Ordu Üniversitesi Bilim ve Teknoloji Dergisi
Ordu Üniv. Bil. Tek. Derg.

Ordu University Journal of Science and Technology
Ordu Univ. J. Sci. Tech.

E-ISSN: 2146-6459 2024, 14 (2), 249-260 Araştırma Makalesi / Research Article
https://doi.org/10.54370/ordubtd.1512101

# Transmuted Unit Exponentiated Half-Logistic Distribution and its Applications

Murat Genç[1], Ömer Özbilen[2]

[1]Tarsus University, Faculty of Economics and Administrative Sciences, Department of Management Information Systems, Mersin
[2]Mersin University, Faculty of Education, Department of Primary Mathematics Teaching, Mersin

Geliş Tarihi / Received Date: 07.07.2024 Kabul Tarihi / Accepted Date: 06.11.2024

**Abstract**

A novel distribution, termed the transmuted unit exponentiated half-logistic distribution, has been proposed using the unit exponential half-logistic distribution, a member of the proportional hazard rate model family, as the base distribution. The statistical characteristics of the proposed distribution, including moments, moment-generating function, quantile function, and stress-strength reliability, have been thoroughly examined in this study. The maximum likelihood estimation method has been discussed for statistical inference of the distribution parameters. A simulation study based on the new distribution has been conducted to investigate the behavior of maximum likelihood estimates under various conditions. In addition, a numerical example has been presented to illustrate the performance of the distribution on a failure-time dataset.

**Keywords:** transmuted family, UEHL distribution, failure times, maximum likelihood estimator, data analysis

# Dönüştürülmüş Birim Üstel Yarı Lojistik Dağılım ve Uygulamaları

**Öz**

Orantılı tehlike hızı model ailesinin bir üyesi olan birim üstel yarı lojistik dağılım temel dağılım olarak kullanılarak, dönüştürülmüş (transmuted) birim üstel yarı lojistik dağılım olarak adlandırılan yeni bir dağılım önerilmiştir. Önerilen dağılımın momentler, moment çıkaran fonksiyon, kantil fonksiyonu ve stres-mukavemet güvenilirliği gibi istatistiksel özellikleri bu çalışmada ayrıntılı olarak incelenmiştir. Dağılım parametrelerinin istatistiksel çıkarımı için maksimum olabilirlik tahmin yöntemi tartışılmıştır. Maksimum olabilirlik tahminlerinin çeşitli koşullar altındaki davranışını araştırmak için yeni dağılıma dayalı bir simülasyon çalışması yapılmıştır. Ayrıca, bir başarısızlık-zaman veri kümesi üzerinde dağılımın performansını göstermek için sayısal bir örnek sunulmuştur.

**Anahtar Kelimeler:** dönüştürülmüş dağılım ailesi, UEHL dağılımı, başarısızlık zamanları, maksimum olabilirlik tahmincisi, veri analizi

*Sorumlu Yazar / Corresponding Author:* Murat Genç muratgenc@tarsus.edu.tr

## Introduction

In several practical applications, the use of proportional data is essential. Random variables, expressed as ratios or percentages, constitute the proportional data. For instance, infection rates of diseases, mortality percentages, response rates to clinical treatments, university admission rates, and percentage of useful volume in the water reservoir of a hydroelectric power plant are examples of proportional data. Random variables corresponding to proportional data can take any value within a unit interval. Therefore, appropriate probability distributions with a well-defined support set within a unit interval are required to model proportional data.

As a distribution with limited support, the omega distribution was proposed by Dombi et al. (2019) and discussed in applications related to reliability theory. The omega distribution belonged to the proportional hazard rate class. Dombi et al. (2019) demonstrated that the asymptotic omega hazard rate function is the hazard rate function of the Weibull distribution. Additionally, Dombi and Jónás (2020) obtained various properties of the omega distribution. Recently, Özbilen and Genç (2022) introduced the unit exponentiated-half logistic (UEHL) distribution inspired by the omega distribution. This distribution corresponds, through a simple transformation, to the exponentiated half-logistic distribution, which has broad applications in reliability theory (Gui, 2017; Seo & Kang, 2015).

In statistics, generating more useful distributions based on transformations of baseline distributions is common (Cordeiro & de Castro, 2011; Gupta et al., 1998; Rahman et al., 2020). In this context, Shaw and Buckley (2007) proposed an interesting method for solving problems related to financial mathematics and named this family the quadratic transmuted family of distributions. Representing the probability density function (PDF) and cumulative distribution function (CDF) of the quadratic transmuted distribution family for a baseline distribution with $f(x)$ and $F(x)$, respectively, we have

$$F_{QT}(x) = (1+\alpha)F(x) - \alpha F(x)^2 \tag{1}$$

and

$$f_{QT}(x) = (1+\alpha)f(x) - 2\alpha f(x)F(x), \tag{2}$$

where $\alpha \in [-1,1]$. Recently, several new distributions have been proposed based on transmuted distributions. Aryal and Tsokos (2009) suggested the transmuted extreme value distribution and studied its applications. Aryal and Tsokos (2011) proposed the transmuted Weibull distribution by applying the quadratic transmuted transformation to the Weibull distribution. Naz et al. (2013) introduced a modified power-generated family of distributions based on transmuted distributions and applied it to reliability analysis. Rahman et al. (2023) proposed a new modified cubic transmuted-G family of distributions. Ahsan-ul-Haq et al. (2023) suggested a new cubic transmuted power-function distribution and examined its properties. Adetunji (2023) introduced the transmuted Ailamujia distribution. Kuş et al. (2023) proposed the compound transmuted family of distributions, which uses the Weibull distribution as a sub-model to model the lifetime of a system composed of random components in series and parallel. Tushar et al. (2024) suggested a new cubic Transmuted Inverse Weibull distribution. Tushar et al. (2024) introduced the second-order transmuted Kumaraswamy distribution. Adetunji and Sabri (2024) proposed a two-parameter Poisson-transmuted exponential distribution for count observations. Additionally, other distributions recently proposed as members of the transmuted family include the transmuted logistic (Samuel, 2019), transmuted Burr Type X (Khan et al., 2020), transmuted modified Weibull (Khan et al., 2018), transmuted Birnbaum-Saunders (Bourguignon et al., 2017), and transmuted Ishita (Gharaibeh & Al-Omari, 2019) distributions.

In this study, we aim to introduce the transmuted UEHL (T-UEHL) distribution, which performs well in modeling failure time data by applying the transmuted family transformation given in Equation (1) to the UEHL distribution. The remaining sections of the study are organized as follows: In Section 2, we introduce the T-UEHL distribution and examine its basic properties. In Section 3, we obtain analytical characteristics of the T-UEHL distribution, including moments, moment-generating function, quantile function, stress-strength reliability, and maximum likelihood estimation. In Section 4, we perform a simulation study to investigate the performance of the proposed maximum likelihood estimators in parameter estimation. In Section 5, we demonstrate the modeling performance of the proposed estimator on real-life failure time data. Finally, in Section 6, we present the study's conclusions.

## T-UEHL Distribution

Recently, the UEHL distribution was proposed by Özbilen and Genç (2022) based on the omega distribution. The UEHL distribution corresponds to a simple transformation of the exponentiated half-logistic distribution and has several applications in reliability theory (Kang & Seo, 2011; Rastogi & Tripathi, 2014).

The PDF and CDF of the UEHL distribution are defined as follows, respectively:

$$f_{UEHL}(x) = 2\lambda\theta x^{\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{(1+x^{\theta})^{\lambda+1}}, 0 < x < 1 \tag{3}$$

and

$$F_{UEHL}(x) = 1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}, 0 < x < 1 \tag{4}$$

where $\theta > 0$ and $\lambda > 0$ are the scale and shape parameters of the distribution, respectively.

If the transformation given in Equations (1) and (2) is applied to the distribution, the PDF and CDF of the transmuted version of the distribution are obtained as follows:

$$f_{T-UEHL}(x) = 2\lambda\theta x^{\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{(1+x^{\theta})^{\lambda+1}}\left[1-\alpha+2\alpha\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right], 0 < x < 1 \tag{5}$$

and

$$F_{T-UEHL}(x) = \left[1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]\left[1+\alpha\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]. \tag{6}$$

A random variable with the CDF given in Equation (6) is called the three-parameter transmuted distribution and is denoted by $T-UEHL(\theta,\lambda,\alpha)$.

Figure 1 displays the PDF plots of the $T-UEHL(\theta,\lambda,\alpha)$ distribution for selected values of the distribution parameter. From Figure 1, the $T-UEHL(\theta,\lambda,\alpha)$ distribution shows right-skewed, decreasing, increasing U-type shapes. Therefore, the $T-UEHL(\theta,\lambda,\alpha)$ distribution can model quite different phenomena depending on the values of the parameters.

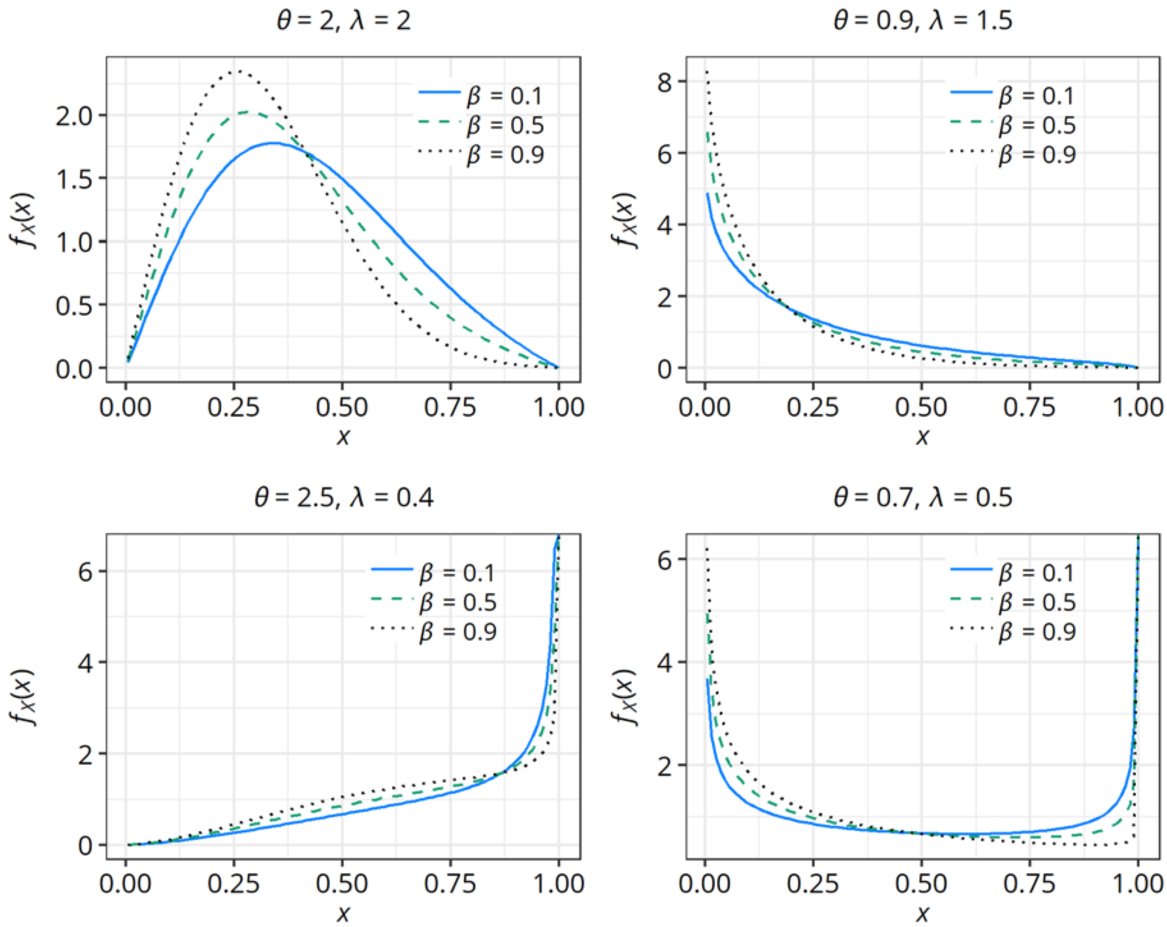


**Figure 1.** PDF's the $T-UEHL(\theta,\lambda)$ Distribution for Several Values of the $\theta$, $\lambda$ and $\beta$.

Also, the survival function and the hazard rate functions of the $T\text{-}UEHL(\theta,\lambda,\alpha)$ distribution are provided, respectively, by

$$S_{T-UEHL}(x)=\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\left[1-\alpha+\alpha\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]$$

and

$$h_{T-UEHL}(x)=\frac{2\lambda\theta x^{\theta-1}}{(1-x^{2\theta})\left(1-\alpha+\alpha\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right)}.$$

## Characteristic Properties of the T-UEHL(θ,λ) Distribution

This section presents the moments, moment-generating function, quantile function, stress-strength reliability, and maximum likelihood estimators that characterize the properties of the $T-UEHL(\theta,\lambda,\alpha)$ distribution relative to other statistical distributions.

### Moments

In statistics, moments are useful in understanding the characteristic properties of statistical distributions. The moments of the $T-UEHL(\theta,\lambda,\alpha)$ distribution can be expressed in terms of simple functions. This is shown in Proposition 1.

**Proposition 1:** Let the random variable $X$ have the $T-UEHL(\theta,\lambda,\alpha)$ distribution with the PDF given in Equation (5). Then for $r \in \{1,2,3,\dots\}$, $r$-th moment of the random variable $X$ is given by:

$$E(X^r)=\lambda\sum_{j=0}^{\infty}\left[(-1)^j\binom{r/\theta+j-1}{j}\{(1-\alpha)B(\lambda+j,1+r/\theta)+2\alpha B(2\lambda+j,1+r/\theta)\}\right].$$

**Proof:** Let $X \sim T-UEHL(\theta,\lambda,\alpha)$, then for $r \in \{1,2,3,\dots\}$ the $r$-th raw moment of $X$ is

$$E(X^r)=2\lambda\theta\int_0^1 x^{r+\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{(1+x^{\theta})^{\lambda+1}}\left[1-\alpha+2\alpha\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]dx. \tag{7}$$

Applying the transformation $u=\left(1-x^{\theta}\right)^{\lambda}/\left(1+x^{\theta}\right)^{\lambda}$ to Equation (7), the expected value in Equation (7) is found to be

$$\begin{aligned}E(X^r)&=\int_0^1\left(1-u^{1/\lambda}\right)^{r/\theta}\left(1+u^{1/\lambda}\right)^{-r/\theta}(1-\alpha+2\alpha u)\,du\\&=(1-\alpha)\int_0^1\left(1-u^{1/\lambda}\right)^{r/\theta}\left(1+u^{1/\lambda}\right)^{-r/\theta}\,du\\&+2\alpha\int_0^1 u\left(1-u^{1/\lambda}\right)^{r/\theta}\left(1+u^{1/\lambda}\right)^{-r/\theta}\,du.\end{aligned} \tag{8}$$

Applying the binomial series expansion $\left(1+u^{1/\lambda}\right)^{-r/\theta}=\sum_{j=0}^{\infty}(-1)^j\binom{r/\theta+j-1}{j}u^{j/\lambda}$ to Equation (8), we obtain

$$\begin{aligned}E(X^r)&=\lambda(1-\alpha)\sum_{j=0}^{\infty}(-1)^j\binom{r/\theta+j-1}{j}B(\lambda(1+j/\lambda),1+r/\theta)\\&+2\lambda\alpha\sum_{j=0}^{\infty}(-1)^j\binom{r/\theta+j-1}{j}B(\lambda(2+j/\lambda),1+r/\theta),\end{aligned} \tag{9}$$

where $B(a,b)=\int_0^1 v^{a-1}(1-v)^{b-1}\,dv$ is the beta function. Thus, by making the necessary simplifications in Equation (9), the $r$-th moment of the random variable $X$ is found to be

$$E(X^r)=\lambda\sum_{j=0}^{\infty}\left[(-1)^j\binom{r/\theta+j-1}{j}\{(1-\alpha)B(\lambda+j,1+r/\theta)+2\alpha B(2\lambda+j,1+r/\theta)\}\right].$$

**Moment Generating Function**

The moment generating function serves as the foundation for a different approach to analytical results compared to working directly with probability density functions. The moments of a random variable $X$ with the $T-UEHL(\theta,\lambda,\alpha)$ distribution were obtained in Equation (9). Utilizing the series expansion of the moment generating function and Equation (9), the moment generating function of the $T-UEHL(\theta,\lambda,\alpha)$ distribution is expressed as the following formula:

$$M_X(t) = E(e^{tx}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i)$$
$$= \lambda \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^i}{i!} \left[ (-1)^j \binom{i/\theta + j - 1}{j} \{(1-\alpha) B(\lambda + j, 1 + i/\theta) + 2\alpha B(2\lambda + j, 1 + i/\theta)\} \right].$$

**Quantile Function**

The quantile function of the $T - UEHL(\theta, \lambda, \alpha)$ distribution is given by

$$Q_{T-UEHL}(u; \theta, \lambda) = F_{T-UEHL}^{-1}(v(u); \theta, \lambda, \alpha) = \left( \frac{1 - (1-v)^{1/\lambda}}{1 + (1-v)^{1/\lambda}} \right)^{1/\theta}, \qquad (10)$$

where the function $v(\cdot)$ is defined as $v(u) = \left(1 + \alpha - \sqrt{\alpha^2 + 2(\alpha - 2u) + 1}\right)/2\alpha$. Thus, based on Equation (10), the steps of the random number generation algorithm for the $T - UEHL(\theta, \lambda, \alpha)$ distribution can be given as follows:

Generate $u \sim U(0,1)$.

Compute $v = \left(1 + \alpha - \sqrt{\alpha^2 + 2(\alpha - 2u) + 1}\right)/2\alpha$.

Compute $x = \left(\frac{1-(1-v)^{1/\lambda}}{1+(1-v)^{1/\lambda}}\right)^{1/\theta}$.

The random variable *U* in Step A1 follows the standard uniform distribution. In the case A3, it becomes $X \sim T - UEHL(\theta, \lambda, \alpha)$.

Furthermore, utilizing Equation (10), the median of the random variable $X$ with the $T - UEHL(\theta, \lambda, \alpha)$ distribution depending on the $\theta$, $\lambda$ ve $\alpha$ parameters can be expressed as

$$Q_{T-UEHL}(v(0.5); \theta, \lambda, \alpha) = \left( \frac{1 - \left(1 - \left(1 + \alpha - \sqrt{\alpha^2 + 2(\alpha - 2u) + 1}\right)/2\alpha\right)^{1/\lambda}}{1 + \left(1 - \left(1 + \alpha - \sqrt{\alpha^2 + 2(\alpha - 2u) + 1}\right)/2\alpha\right)^{1/\lambda}} \right)^{1/\theta}.$$

**Stress-Strength Reliability**

Given stress and strength random variables $Y$ and $X$, we aim to calculate the $R = P(Y < X)$ stress-strength reliability for the $T - UEHL(\theta, \lambda, \alpha)$ distribution.

**Proposition 2.** Let $Y \sim T - UEHL(\theta, \lambda_1, \alpha_1)$ and $X \sim T - UEHL(\theta, \lambda_2, \alpha_2)$ be independent stress and strength random variables following $T - UEHL$ distribution with given parameters. Then, the stress-strength reliability is as follows:

$$R = 1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}(\alpha_1(1-\alpha_2) + \alpha_2(1-\alpha_1) - 1) - \frac{2\lambda_2\alpha_2}{\lambda_1 + 2\lambda_2}(1-\alpha_1) - \frac{\lambda_2\alpha_1}{2\lambda_1 + \lambda_2}(1-\alpha_2). \quad (11)$$

**Proof:** By definition, the stress-strength reliability can be written as:

$$R = 2\lambda_2\theta \int_0^1 \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_1}\right] \left[1 + \alpha_1 \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_1}\right] x^{\theta-1} \frac{(1-x^\theta)^{\lambda_2-1}}{(1+x^\theta)^{\lambda_2+1}} \left[1 - \alpha_2 + 2\alpha_2 \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2}\right] dx. \quad (12)$$

Substituting the transformation $u = \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2}$ into Equation (12), we obtain

$$R = \int_0^1 \left(1 - u^{\lambda_1/\lambda_2}\right)\left(1 + \alpha_1 u^{\lambda_1/\lambda_2}\right)(1 - \alpha_2 + 2\alpha_2 u)\, du. \quad (13)$$

Performing algebraic manipulations and integration in Equation (13) completes the proof.

**Maximum Likelihood Estimation**

Let $X_1, X_2, \dots, X_n$ be an identically independent distributed sample from $T - UEHL(\theta, \lambda, \alpha)$, then, the likelihood and log-likelihood functions are written, respectively, as

$$L(\theta, \lambda, \alpha) = \prod_{i=1}^{n} \left( 2\lambda\theta x_i^{\theta-1} \frac{\left(1 - x_i^{\theta}\right)^{\lambda-1}}{\left(1 + x_i^{\theta}\right)^{\lambda+1}} \left( 1 - \alpha + 2\alpha \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda} \right) \right) \quad (14)$$

and

$$\begin{aligned} \ell(\theta, \lambda, \alpha) &= n \log 2 + n \log \lambda + n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log(x_i) \\ &+ (\lambda - 1) \sum_{i=1}^{n} \log\left(1 - x_i^{\theta}\right) - (\lambda + 1) \sum_{i=1}^{n} \log\left(1 + x_i^{\theta}\right) + \sum_{i=1}^{n} \log \left( 1 - \alpha + 2\alpha \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda} \right). \end{aligned} \quad (15)$$

By taking the derivatives of the log-likelihood function with respect to the distribution parameters, the log-likelihood equations are obtained as

$$\begin{aligned} \frac{\partial \ell(\theta, \lambda, \alpha)}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i - (\lambda - 1) \sum_{i=1}^{n} \frac{x_i^{\theta} \log x_i}{1 - x_i^{\theta}} - (\lambda + 1) \sum_{i=1}^{n} \frac{x_i^{\theta} \log x_i}{1 + x_i^{\theta}} \\ &- 4\lambda\alpha \sum_{i=1}^{n} \frac{x_i^{\theta} \log x_i \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda}}{(1 - x^{2\theta}) \left( 1 - \alpha + 2\alpha \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda} \right)} = 0, \end{aligned} \quad (16)$$

$$\frac{\partial \ell(\theta, \lambda, \alpha)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right) + 2\alpha \sum_{i=1}^{n} \frac{\left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda} \log \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)}{1 - \alpha + 2\alpha \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda}} = 0 \quad (17)$$

and

$$\frac{\partial \ell(\theta, \lambda, \alpha)}{\partial \alpha} = \sum_{i=1}^{n} \frac{2 \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda} - 1}{1 - \alpha + 2\alpha \left( \frac{1 - x_i^{\theta}}{1 + x_i^{\theta}} \right)^{\lambda}} = 0. \quad (18)$$

Since Equations (16), (17), and (18) do not have a closed-form solution, a solution must be obtained based on an iterative method. In this study, the optim function in the stats package in the R program was used to solve the system of equations.

## Simulation Study

Simulation studies are commonly conducted to evaluate the performance of the properties of MLE. Therefore, in this section, the MLEs of the parameters of the $T-UEHL(\theta,\lambda,\alpha)$ distribution, which were discussed in Section 3, are examined using a simulation study. In this context, random samples of size $N$ = 20, 50, 100, 300, 500 were generated from the $T-UEHL(\theta,\lambda,\alpha)$ distribution for fixed $\alpha$ values and true $\theta$ and $\lambda$ values. The simulations were repeated 2000 times to ensure the reliability of the estimation. Additionally, the parameter $\alpha$ was considered a hyperparameter in the simulation studies, and the MLEs of the parameters $\theta$ and $\lambda$ were obtained accordingly.

**Table 1.** Bias And Mses of MLE Estimators for Selected Parameter Values

| | | | | Bias | | MSE | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $\lambda$ | $\alpha$ | $n$ | $\theta$ | $\lambda$ | $\theta$ | $\lambda$ |
| 0.5 | 0.9 | 0.5 | 20 | 0.00972 | 0.43485 | 0.01450 | 0.48166 |
| | | | 50 | -0.01714 | 0.30067 | 0.00528 | 0.15694 |
| | | | 100 | -0.02643 | 0.25980 | 0.00302 | 0.09551 |
| | | | 300 | -0.03256 | 0.23894 | 0.00178 | 0.06542 |
| | | | 500 | -0.03294 | 0.23809 | 0.00152 | 0.06184 |
| 1.5 | 0.7 | 0.7 | 20 | 0.04505 | 0.50477 | 0.14332 | 0.48358 |
| | | | 50 | -0.03952 | 0.38661 | 0.05062 | 0.20376 |
| | | | 100 | -0.06895 | 0.35003 | 0.02755 | 0.14558 |
| | | | 300 | -0.08842 | 0.33129 | 0.01482 | 0.11663 |
| | | | 500 | -0.08968 | 0.33047 | 0.01228 | 0.11348 |
| 3 | 2 | 0.3 | 20 | 0.07919 | 0.74248 | 0.42467 | 2.50725 |
| | | | 50 | -0.06772 | 0.36977 | 0.14719 | 0.48225 |
| | | | 100 | -0.11683 | 0.26725 | 0.08060 | 0.20845 |
| | | | 300 | -0.14900 | 0.21356 | 0.04291 | 0.08492 |
| | | | 500 | -0.15116 | 0.21035 | 0.03535 | 0.06840 |
| 0.5 | 2.5 | 0.5 | 20 | 0.00600 | 1.45497 | 0.01070 | 7.93182 |
| | | | 50 | -0.01834 | 0.79329 | 0.00392 | 1.49445 |
| | | | 100 | -0.02633 | 0.62237 | 0.00237 | 0.71464 |
| | | | 300 | -0.03154 | 0.53211 | 0.00151 | 0.37473 |
| | | | 500 | -0.03192 | 0.52573 | 0.00133 | 0.33229 |

Table 1 shows the MLEs of the parameters of the $T-UEHL(\theta,\lambda,\alpha)$ distribution. The table indicates that a negative bias emerges for the parameter $\theta$ as the sample size increases. It is also observed that the bias of $\theta$ decreases as the sample size increases, as expected. This interpretation also holds for the MSE results of the parameters.

## Real Data Analysis: Failure Times

In this section, we examine the performance of the $T-UEHL(\theta,\lambda,\alpha)$ distribution on real data obtained from a lifetime test of light-emitting diodes (LEDs). Specifically, we focus on the failure times of these LEDs. The dataset we consider has also been analyzed by many authors (Birbiçer and Genç, 2023; Cheng and Wang, 2012; Dey, Wang and Nassar, 2022). The dataset, which has been divided by 10, is as follows: {0.018, 0.019, 0.019, 0.034, 0.036, 0.040, 0.044, 0.044, 0.045, 0.046, 0.047, 0.053, 0.057, 0.057, 0.063, 0.065, 0.070, 0.071, 0.071, 0.075, 0.076, 0.076, 0.079, 0.080, 0.085, 0.098, 0.101, 0.107, 0.112, 0.114, 0.115, 0.117, 0.120, 0.123, 0.124, 0.125, 0.126, 0.132, 0.133, 0.133, 0.139, 0.142, 0.150, 0.155, 0.158, 0.159, 0.162, 0.168, 0.170, 0.179, 0.200, 0.201, 0.204, 0.254, 0.361, 0.376, 0.465, 0.897}. The dataset exhibits highly positive skewness and high kurtosis. We compare the performance of the $T-UEHL(\theta,\lambda,\alpha)$ distribution with that of the Beta, Kumaraswamy (Kumaraswamy, 1980), UEHL (Özbilen and Genç, 2022), DUS-Kumaraswamy (Karakaya et al., 2021), and DUS-UEHL (Genç and Özbilen, 2023) distributions on this dataset. The distribution parameters are estimated using the maximum likelihood method. We assess the goodness of fit of the models using the Anderson-Darling test (AD stat) and the corresponding p-value (AD p-value). To compare the models' performance, we compute the Akaike information criterion (AIC) and Bayesian information criterion (BIC) as

$$\mathrm{AIC} = 2k - 2\ell(\cdot) \quad \text{and} \quad \mathrm{BIC} = k\log n - 2\ell(\cdot).$$

Here, $k$ represents the number of parameters, $n$ denotes the number of observations, and $\ell(\cdot)$ represents the likelihood function. The analysis results are presented in Table 4.

**Table 2.** Fitted Models and Comparison Criteria for LED Failure Times

| | $\theta$ | $\lambda$ | $\alpha$ | AIC | BIC | AD (stat) | AD (p-value) |
|---|---|---|---|---|---|---|---|
| Beta | 1.2351 | 7.1830 | - | -107.1718 | -103.0509 | 2.5721 | 0.0456 |
| Kumaraswamy | 1.0485 | 6.3951 | - | -105.8472 | -101.7263 | 2.7912 | 0.0352 |
| UEHL | 1.1806 | 4.6215 | - | -113.1899 | -109.0691 | 2.1490 | 0.0764 |
| DUS-Kumaraswamy | 0.8966 | 6.4437 | - | -104.4895 | -100.3686 | 2.6978 | 0.0393 |
| DUS-UEHL | 1.0346 | 4.7631 | - | -112.0986 | -107.9777 | 2.0765 | 0.0836 |
| T-UEHL | 1.3153 | 3.5409 | 0.8561 | -118.5605 | -112.3792 | 1.4384 | 0.1919 |

According to the Anderson-Darling test in Table 2, it can be concluded that the selected distributions are suitable for modeling the dataset. Considering the AIC and BIC values, the best fit among the compared models is achieved with the $T-UEHL(\theta,\lambda,\alpha)$ distribution. Therefore, in modeling LED failure times, the $T-UEHL(\theta,\lambda,\alpha)$ distribution stands out in terms of performance compared to other distributions.

## Conclusion

In this study, a three-parameter $T-UEHL(\theta,\lambda,\alpha)$ distribution is proposed for modeling failure time data based on a transmuted family type transformation on the CDF of the unit exponential half-logistic distribution. The proposed distribution's moments, moment-generating function, quantile function, and stress-strength reliability have been obtained analytically. A simulation study demonstrating the performance of maximum likelihood estimates for the $T-UEHL(\theta,\lambda,\alpha)$ distribution has been conducted. Real data analysis on time-to-failure data reveals that the $T-UEHL(\theta,\lambda,\alpha)$ model outperforms other well-known models in terms of the criteria AIC and BIC. Specifically, the $T-UEHL(\theta,\lambda,\alpha)$ distribution is compared with the beta, Kumaraswamy, UEHL, DUS-Kumaraswamy and DUS-UEHL distributions with bounded supports in the literature and it is shown that the proposed distribution has better performance on the real dataset.

The findings of this article can be extended by applying the transmuted family type transformation to the omega distribution. In this context, investigating the impact of other support parameters contained within the transmuted family type transformation and the omega distribution could be a new research topic.

**Author Contribution**

*Murat Genç*; methodology, software, formal analysis, investigation, visualization, writing - original draft. *Ömer Özbilen;* conceptualization, methodology, validation, writing - review & editing. The authors wrote, read and approved the paper together.

**Ethics**

There are no ethical issues related to the publication of this article.

**Conflict of Interest**

The authors declare that there is no conflict of interest.

**ORCID**

*Murat Genç* https://orcid.org/0000-0002-6335-3044

*Ömer Özbilen* https://orcid.org/0000-0001-6110-1911

## References

Adetunji, A. (2023). Transmuted Ailamujia distribution with applications to lifetime observations. *Asian Journal of Probability and Statistics, 21*(1), 1-11. https://doi.org/10.9734/AJPAS/2023/v21i1452

Adetunji, A. A., & Sabri, S. R. (2024). A new two-parameter Poisson-transmuted exponential distribution: Properties and applications in count observations. *AIP Conference Proceedings, 3016*(1). https://doi.org/10.1063/5.0192459

Ahsan-ul-Haq, M., Aldahlan, M. A., Zafar, J., Gómez, H. W., Afify, A. Z., & Mahran, H. A. (2023). A new cubic transmuted power-function distribution: Properties, inference, and applications. *Plos one, 18*(2), e0281419. https://doi.org/10.1371/journal.pone.0281419

Aryal, G. R., & Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods & Applications, 71*(12), e1401-e1407. https://doi.org/10.1016/j.na.2009.01.168

Aryal, G. R., & Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of pure and applied mathematics, 4*(2), 89-102. https://www.ejpam.com/index.php/ejpam/article/view/1170

Birbiçer, İ., Genç, A. İ. (2023). On parameter estimation of the standard omega distribution. *Journal of Applied Statistics, 50*(15), 3108-3124. https://doi.org/10.1080/02664763.2022.2101045

Bourguignon, M., Leão, J., Leiva, V., Santos-Neto, M. (2017). The transmuted birnbaum-saunders distribuition. *Revstat Statistical Journal, 15*(4), 601-628. https://doi.org/10.57805/revstat.v15i4.229

Cheng, Y.-F., Wang, F.-K. (2012). Estimating the Burr XII parameters in constant-stress partially accelerated life tests under multiple censored data. *Communications in Statistics-Simulation and Computation, 41*(9), 1711-1727. https://doi.org/10.1080/03610918.2011.617478

Cordeiro, G. M., de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation, 81*(7), 883-898. https://doi.org/10.1080/00949650903530745

Dey, S., Wang, L., Nassar, M. (2022). Inference on Nadarajah-Haghighi distribution with constant stress partially accelerated life tests under progressive type-II censoring. *Journal of Applied Statistics, 49*(11), 2891-2912. https://doi.org/10.1080/02664763.2021.1928014

Dombi, J., Jónás, T. (2020). On an alternative to four notable distribution functions with applications in engineering and the business sciences. *Acta Polytechnica Hungarica, 17*(1), 231-252. http://dx.doi.org/10.12700/APH.17.1.2020.1.13

Dombi, J., Jonas, T., Toth, Z. E., Arva, G. (2019). The omega probability distribution and its applications in reliability theory. *Quality and Reliability Engineering International, 35*(2), 600-626. https://doi.org/10.1002/qre.2425

Genç, M., Özbilen, Ö. (2023). An extension of the UEHL distribution based on the DUS transformation. *Journal of New Theory, 44*, 20-30. https://doi.org/10.53570/jnt.1317652

Gharaibeh, M. M., Al-Omari, A. I. (2019). Transmuted Ishita distribution and its applications. *Journal of Statistics Applications Probability, 8*(2), 67-81. https://doi.org/10.18576/jsap/080201

Gui, W. (2017). Exponentiated half logistic distribution: Different estimation methods and joint confidence regions. *Communications in Statistics-Simulation and Computation, 46*(6), 4600-4617. https://doi.org/10.1080/03610918.2015.1122053

Gupta, R. C., Gupta, P. L., Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and methods, 27*(4), 887-904. https://doi.org/10.1080/03610929808832134

Kang, S. B., & Seo, J. I. (2011). Estimation in an exponentiated half logistic distribution under progressively Type-2 censoring. *Communications for Statistical Applications and Methods, 18*(5), 657-666. http://dx.doi.org/10.5351/CKSS.2011.18.5.657

Karakaya, K., Kınacı, İ., Kuş, C., & Akdoğan, Y. (2021). On the DUS-kumaraswamy distribution. *Istatistik Journal of The Turkish Statistical Association, 13*(1), 29-38. https://dergipark.org.tr/en/pub/ijtsa/issue/62665/807304

Khan, M. S., King, R., & Hudson, I. L. (2018). Transmuted modified Weibull distribution: Properties and application. *European Journal of Pure and Applied Mathematics, 11*(2), 362-374. https://doi.org/10.29020/nybg.ejpam.v11i2.3208

Khan, M. S., King, R., & Hudson, I. L. (2020). Transmuted Burr type X distribution with covariates regression modeling to analyze reliability data. *American Journal of Mathematical and Management Sciences, 39*(2), 99-121. https://doi.org/10.1080/01966324.2019.1605320

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology, 46*(1-2), 79-88. https://doi.org/10.1016/0022-1694(80)90036-0

Kuş, C., Karakaya, K., Tanış, C., Akdoğan, Y., Sert, S. & Kalkan, F. (2023). Compound transmuted family of distributions: Properties and applications. *Ricerche di Matematica*, 1-20. https://doi.org/10.1007/s11587-023-00808-7

Naz, S., Al-Essa, L. A., Bakouch, H. S., & Chesneau, C. (2013). A transmuted modified power-generated family of distributions with practice on submodels in insurance and reliability. *Symmetry, 15*(7), 1458. https://doi.org/10.3390/sym15071458

Özbilen, Ö., & Genç, A. İ. (2022). A bivariate extension of the omega distribution for two-dimensional proportional data. *Mathematica Slovaca, 72*(6), 1605-1622. https://doi.org/10.1515/ms-2022-0111

Rahman, M. M., Al-Zahrani, B., Shahbaz, S. H., & Shahbaz, M. Q. (2020). Transmuted probability distributions: A review. *Pakistan Journal of Statistics and Operation Research*, *16*(1), 83-94. https://doi.org/10.18187/pjsor.v16i1.3217

Rahman, M. M., Gemeay, A. M., Islam Khan, M. A., Meraou, M. A., Bakr, M., Muse, A. H., Balogun, O. S. (2023). A new modified cubic transmuted-G family of distributions: Properties and different methods of estimation with applications to real-life data. *AIP Advances, 12*(9). https://doi.org/10.1063/5.0170178

Rastogi, M. K., & Tripathi, Y. M. (2014). Parameter and reliability estimation for an exponentiated half-logistic distribution under progressive type II censoring. *Journal of Statistical Computation and Simulation, 84*(8), 1711-1727. https://doi.org/10.1080/00949655.2012.762366

Samuel, A. F. (2019). On the performance of transmuted logistic distribution: statistical properties and application. *Budapest International Research in Exact Sciences (BirEx) Journal, 1*(3), 26-34. https://doi.org/10.33258/birex.v1i3.341

Seo, J. I., & Kang, S. B. (2015). Notes on the exponentiated half logistic distribution. *Applied Mathematical Modelling, 39*(21), 6491-6500. https://doi.org/10.1016/j.apm.2015.01.039

Shaw, W. T., & Buckley, I. R. (2007). *The Alchemy of Probability Distributions: Beyond Gram Charlier Cornish Fisher Expansions, and Skew-Normal or Kurtotic-Normal Distributions.* Technical report, Financial Mathematics Group, King's College, London, U.K.

Tushar, M. T., Rashedi, K. A., Alshammari, T. S., & Rahman, M. M. (2024). Second order transmuted kumaraswamy distribution and its related results. *European Journal of Pure and Applied Mathematics, 17*(2), 616-637. https://doi.org/10.29020/nybg.ejpam.v17i2.5069

Tushar, M. T., Shahbaz, S. H., Rahman, M. M., & Shahbaz, M. Q. (2024). A new cubic transmuted inverse weibull distribution: Theory and applications. *Pakistan Journal of Statistics and Operation Research*, 85-98. https://doi.org/10.18187/pjsor.v20i1.4448