circ (\lambda_H^1 \otimes H))) \circ (((\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)) \otimes \mu_H^2)\\
&\circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by coassociativity of } \delta_H\text{, naturality of } c \text{ and definition of } \Gamma_{H_1})\\
=&(H \otimes (\mu_H^1 \circ (\lambda_H^1 \otimes H))) \circ (((\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes \mu_H^2) \circ (H \otimes c_{H,H}\\
&\otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by (i) of Definition 3.6)}\\
=&(\Gamma_{H_1} \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by coassociativity of } \delta_H \text{ and naturality of } c).
\end{aligned}$$
□

Theorem 3.17. *If* $\mathbb{H}$ *is an object in* sHBr*, then* $(H_1, \Gamma_{H_1}, \lambda_H^2)$ *is a brace triple.*

Proof. It is a consequence of the following facts: By (i) of Definition 3.6 and previous lemma, conditions (i) and (ii) of Definition 3.1 hold. Moreover, it is well known that (H_1, Γ_{H_1}) is a left H_2-module algebra. This property together with (2.10) implies that axioms (iii), (iv) and (v) of Definition 3.1 also hold. Identities (vi), (vi.1) and (vi.2) of Definition 3.1 follow by (ii) of Definition 3.6 and equations (2.3) and (2.4). The remaining axioms, (vi.3), (vi.4) and (vi.5) of Definition 3.1, are consequence of (ii.1) of Definition 3.6, equation (2.11) and (ii.2) of Definition 3.6, respectively. □

As a consequence of the previous theorem, we obtain a functor $G\colon \mathsf{sHBr} \longrightarrow \mathsf{BT}$ acting on objects by $G(\mathbb{H}) = (H_1, \Gamma_{H_1}, \lambda_H^2)$ and on morphisms by the identity. To see that G_1 is well-defined on morphisms, we have to compute that if $f\colon \mathbb{H} \to \mathbb{B}$ is a morphism in sHBr, then $f \circ \Gamma_{H_1} = \Gamma_{B_1} \circ (f \otimes f)$. Indeed:

$$\begin{aligned}
&f \circ \Gamma_{H_1}\\
=&f \circ \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H) \text{ (by definition of } \Gamma_{H_1})\\
=&\mu_B^1 \circ ((f \circ \lambda_H^1) \otimes (f \circ \mu_H^2)) \circ (\delta_H \otimes H) \text{ (by the condition of algebra morphism for } f\colon H_1 \to B_1)\\
=&\mu_B^1 \circ (\lambda_B^1 \otimes \mu_B^2) \circ (((f \otimes f) \circ \delta_H) \otimes f) \text{ (by the condition of algebra morphism for } f\colon H_2 \to B_2\\
&\text{and (2.2))}\\
=&\mu_B^1 \circ (\lambda_B^1 \otimes \mu_B^2) \circ (\delta_B \otimes B) \circ (f \otimes f) \text{ (by the condition of coalgebra morphism for } f)\\
=&\Gamma_{B_1} \circ (f \otimes f) \text{ (by definition of } \Gamma_{B_1}).
\end{aligned}$$

Next theorem is the main result of this section. We will prove that functors F and G induce a categorical isomorphism between sHBr and BT.

Theorem 3.18. *The categories* sHBr *and* BT *are isomorphic.*

Proof. First of all, it results clear that $G \circ F = \mathsf{id}_{\mathsf{BT}}$. Indeed, consider (H, γ_H, T_H) a brace triple, we obtain that:

$$\begin{aligned}
&(G \circ F)((H, \gamma_H, T_H))\\
=&G(\mathbb{H}_{\mathsf{BT}}) \text{ (by definition of functor } F)\\
=&(H, \Gamma_H^{\mathsf{BT}}, T_H) \text{ (by definition of functor } G)\\
=&(H, \gamma_H, T_H) \text{ (by (3.5))}.
\end{aligned}$$

On the other side, consider $\mathbb{H}$ an object in sHBr. We have that:

$$
\begin{aligned}
&(F \circ G)(\mathbb{H})\\
=&F((H_1, \Gamma_{H_1}, \lambda_H^2)) \text{ (by definition of functor } G)\\
=&(H_1, H_{\mathsf{BT}}) \text{ (by definition of functor } F),
\end{aligned}
$$

where, in this particular case,

$$
\begin{aligned}
&\mu_H^{\mathsf{BT}}\\
=&\mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (\delta_H \otimes H) \text{ (by definition of } \mu_H^{\mathsf{BT}})\\
=&\mu_H^2 \text{ (by (2.10))}.
\end{aligned}
$$

Therefore, $H_{\mathsf{BT}} = H_2$, and then $F \circ G = \mathsf{id}_{\mathsf{sHBr}}$. □

Corollary 3.19. *Categories* cocHBr *and* cocBT *are isomorphic.*

Proof. It is enough to take into account Remarks 3.5 and 3.9 and the previous theorem. The isomorphism in this case is given by functors F' and G' which are the restrictions of F and G to cocBT and cocHBr, respectively.

$$
\begin{tikzcd}
\mathsf{BT} \arrow[r, shift left, "F"] \arrow[r, phantom, "\simeq"] & \mathsf{HBr} \arrow[l, shift left, "G"] \\
\mathsf{cocBT} \arrow[u, hook] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\simeq"] & \mathsf{cocHBr} \arrow[u, hook] \arrow[l, shift left, "G'"]
\end{tikzcd}
$$

□

4. Post-Hopf algebras and Hopf braces

In this section we introduce the notion of post-Hopf algebra in the braided monoidal context. In particular, for the category of vector spaces over a field $\mathbb{K}$, we obtain the concept of post-Hopf algebra presented in [14], where the authors get an equivalence between Hopf braces and these objects under cocommutativity assumption. Besides being working in a more general setting, in this section we prove that the categories of finite cocommutative Hopf braces and cocommutative post-Hopf algebras satisfying condition (4.13) are isomoprhic. As a consequence of this result together with Corollary 3.19, we also deduce that finite cocommutative brace triples are isomorphic to post-Hopf algebras verifying (4.13).

Definition 4.1. A post-Hopf algebra in C is a pair (H, m_H) where H is a finite Hopf algebra in C and $m_H\colon H \otimes H \to H$ is a morphism in C that satisfies the following conditions:

(i) m_H is a coalgebra morphism, which means that the following equalities are satisfied:
  (i.1) $\delta_H \circ m_H = (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,
  (i.2) $\varepsilon_H \circ m_H = \varepsilon_H \otimes \varepsilon_H$.
(ii) $m_H \circ (H \otimes m_H) = m_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes H)$, which is called the "weighted" associativity.
(iii) $m_H \circ (H \otimes \mu_H) = \mu_H \circ (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)$.
(iv) The morphism

$$\alpha_H := (H^* \otimes m_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K))\colon H \to H^* \otimes H$$

is convolution invertible in $\mathrm{Hom}(H, H^* \otimes H)$, which means that there exists a morphism $\beta_H\colon H \to H^* \otimes H$ such that

$$(H^* \otimes b_H(K) \otimes H) \circ (\alpha_H \otimes \beta_H) \circ \delta_H = \varepsilon_H \otimes a_H(K) = (H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ \delta_H.$$

Remark 4.2. Given a post-Hopf algebra (H, m_H), conditions (i) and (iii) of Definition 4.1 imply that

$$m_H \circ (H \otimes \eta_H) = \varepsilon_H \otimes \eta_H \tag{4.1}$$

holds by Theorem 2.8.

Definition 4.3. Let (H, m_H) and (B, m_B) be post-Hopf algebras in C and let $f\colon H \to B$ be a morphism in C. We will say that f is a post-Hopf algebra morphism if f is a Hopf algebra morphism and the condition

$$f \circ m_H = m_B \circ (f \otimes f) \tag{4.2}$$

holds.

Post-Hopf algebras and their morphisms form a category and we will denote it by $\mathsf{Post\text{-}Hopf}$. When H is cocommutative, we will say that (H, m_H) is a cocommutative post-Hopf algebra in C. Cocommutative post-Hopf algebras constitute a full subcategory of $\mathsf{Post\text{-}Hopf}$ which we will denote by $\mathsf{cocPost\text{-}Hopf}$.

Lemma 4.4. *Let (H, m_H) be an object in $\mathsf{Post\text{-}Hopf}$. It is verified that*

$$m_H \circ c^{-1}_{H,H} = (b_H(K) \otimes H) \circ (H \otimes \alpha_H). \tag{4.3}$$

Therefore,

$$m_H = (b_H(K) \otimes H) \circ (H \otimes \alpha_H) \circ c_{H,H}. \tag{4.4}$$

Proof. Let's start proving (4.3):

$$\begin{aligned}
&(b_H(K) \otimes H) \circ (H \otimes \alpha_H)\\
=&(b_H(K) \otimes m_H) \circ (H \otimes ((c_{H,H^*} \otimes H) \circ (H \otimes a_H(K)))) \text{ (by definition of } \alpha_H)\\
=&(b_H(K) \otimes (m_H \circ c^{-1}_{H,H})) \circ (H \otimes a_H(K) \otimes H) \text{ (by (2.12))}\\
=&m_H \circ c^{-1}_{H,H} \text{ (by the adjunction properties).}
\end{aligned}$$

So, composing on the right with $c_{H,H}$, we obtain (4.4). □

Lemma 4.5. *Let (H, m_H) be an object in $\mathsf{Post\text{-}Hopf}$, then*

$$m_H \circ (\eta_H \otimes H) = id_H. \tag{4.5}$$

Proof. First of all, note that the morphism $m_H \circ (\eta_H \otimes H)$ is idempotent. Indeed,

$$\begin{aligned}
&m_H \circ (H \otimes m_H) \circ (\eta_H \otimes \eta_H \otimes H)\\
=&m_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H) \circ (\eta_H \otimes \eta_H)) \otimes H) \text{ (by (ii) of Definition 4.1)}\\
=&m_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\eta_H \otimes \eta_H \otimes \eta_H)) \otimes H) \text{ (by the condition of coalgebra}\\
&\text{morphism for } \eta_H)\\
=&m_H \circ ((m_H \circ (\eta_H \otimes \eta_H)) \otimes H) \text{ (by unit property)}\\
=&(\varepsilon_H \circ \eta_H) \otimes (m_H \circ (\eta_H \otimes H)) \text{ (by (4.1))}\\
=&m_H \circ (\eta_H \otimes H) \text{ (by (co)unit property).}
\end{aligned}$$

Therefore, the equality

$$(b_H(K) \otimes b_H(K) \otimes H) \circ (H \otimes ((\alpha_H \otimes \alpha_H) \circ (\eta_H \otimes \eta_H))) = (b_H(K) \otimes H) \circ (H \otimes (\alpha_H \circ \eta_H)) \tag{4.6}$$

holds because

$$\begin{aligned}
&(b_H(K) \otimes b_H(K) \otimes H) \circ (H \otimes ((\alpha_H \otimes \alpha_H) \circ (\eta_H \otimes \eta_H)))\\
=&m_H \circ (H \otimes m_H) \circ (\eta_H \otimes \eta_H \otimes H) \text{ (by (4.3) and naturality of } c)\\
=&m_H \circ (\eta_H \otimes H) \text{ (by the idempotent character of } m_H \circ (\eta_H \otimes H))\\
=&(b_H(K) \otimes H) \circ (H \otimes (\alpha_H \circ \eta_H)) \text{ (by (4.3) and naturality of } c).
\end{aligned}$$

Then, using the previous equalities and the finite character of H, we have that

$$
\begin{aligned}
&(H^* \otimes b_H(K) \otimes H) \circ (\alpha_H \otimes \alpha_H) \circ (\eta_H \otimes \eta_H) \\
=&(H^* \otimes b_H(K) \otimes b_H(K) \otimes H) \circ (a_H(K) \otimes (\alpha_H \circ \eta_H) \otimes (\alpha_H \circ \eta_H)) \text{ (by the adjunction properties)} \\
=&(H^* \otimes b_H(K) \otimes H) \circ (a_H(K) \otimes (\alpha_H \circ \eta_H)) \text{ (by (4.6))} \\
=&\alpha_H \circ \eta_H \text{ (by the adjunction properties).}
\end{aligned}
$$

As a consequence, if β_H is the convolution inverse of α_H in $\mathrm{Hom}(H, H^* \otimes H)$, we deduce the following:

$$
\begin{aligned}
&(H^* \otimes b_H(K) \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H \otimes \alpha_H) \circ ((\delta_H \circ \eta_H) \otimes \eta_H) \\
=&(H^* \otimes b_H(K) \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H \otimes \alpha_H) \circ (\eta_H \otimes \eta_H \otimes \eta_H) \text{ (by the condition of coalgebra morphism for } \eta_H) \\
=&(H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ (\eta_H \otimes \eta_H) \text{ (by (4.6))} \\
=&(H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ \delta_H \circ \eta_H \text{ (by the condition of coalgebra morphism for } \eta_H) \\
=&(\varepsilon_H \circ \eta_H) \otimes a_H(K) \text{ (by (iv) of Definition 4.1)} \\
=&a_H(K) \text{ (by the (co)unit properties)}
\end{aligned}
$$

and, on the other hand:

$$
\begin{aligned}
&(H^* \otimes b_H(K) \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H \otimes \alpha_H) \circ ((\delta_H \circ \eta_H) \otimes \eta_H) \\
=&(\varepsilon_H \circ \eta_H) \otimes ((H^* \otimes b_H(K) \otimes H) \circ (a_H(K) \otimes (\alpha_H \circ \eta_H))) \text{ (by (iv) of Definition 4.1)} \\
=&\alpha_H \circ \eta_H \text{ (by (co)unit properties and the adjunction properties).}
\end{aligned}
$$

So, by the two previous equalities, we obtain that

$$\alpha_H \circ \eta_H = a_H(K). \tag{4.7}$$

Therefore, we conclude the proof as follows:

$$
\begin{aligned}
&id_H \\
=&(b_H(K) \otimes H) \circ (H \otimes a_H(K)) \text{ (by the adjunction properties)} \\
=&(b_H(K) \otimes H) \circ (H \otimes (\alpha_H \circ \eta_H)) \text{ (by (4.7))} \\
=&(b_H(K) \otimes m_H) \circ (H \otimes ((c_{H,H^*} \otimes H) \circ (\eta_H \otimes a_H(K)))) \text{ (by definition of } \alpha_H) \\
=&(b_H(K) \otimes (m_H \circ c^{-1}_{H,H})) \circ (H \otimes a_H(K) \otimes \eta_H) \text{ (by (2.12))} \\
=&m_H \circ (\eta_H \otimes H) \text{ (by naturality of } c \text{ and the adjunction properties).} \qquad \square
\end{aligned}
$$

The goal of the following results will consist of building a post-Hopf algebra from a brace triple. Suppose that (H, γ_H, T_H) is a brace triple in C with H finite. Note that conditions (ii), (iii) and (iv) of Definition 3.1 imply that γ_H satisfies (i), (iii) and (ii) of Definition 4.1, respectively. So, in order to construct a post-Hopf algebra from a brace triple, it is enough to prove that $\alpha_H = (H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K))$ is convolution invertible in $\mathrm{Hom}(H, H^* \otimes H)$.

Theorem 4.6. *Let (H, γ_H, T_H) be a brace triple in C with H finite. The morphism $\alpha_H = (H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K))$ is invertible for the convolution in* $\mathrm{Hom}(H, H^* \otimes H)$ *with inverse*

$$\beta_H := \alpha_H \circ T_H^{-1}.$$

Proof. On the one side,

$(H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ \delta_H$

$=(H^* \otimes b_H(K) \otimes H) \circ (((H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (T_H^{-1} \otimes a_H(K))) \otimes ((H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K)))) \circ \delta_H$ (by definition of α_H and β_H)

$=(H^* \otimes (\gamma_H \circ c_{H,H}^{-1} \circ (\gamma_H \otimes H))) \circ (((c_{H,H^*} \otimes H) \circ (T_H^{-1} \otimes a_H(K))) \otimes H) \circ \delta_H$ (by (2.12) and the adjunction properties)

$=(H^* \otimes (\gamma_H \circ (H \otimes \gamma_H))) \circ (((H^* \otimes c_{H,H}^{-1}) \circ (c_{H,H^*} \otimes H) \circ (T_H^{-1} \otimes c_{H,H^*})) \otimes H) \circ (\delta_H \otimes a_H(K))$ (by naturality of c)

$=(H^* \otimes \gamma_H) \circ (((H^* \otimes \mu_H^{\mathsf{BT}}) \circ (c_{H,H^*} \otimes H) \circ (H \otimes c_{H,H^*})) \otimes H) \circ ((c_{H,H}^{-1} \circ (T_H^{-1} \otimes H) \circ \delta_H) \otimes a_H(K))$ (by naturality of c and (iv) of Definition 3.1)

$=(H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ ((\mu_H^{\mathsf{BT}} \circ c_{H,H}^{-1} \circ (T_H^{-1} \otimes H) \circ \delta_H) \otimes a_H(K))$ (by naturality of c)

$=(H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ ((\eta_H \circ \varepsilon_H) \otimes a_H(K))$ (by (2.1) for H_{BT}^{cop})

$=\varepsilon_H \otimes a_H(K)$ (by naturality of c and (v) of Definition 3.1).

On the other side,

$(H^* \otimes b_H(K) \otimes H) \circ (\alpha_H \otimes \beta_H) \circ \delta_H$

$=(H^* \otimes b_H(K) \otimes H) \circ (((H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K))) \otimes ((H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (T_H^{-1} \otimes a_H(K)))) \circ \delta_H$ (by definition of α_H and β_H)

$=(H^* \otimes (\gamma_H \circ c_{H,H}^{-1})) \circ (((H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ (H \otimes a_H(K))) \otimes T_H^{-1}) \circ \delta_H$ (by (2.12) and the adjunction properties)

$=(H^* \otimes (\gamma_H \circ (H \otimes \gamma_H) \circ (c_{H,H}^{-1} \otimes H))) \circ (((c_{H,H^*} \otimes H) \circ (H \otimes c_{H,H^*})) \otimes H) \circ (((H \otimes T_H^{-1}) \circ \delta_H) \otimes a_H(K))$ (by naturality of c)

$=(H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ ((\mu_H^{\mathsf{BT}} \circ (T_H^{-1} \otimes H) \circ c_{H,H}^{-1} \circ \delta_H) \otimes a_H(K))$ (by naturality of c and (iv) of Definition 3.1)

$=(H^* \otimes \gamma_H) \circ (c_{H,H^*} \otimes H) \circ ((\eta_H \circ \varepsilon_H) \otimes a_H(K))$ (by (2.1) for H_{BT}^{cop})

$=\varepsilon_H \otimes a_H(K)$ (by naturality of c and (v) of Definition 3.1). □

Previous theorem can be interpreted in a functorial way as follows: If we denote by BT^{f} the subcategory of brace triples whose underlying Hopf algebra is finite, then there exists a functor $P\colon \mathsf{BT}^{\mathsf{f}} \longrightarrow \mathsf{Post\text{-}Hopf}$ acting on objects by $P((H, \gamma_H, T_H)) = (H, \gamma_H)$ and on morphisms by the identity.

Theorem 4.7. *Let* (H, m_H) *be an object in* $\mathsf{cocPost\text{-}Hopf}$*, then*

$$\widehat{H} = (H, \eta_H, \widehat{\mu}_H, \varepsilon_H, \delta_H)$$

is a bialgebra in C*, where* $\widehat{\mu}_H := \mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)$.

Proof. Note that we already know that $(H, \varepsilon_H, \delta_H)$ is a coalgebra in C and that η_H is a coalgebra morphism. Then, firstly, we have to compute that $(H, \eta_H, \widehat{\mu}_H)$ is an algebra in C. Indeed, let's start proving the unit property. On the one hand,

$\widehat{\mu}_H \circ (\eta_H \otimes H)$

$=\mu_H \circ (H \otimes m_H) \circ ((\delta_H \circ \eta_H) \otimes H)$ (by definition of $\widehat{\mu}_H$)

$=\mu_H \circ (H \otimes m_H) \circ (\eta_H \otimes \eta_H \otimes H)$ (by the condition of coalgebra morphism for η_H)

$=m_H \circ (\eta_H \otimes H)$ (by unit properties)

$=id_H$ (by (4.5))

and, on the other hand,

$$\begin{aligned}
&\widehat{\mu}_H \circ (H \otimes \eta_H)\\
=&\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes \eta_H) \text{ (by definition of } \widehat{\mu}_H)\\
=&\mu_H \circ (H \otimes \varepsilon_H \otimes \eta_H) \circ \delta_H \text{ (by (4.1))}\\
=&id_H \text{ (by (co)unit properties).}
\end{aligned}$$

The associativity of $\widehat{\mu}_H$ follows by:

$$\begin{aligned}
&\widehat{\mu}_H \circ (\widehat{\mu}_H \otimes H)\\
=&\mu_H \circ (H \otimes m_H) \circ ((\delta_H \circ \mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes H) \text{ (by definition of } \widehat{\mu}_H)\\
=&\mu_H \circ (\mu_H \otimes (m_H \circ (\mu_H \otimes H))) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (\delta_H \otimes (\delta_H \circ m_H) \otimes H)\\
&\circ (\delta_H \otimes H \otimes H) \text{(by the condition of coalgebra morphism for } \mu_H)\\
=&\mu_H \circ (\mu_H \otimes (m_H \circ (\mu_H \otimes H))) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (\delta_H \otimes ((m_H \otimes m_H) \circ (H \otimes c_{H,H}\\
&\otimes H) \circ (\delta_H \otimes \delta_H)) \otimes H) \circ (\delta_H \otimes H \otimes H) \text{ (by (i.1) of Definition 4.1)}\\
=&\mu_H \circ ((\mu_H \circ (H \otimes m_H)) \otimes (m_H \circ (\mu_H \otimes H) \circ (H \otimes m_H \otimes H))) \circ (((H \otimes H \otimes c_{H,H} \otimes H\\
&\otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (((\delta_H \otimes \delta_H) \circ \delta_H) \otimes \delta_H)) \otimes H) \text{ (by naturality of } c)\\
=&\mu_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes (m_H \circ (\mu_H \otimes H)))\\
&\circ (H \otimes ((c_{H,H} \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H \otimes H)\\
&\text{(by coassociativity and cocommutativity of } \delta_H)\\
=&\mu_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes (m_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes H)))\\
&\circ (((H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes H) \text{ (by naturality of } c)\\
=&\mu_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes (m_H \circ (H \otimes m_H))) \circ (((H \otimes c_{H,H} \otimes H)\\
&\circ (\delta_H \otimes \delta_H)) \otimes H) \text{ (by (ii) of Definition 4.1)}\\
=&\mu_H \circ (H \otimes (\mu_H \circ (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H))) \circ (H \otimes H \otimes H \otimes m_H)\\
&\circ (\delta_H \otimes \delta_H \otimes H) \text{ (by coassociativity of } \delta_H \text{ and associativity of } \mu_H)\\
=&\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes (\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H))) \text{ (by (iii) of Definition 4.1)}\\
=&\widehat{\mu}_H \circ (H \otimes \widehat{\mu}_H) \text{ (by definition of } \widehat{\mu}_H).
\end{aligned}$$

Finally, we will prove that $\widehat{\mu}_H$ is a coalgebra morphism. By the condition of coalgebra morphism for μ_H, (i.2) of Definition 4.1 and the counit property, it is straightforward to compute that $\varepsilon_H \circ \widehat{\mu}_H = \varepsilon_H \otimes \varepsilon_H$. Moreover,

$$\begin{aligned}
&\delta_H \circ \widehat{\mu}_H\\
=&\delta_H \circ \mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H) \text{ (by definition of } \widehat{\mu}_H)\\
=&(\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (\delta_H \circ m_H)) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra}\\
&\text{morphism for } \mu_H)\\
=&(\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes ((m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)))\\
&\circ (\delta_H \otimes H) \text{ (by (i.1) of Definition 4.1)}\\
=&((\mu_H \circ (H \otimes m_H)) \otimes (\mu_H \circ (H \otimes m_H))) \circ (H \otimes ((H \otimes c_{H,H} \otimes H) \circ (c_{H,H} \otimes c_{H,H})) \otimes H)\\
&\circ (((\delta_H \otimes \delta_H) \circ \delta_H) \otimes \delta_H) \text{ (by naturality of } c)\\
=&((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes (\mu_H \circ (H \otimes m_H))) \circ (H \otimes ((c_{H,H} \otimes H) \circ (H \otimes c_{H,H})\\
&\circ (\delta_H \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by cocommutativity and coassociativity of } \delta_H)\\
=&((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes (\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H))) \circ (H \otimes c_{H,H} \otimes H)\\
&\circ (\delta_H \otimes \delta_H) \text{ (by naturality of } c)\\
=&(\widehat{\mu}_H \otimes \widehat{\mu}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by definition of } \widehat{\mu}_H). \qquad \square
\end{aligned}$$

Corollary 4.8. *If (H, m_H) is an object in* cocPost-Hopf*, then (H, m_H) is a left $\widehat{H}$-module algebra-coalgebra, i.e., a left $\widehat{H}$-module bialgebra.*

Proof. It is a consequence of the following facts: Thanks to conditions (ii) of Definition 4.1 and (4.5), (H, m_H) is a left $\widehat{H}$-module. Moreover, by (4.1) and (iii) of Definition 4.1, η_H and μ_H are morphisms of left $\widehat{H}$-modules, respectively. To finish, m_H is a coalgebra morphism by (i) of Definition 4.1 which implies that (H, m_H) is a left $\widehat{H}$-module coalgebra. □

Along the following results, we are going to study some properties about the morphism

$$\begin{aligned}\widehat{\lambda}_H &:= (b_H(K) \otimes H) \circ (c_{H^*,H} \otimes H) \circ (H^* \otimes c_{H,H}) \circ (\beta_H \otimes \lambda_H) \circ \delta_H \\ &= (b_H(K) \otimes H) \circ (H \otimes \beta_H) \circ c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \text{ (by naturality of } c)\end{aligned}$$

with the final objective of proving that it is the antipode for $\widehat{H}$. We are going to denote by $\hat{*}$ the convolution in $\mathrm{Hom}(H, \widehat{H})$.

Remark 4.9. First of all, note that

$$\widehat{\lambda}_H = (b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ \delta_H \tag{4.8}$$

when (H, m_H) is a cocommutative post-Hopf algebra. Indeed,

$$\begin{aligned}\widehat{\lambda}_H &= (b_H(K) \otimes H) \circ (c_{H^*,H} \otimes H) \circ (H^* \otimes c_{H,H}) \circ (\beta_H \otimes \lambda_H) \circ \delta_H \\ &= (b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ c_{H,H} \circ \delta_H \text{ (by naturality of } c) \\ &= (b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ \delta_H \text{ (by cocommutativity of } \delta_H).\end{aligned}$$

Lemma 4.10. *If (H, m_H) is a cocommutative post-Hopf algebra in* C*, then*

$$m_H \circ (H \otimes \widehat{\lambda}_H) \circ \delta_H = \lambda_H. \tag{4.9}$$

As a consequence,

$$id_H \,\hat{*}\, \widehat{\lambda}_H = \varepsilon_H \otimes \eta_H. \tag{4.10}$$

Proof. Let's start proving (4.9):

$$\begin{aligned}& m_H \circ (H \otimes \widehat{\lambda}_H) \circ \delta_H \\ =& m_H \circ (H \otimes ((b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ \delta_H)) \circ \delta_H \text{ (by (4.8))} \\ =& (b_H(K) \otimes H) \circ (H \otimes \alpha_H) \circ c_{H,H} \circ (H \otimes ((b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ \delta_H)) \circ \delta_H \text{ (by (4.4))} \\ =& ((b_H(K) \circ (\lambda_H \otimes H^*)) \otimes (b_H(K) \circ c_{H^*,H}) \otimes H) \circ (H \otimes c_{H^*,H^*} \otimes c_{H,H}) \circ (H \otimes H^* \otimes c_{H,H^*} \\ & \otimes H) \circ (((c_{H^*,H} \otimes H) \circ (H^* \otimes c_{H,H}) \circ (\alpha_H \otimes H)) \otimes \beta_H) \circ (H \otimes \delta_H) \circ \delta_H \text{ (by naturality of } c) \\ =& ((b_H(K) \circ (\lambda_H \otimes H^*)) \otimes H) \circ (H \otimes ((H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H))) \circ (H \otimes c_{H,H}) \\ & \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \text{ (by naturality of } c) \\ =& ((b_H(K) \circ (\lambda_H \otimes H^*)) \otimes H) \circ (H \otimes ((H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ \delta_H)) \circ \delta_H \\ & \text{(by naturality of } c \text{ and cocommutativity and coassociativity of } \delta_H) \\ =& ((b_H(K) \circ (\lambda_H \otimes H^*)) \otimes H) \circ (H \otimes (\varepsilon_H \otimes a_H(K))) \circ \delta_H \text{ (by (iv) of Definition 4.1)} \\ =& \lambda_H \text{ (by counit property and the adjunction properties).}\end{aligned}$$

From the previous identity we obtain the following:

$$\begin{aligned}& id_H \,\hat{*}\, \widehat{\lambda}_H \\ =& \widehat{\mu}_H \circ (H \otimes \widehat{\lambda}_H) \circ \delta_H \text{ (by definition of } \hat{*}) \\ =& \mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes \widehat{\lambda}_H) \circ \delta_H \text{ (by definition of } \widehat{\mu}_H) \\ =& \mu_H \circ (H \otimes (m_H \circ (H \otimes \widehat{\lambda}_H) \circ \delta_H)) \circ \delta_H \text{ (by coassociativity of } \delta_H) \\ =& id_H * \lambda_H \text{ (by (4.9))} \\ =& \varepsilon_H \otimes \eta_H \text{ (by (2.1)).}\end{aligned}$$

□

The aim of the following results will be to prove that the convolution in the opposite direction is also the identity element, i.e., $\widehat{\lambda}_H \,\tilde{*}\, id_H = \varepsilon_H \otimes \eta_H$.

Lemma 4.11. *If (H, m_H) is a cocommutative post-Hopf algebra in C, then*

$$\widetilde{\alpha}_H := (b_H(K) \otimes H) \circ (H \otimes \alpha_H) \colon H \otimes H \to H$$

is a coalgebra morphism.

Proof. From (i.1) of Definition 4.1, (2.7) and the naturality of c we can deduce that

$$\delta_H \circ m_H \circ c_{H,H} = ((m_H \circ c_{H,H}) \otimes (m_H \circ c_{H,H})) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \tag{4.11}$$

holds. Therefore, we obtain that:

$$\begin{aligned}
&\delta_H \circ \widetilde{\alpha}_H \\
=&(b_H(K) \otimes \delta_H) \circ (H \otimes \alpha_H) \text{ (by definition of } \widetilde{\alpha}_H) \\
=&\delta_H \circ m_H \circ c_{H,H} \text{ (by (2.7) and (4.3))} \\
=&((m_H \circ c_{H,H}) \otimes (m_H \circ c_{H,H})) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by (4.11))} \\
=&(\widetilde{\alpha}_H \otimes \widetilde{\alpha}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by (2.7), (4.3) and definition of } \widetilde{\alpha}_H).
\end{aligned}$$

Moreover, by (4.3) and (i.2) of Definition 4.1, it is easy to prove that $\varepsilon_H \circ \widetilde{\alpha}_H = \varepsilon_H \otimes \varepsilon_H$. □

Let (H, m_H) be a post-Hopf algebra in C and consider now the morphism

$$\widetilde{\beta}_H := (b_H(K) \otimes H) \circ (H \otimes \beta_H) \colon H \otimes H \to H.$$

Lemma 4.12. *Let (H, m_H) be a post-Hopf algebra in C. It is satisfied that*

$$\varepsilon_H \circ \widetilde{\beta}_H = \varepsilon_H \otimes \varepsilon_H. \tag{4.12}$$

Proof.

$$\begin{aligned}
&\varepsilon_H \circ \widetilde{\beta}_H \\
=&((\varepsilon_H \circ \widetilde{\beta}_H) \otimes \varepsilon_H) \circ (H \otimes \delta_H) \text{ (by counit properties)} \\
=&(b_H(K) \otimes \varepsilon_H \otimes \varepsilon_H) \circ (H \otimes \beta_H \otimes H) \circ (H \otimes \delta_H) \text{ (by definition of } \widetilde{\beta}_H) \\
=&(b_H(K) \otimes (\varepsilon_H \circ \widetilde{\alpha}_H)) \circ (H \otimes \beta_H \otimes H) \circ (H \otimes \delta_H) \text{ (by the condition of coalgebra} \\
&\text{morphism for } \widetilde{\alpha}_H) \\
=&(b_H(K) \otimes \varepsilon_H) \circ (H \otimes ((H^* \otimes b_H(K) \otimes H) \circ (\beta_H \otimes \alpha_H) \circ \delta_H)) \text{ (by definition of } \widetilde{\alpha}_H) \\
=&((b_H(K) \otimes \varepsilon_H) \circ (H \otimes a_H(K))) \otimes \varepsilon_H \text{ (by (iv) of Definition 4.1)} \\
=&\varepsilon_H \otimes \varepsilon_H \text{ (by the adjunction properties).}
\end{aligned}$$

□

Let (H, m_H) be a post-Hopf algebra and suppose that $\widetilde{\beta}_H$ satisfies that

$$\delta_H \circ \widetilde{\beta}_H = (\widetilde{\beta}_H \otimes \widetilde{\beta}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H). \tag{4.13}$$

Note that if (4.13) holds, then $\widetilde{\beta}_H$ is a coalgebra morphism by the previous lemma.

Lemma 4.13. *Let (H, m_H) be a cocommutative post-Hopf algebra in C, then*

$$\varepsilon_H \circ \widehat{\lambda}_H = \varepsilon_H. \tag{4.14}$$

In addition, if the identity (4.13) holds, then

$$\delta_H \circ \widehat{\lambda}_H = (\widehat{\lambda}_H \otimes \widehat{\lambda}_H) \circ \delta_H, \tag{4.15}$$

i.e. $\widehat{\lambda}_H$ is a coalgebra morphism,

$$\widehat{\lambda}_H \circ \widehat{\lambda}_H = id_H, \tag{4.16}$$

and

$$\widehat{\lambda}_H \,\tilde{*}\, id_H = \varepsilon_H \otimes \eta_H. \tag{4.17}$$

Proof. First of all, note that it is straightforward to prove that $\varepsilon_H \circ \widehat{\lambda}_H = \varepsilon_H$ using (4.8), (4.12), (2.4) and counit property. Moreover

$$
\begin{aligned}
&\delta_H \circ \widehat{\lambda}_H \\
=&\delta_H \circ \widetilde{\beta}_H \circ (\lambda_H \otimes H) \circ \delta_H \text{ (by (4.8) and definition of } \widetilde{\beta}_H) \\
=&(\widetilde{\beta}_H \otimes \widetilde{\beta}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\lambda_H \otimes H) \circ \delta_H \text{ (by (4.13))} \\
=&(\widetilde{\beta}_H \otimes \widetilde{\beta}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes \delta_H) \circ \delta_H \\
&\text{(by (2.3) and cocommutativity of } \delta_H) \\
=&((\widetilde{\beta}_H \circ (\lambda_H \otimes H)) \otimes (\widetilde{\beta}_H \circ (\lambda_H \otimes H))) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&\text{(by naturality of } c \text{ and coassociativity of } \delta_H) \\
=&((\widetilde{\beta}_H \circ (\lambda_H \otimes H) \circ \delta_H) \otimes (\widetilde{\beta}_H \circ (\lambda_H \otimes H) \circ \delta_H)) \circ \delta_H \\
&\text{(by cocommutativity and coassociativity of } \delta_H) \\
=&(\widehat{\lambda}_H \otimes \widehat{\lambda}_H) \circ \delta_H \text{ (by (4.8) and definition of } \widetilde{\beta}_H).
\end{aligned}
$$

As a consequence, we can prove that $\widehat{\lambda}_H \circ \widehat{\lambda}_H = id_H$. Indeed,

$$
\begin{aligned}
&\widehat{\lambda}_H \circ \widehat{\lambda}_H \\
=&\widehat{\mu}_H \circ ((\eta_H \circ \varepsilon_H) \otimes (\widehat{\lambda}_H \circ \widehat{\lambda}_H)) \circ \delta_H \text{ (by (co)unit properties)} \\
=&\widehat{\mu}_H \circ ((id_H \,\hat{*}\, \widehat{\lambda}_H) \otimes (\widehat{\lambda}_H \circ \widehat{\lambda}_H)) \circ \delta_H \text{ (by (4.10))} \\
=&\widehat{\mu}_H \circ (H \otimes (\widehat{\mu}_H \circ (\widehat{\lambda}_H \otimes (\widehat{\lambda}_H \circ \widehat{\lambda}_H)) \circ \delta_H)) \circ \delta_H \\
&\text{(by coassociativity of } \delta_H \text{ and associativity of } \widehat{\mu}_H) \\
=&\widehat{\mu}_H \circ (H \otimes ((id_H \,\hat{*}\, \widehat{\lambda}_H) \circ \widehat{\lambda}_H)) \circ \delta_H \text{ (by (4.15))} \\
=&\widehat{\mu}_H \circ (H \otimes (\eta_H \circ \varepsilon_H \circ \widehat{\lambda}_H)) \circ \delta_H \text{ (by (4.10))} \\
=&\widehat{\mu}_H \circ (H \otimes (\eta_H \circ \varepsilon_H)) \circ \delta_H \text{ (by (4.14))} \\
=&id_H \text{ (by (co)unit properties).}
\end{aligned}
$$

To finish, we will see that $\widehat{\lambda}_H \,\hat{*}\, id_H = \varepsilon_H \otimes \eta_H$. Indeed,

$$
\begin{aligned}
&\widehat{\lambda}_H \,\hat{*}\, id_H \\
=&\widehat{\mu}_H \circ (\widehat{\lambda}_H \otimes H) \circ \delta_H \text{ (by definition of } \hat{*}) \\
=&\widehat{\mu}_H \circ (\widehat{\lambda}_H \otimes (\widehat{\lambda}_H \circ \widehat{\lambda}_H)) \circ \delta_H \text{ (by (4.16))} \\
=&(id_H \,\hat{*}\, \widehat{\lambda}_H) \circ \widehat{\lambda}_H \text{ (by (4.15))} \\
=&\eta_H \circ \varepsilon_H \circ \widehat{\lambda}_H \text{ (by (4.10))} \\
=&\varepsilon_H \otimes \eta_H \text{ (by (4.14)).}
\end{aligned}
$$
□

Theorem 4.14. *Let (H, m_H) be a cocommutative post-Hopf algebra in C. If the identity (4.13) holds, then $\widehat{H} = (H, \eta_H, \widehat{\mu}_H, \varepsilon_H, \delta_H, \widehat{\lambda}_H)$ is a cocommutative Hopf algebra in C. This particular Hopf algebra structure is called the subadjacent Hopf algebra of (H, m_H).*

Proof. It is a direct consequence of Theorem 4.7 and equalities (4.10) and (4.17). □

So, we have deduced that it is possible to obtain from any cocommutative post-Hopf algebra satisfying (4.13) another Hopf algebra structure whose underlying coalgebra is the same as that of H. Therefore, at this point it is natural to wonder whether $\widehat{\mathbb{H}} = (H, \widehat{H})$ is a Hopf brace in C. The following theorem solves this question.

Theorem 4.15. *Let* (H, m_H) *be a cocommutative post-Hopf algebra in* C*. If the identity* (4.13) *is satisfied, then*

$$\widehat{\mathbb{H}} = (H, \widehat{H})$$

is a cocommutative Hopf brace in C*.*

Proof. By Theorem 4.14, to prove that $\widehat{\mathbb{H}} = (H, \widehat{H})$ is a Hopf brace we only have to show that (iii) of Definition 2.10 holds. Note that

$$\widehat{\Gamma}_H = m_H. \tag{4.18}$$

Indeed,

$$\begin{aligned}
&\widehat{\Gamma}_H \\
=&\mu_H \circ (\lambda_H \otimes \widehat{\mu}_H) \circ (\delta_H \otimes H) \text{ (by definition of } \widehat{\Gamma}_H) \\
=&\mu_H \circ (\lambda_H \otimes (\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H))) \circ (\delta_H \otimes H) \text{ (by definition of } \widehat{\mu}_H) \\
=&\mu_H \circ ((\lambda_H * id_H) \otimes m_H) \circ (\delta_H \otimes H) \text{ (by coassociativity of } \delta_H \text{ and associativity of } \mu_H) \\
=&\mu_H \circ ((\eta_H \circ \varepsilon_H) \otimes m_H) \circ (\delta_H \otimes H) \text{ (by (2.1))} \\
=&m_H \text{ (by (co)unit property).}
\end{aligned}$$

So, we obtain the following:

$$\begin{aligned}
&\mu_H \circ (\widehat{\mu}_H \otimes \widehat{\Gamma}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H) \\
=&\mu_H \circ ((\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H)) \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H) \\
&\text{(by definition of } \widehat{\mu}_H \text{ and (4.18))} \\
=&\mu_H \circ (H \otimes (\mu_H \circ (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H))) \circ (\delta_H \otimes H \otimes H) \\
&\text{(by associativity of } \mu_H \text{ and coassociativity of } \delta_H) \\
=&\mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes \mu_H) \text{ (by (iii) of Definition 4.1)} \\
=&\widehat{\mu}_H \circ (H \otimes \mu_H) \text{ (by definition of } \widehat{\mu}_H). \qquad \square
\end{aligned}$$

It is possible to interpret the previous result in the following sense: If $\mathsf{cocPost\text{-}Hopf}^{\star}$ denotes the full subcategory of cocommutative post-Hopf algebras such that (4.13) holds, and $\mathsf{cocHBr}^{\mathsf{f}}$ denotes the category of finite cocommutative Hopf braces, then a functor $Q\colon \mathsf{cocPost\text{-}Hopf}^{\star} \longrightarrow \mathsf{cocHBr}^{\mathsf{f}}$ exists which acts on objects by $Q((H, m_H)) = \widehat{\mathbb{H}}$ and on morphisms by the identity. To see that Q is well-defined on morphisms, we have to prove that if $f\colon (H, m_H) \to (B, m_B)$ is a morphism in $\mathsf{cocPost\text{-}Hopf}$, then f is a morphism of Hopf braces between $\widehat{\mathbb{H}}$ and $\widehat{\mathbb{B}}$. Indeed:

$$\begin{aligned}
&f \circ \widehat{\mu}_H \\
=&f \circ \mu_H \circ (H \otimes m_H) \circ (\delta_H \otimes H) \text{ (by definition of } \widehat{\mu}_H) \\
=&\mu_B \circ (f \otimes f) \circ (H \otimes m_H) \circ (\delta_H \otimes H) \text{ (by the condition of algebra morphism for } f\colon H \to B) \\
=&\mu_B \circ (B \otimes m_B) \circ (((f \otimes f) \circ \delta_H) \otimes f) \text{ (by (4.2))} \\
=&\mu_B \circ (B \otimes m_B) \circ (\delta_B \otimes B) \circ (f \otimes f) \text{ (by the condition of coalgebra morphism for } f) \\
=&\widehat{\mu}_B \circ (f \otimes f) \text{ (by definition of } \widehat{\mu}_B).
\end{aligned}$$

The following theorem is the main result of this section.

Theorem 4.16. *The categories* $\mathsf{cocPost\text{-}Hopf}^{\star}$ *and* $\mathsf{cocBT}^{\mathsf{f}}$ *are isomorphic.*

Proof. At first, let's consider the following commutative diagram of functors:

$$\begin{tikzcd}
\mathsf{cocPost\text{-}Hopf}^{\star} \arrow[r, "Q"] & \mathsf{cocHBr}^{\mathsf{f}} \arrow[d, shift left, "G''"] \arrow[d, phantom, "\wr"] \\
& \mathsf{cocBT}^{\mathsf{f}}, \arrow[ul, "P'"] \arrow[u, shift left, "F''"]
\end{tikzcd}$$

where F'' and G'' are the restrictions of functors F' and G' introduced in Corollary 3.19 to the subcategories of finite objects, and P' is the restriction of functor P to $\mathsf{cocBT}^{\mathsf{f}}$.

To begin with, we are going to see that P' is well-defined on objects. That is to say, we have to prove that if (H, γ_H, T_H) is a finite cocommutative brace triple, then the post-Hopf algebra (H, γ_H) satisfies (4.13). In this situation, by Theorem 4.6 and (3.6),

$$\beta_H = \alpha_H \circ T_H, \tag{4.19}$$

and then,

$$\widetilde{\beta}_H = \widetilde{\alpha}_H \circ (H \otimes T_H). \tag{4.20}$$

Therefore, (4.13) follows by:

$$\begin{aligned}
&\delta_H \circ \widetilde{\beta}_H \\
=&\delta_H \circ \widetilde{\alpha}_H \circ (H \otimes T_H) \text{ (by } \widetilde{\beta}_H = \widetilde{\alpha}_H \circ (H \otimes T_H)) \\
=&(\widetilde{\alpha}_H \otimes \widetilde{\alpha}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (\delta_H \circ T_H)) \text{ (by Lemma 4.11)} \\
=&(\widetilde{\alpha}_H \otimes \widetilde{\alpha}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes ((T_H \otimes T_H) \circ \delta_H)) \text{ (by (vi.1) of Definition 3.1} \\
&\text{and cocommutativity of } \delta_H) \\
=&(\widetilde{\beta}_H \otimes \widetilde{\beta}_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by naturality of } c \text{ and (4.20)).}
\end{aligned}$$

Taking into account functors P', Q and G'', on the one hand, we have that

$$\begin{aligned}
&(P' \circ (G'' \circ Q))((H, m_H)) \\
=&(P' \circ G'')(\widehat{\mathbb{H}}) \text{ (by definition of } Q) \\
=&P'((H, m_H, \widehat{\lambda}_H)) \text{ (by definition of } G'' \text{ and (4.18))} \\
=&(H, m_H) \text{ (by definition of } P').
\end{aligned}$$

So, $P' \circ (G'' \circ Q) = \mathsf{id}_{\mathsf{cocPost\text{-}Hopf}^{\star}}$. On the other hand,

$$\begin{aligned}
&((G'' \circ Q) \circ P')((H, \gamma_H, T_H)) \\
=&(G'' \circ Q)((H, \gamma_H)) \text{ (by definition of } P') \\
=&G''(\widehat{\mathbb{H}}) \text{ (by definition of } Q) \\
=&(H, \gamma_H, \widehat{\lambda}_H) \text{ (by definition of } G'' \text{ and (4.18))} \\
=&(H, \gamma_H, T_H),
\end{aligned}$$

where the last equality is due to the fact that $\widehat{\lambda}_H = T_H$. Indeed, firstly note that

$$\begin{aligned}
&\widehat{\lambda}_H \\
=&(b_H(K) \otimes H) \circ (\lambda_H \otimes \beta_H) \circ \delta_H \text{ (by (4.8))} \\
=&(b_H(K) \otimes H) \circ (\lambda_H \otimes (\alpha_H \circ T_H)) \circ \delta_H \text{ (by Theorem 4.6 and (3.6))} \\
=&\gamma_H \circ c_{H,H} \circ (\lambda_H \otimes T_H) \circ \delta_H \text{ (by (4.3) and (2.7))} \\
=&\gamma_H \circ (T_H \otimes \lambda_H) \circ \delta_H \text{ (by naturality of } c \text{ and cocommutativity of } \delta_H).
\end{aligned}$$

As a result,

$$
\begin{aligned}
& id_H *_{\mathsf{BT}} \widehat{\lambda}_H \\
=& \mu_H^{\mathsf{BT}} \circ (H \otimes \widehat{\lambda}_H) \circ \delta_H \text{ (by definition of } *_{\mathsf{BT}}) \\
=& \mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes (\gamma_H \circ (T_H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H \\
& \text{(by definition of } \mu_H^{\mathsf{BT}} \text{ and previous equality)} \\
=& \mu_H \circ (H \otimes (\gamma_H \circ ((\mu_H \circ (H \otimes \gamma_H) \circ (\delta_H \otimes H)) \otimes H))) \circ (\delta_H \otimes ((T_H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H \\
& \text{(by (iv) of Definition 3.1)} \\
=& \mu_H \circ (H \otimes (\gamma_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes (\gamma_H \circ (H \otimes T_H) \circ \delta_H) \otimes H))) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
& \text{(by coassociativity of } \delta_H) \\
=& \mu_H \circ (H \otimes (\gamma_H \circ ((id_H * \lambda_H) \otimes \lambda_H) \circ \delta_H)) \circ \delta_H \\
& \text{(by (vi.4) of Definition 3.1 and coassociativity of } \delta_H) \\
=& id_H * \lambda_H \text{ (by (2.1), counit property and (v) of Definition 3.1)} \\
=& \varepsilon_H \otimes \eta_H \text{ (by (2.1))},
\end{aligned}
$$

which implies, due to the uniqueness of the antipode for the Hopf algebra H_{BT}, that $\widehat{\lambda}_H = T_H$. Hence, $(G'' \circ Q) \circ P' = \mathsf{id}_{\mathsf{cocBT^f}}$. □

Corollary 4.17. *Categories* $\mathsf{cocBT^f}$, $\mathsf{cocHBr^f}$ *and* $\mathsf{cocPost\text{-}Hopf}^\star$ *are isomorphic.*

Proof. It is a direct consequence of the previous theorem and Corollary 3.19. □

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