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An application of stochastic maximum principle for a constrained system with memory

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Abstract. In this research article, we study a stochastic control problem in a theoretical frame to solve a constrained task under memory impact. The nature of memory is modeled by Stochastic Differential Delay Equations and our state process evolves according to a jump-diffusion process with time-delay. We work on two specific types of constraints, which are described in the stochastic control problem as running gain components. We develop two theorems for corresponding deterministic and stochastic Lagrange multipliers. Furthermore, these theorems are applicable to a wide range of continuous-time stochastic optimal control problems in a diversified scientific area such as Operations Research, Biology, Computer Science, Engineering and Finance. Here, in this work, we apply our results to a financial application to investigate the optimal consumption process of a company via its wealth process with historical performance. We utilize the stochastic maximum principle, which is one of the main methods of continuous-time Stochastic Optimal Control theory. Moreover, we compute a real-valued Lagrange multiplier and clarify the relation between this value and the specified constraint.

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1. Introduction and Unconstrained Control Problem

Stochastic Optimal Control theory is one of the main fields of sequential decision-making under uncertainty. Its fundamental goal is to determine the optimal control processes and the optimal value function for a specified control task, see [21,22,35]. The state process of a control problem is generally represented by a diffusion process, a jump-diffusion process or by a larger model such as a regime-switching process, see [2, 11, 15, 25, 28, 30, 33]. These processes meet specific mathematical requirements of each problem in a wide range of scientific disciplines such as finance, insurance, biology computer science, engineering etc. Whenever the uncertainty in an application can be expressed as a continuous-time process, diffusion processes can be used effectively. On the other side, in real-life applications, we usually require discontinuous formulations and in those cases, jump-diffusion processes and regime-switching models well-describe sudden changes in the process as well as in the environment.

Especially, in financial applications, the state processes may represent the price process of a risky asset, the wealth process of a company, the surplus process of an insurance policy, etc. Furthermore, since stochastic control theory provides quite strong tools to handle uncertainty and to develop optimal feedback controls, it is widely utilized in quantitative finance, see [6, 9, 12, 16, 26, 27, 34]. In this work, we use a jump-diffusion model to present the wealth process of a company and it is well known that such models efficiently describe the abrupt changes in the dynamics of a risky asset (for a broad literature, see [3]). The probabilistic literature for jump processes has been extensively developed and applied in financial mathematics so far, see also [1].

Moreover, in our article, we study a stochastic control problem with *memory* and *constraints*. The memory component is represented by a *time-delay* term, $\delta > 0$, in the dynamics of a Stochastic Differential Delay Equation (**SDDE**) (for a comprehensive theory of such equations, see [17]). Moreover, SDDEs express real-life financial phenomena more realistically with a meaning of historical performance of risky assets, economic inertia, time lag in financial operations. Hence, such systems have got significant

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attention from the researchers in the Stochastic Optimal Control field so far, see [7, 8, 14, 20, 23, 25, 28] and references therein.

Let us introduce the technical details and mathematical structure of our work:

As we stated, we use a jump-diffusion process with delay as the state process of our control task (for a detailed theory of continuous-time stochastic processes, see [1, 13, 19] and references therein).

Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ be. $\mathcal{B}_0$ represents a Borel σ-field generated by the open subset O of $\mathbb{R}_0$, whose closure does not include the point 0.

Let $(N(dt, dz) : t \in [0, T], z \in \mathbb{R}_0)$ be a Poisson random measure on $([0, T] \times \mathbb{R}_0, \mathcal{B}([0, T]) \otimes \mathcal{B}_0)$. The Lévy measure of $N(\cdot, \cdot)$ is defined by ν and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is a compensated Poisson random measure.

Let $(W(t) : t \in [0, T])$ be a Brownian motion. $(\Omega, \mathbb{F}, \mathcal{F}_t, \mathbb{P})$ represents a complete filtered probability space generated by the Brownian motion $W(\cdot)$ and the Poisson random measure $N(\cdot, \cdot)$. We define $\mathbb{F} = (\mathcal{F}_t : t \in [0, T])$ as a right-continuous, $\mathbb{P}$-completed filtration and assume that the Brownian motion and the Poisson random measure are independent of each other and adapted to $\mathbb{F}$.

We follow a controlled jump-diffusion model with a constant delay term $\delta > 0$, which is one of the most general representations of such systems and is introduced in [20] as follows:

$$\begin{aligned} dX(t) =& b(t, X(t), Y(t), A(t), u(t))dt + \sigma(t, X(t), Y(t), A(t), u(t))dW(t) \\ &+ \int_{\mathbb{R}_0} \eta(t, X(t), Y(t), A(t), u(t), z)\tilde{N}(dt, dz) \\ X(t) =& \theta(t), \qquad t \in [-\delta, 0], \end{aligned} \tag{1}$$

where for $t \in [0, T]$,

$$Y(t) = X(t - \delta), \qquad A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r)dr.$$

The coefficient functions of the model are defined as:

$$\begin{aligned} b &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}, \\ \sigma &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}, \\ \eta &: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \to \mathbb{R}, \end{aligned}$$

and generally, in financial applications, b, σ, and η represent appreciation rate, volatility and jump size of a risky asset correspondingly.

Moreover, for example, while Brownian motion $W(\cdot)$ catches little shocks in the price process of an asset, the Poisson random measure $N(\cdot, \cdot)$ captures the jumps of that process, which occur as a consequence of abrupt changes, sudden news or big sell/buy orders in the financial markets.

In this model, we observe the *memory* component in the dynamics of the system as $Y(\cdot)$ and $A(\cdot)$ terms. Note that for the systems described by SDDEs, rather than an initial value, we need an initial path. $\theta(\cdot)$ represents the initial path and is a continuous, deterministic function. Here, $\rho \geq 0$ is a constant averaging parameter.

We assume that $\mathcal{U}$ is a non-empty subset of $\mathbb{R}$ and represents a set of admissible control values $u(t)$, $t \in [0, T]$. We define an *admissible control process* $u(\cdot)$ as a $\mathcal{U}$-valued, $\mathcal{F}_t$-measurable and càdlàg process such that the Equation (1) has a unique solution $X(\cdot) \in L^2(\xi \times \mathbb{P})$, where ξ represents the Lebesgue measure on $[0, T]$. Let $\mathcal{A}$ denote a family of admissible control processes (for more detail, see [20]).

Moreover, we assume that

$$E\left[\int_0^T |u(t)|^2 \, dt\right] < \infty.$$

For all $u \in \mathcal{A}$, let us define the objective criterion in the classical sense (for a broad survey of the Stochastic Optimal Control theory, see [21, 22, 32] and references therein) as follows:

$$\begin{aligned} J(u) &= J(x, y, a, u) \\ &= E\left[\int_0^T f(t, X(t), Y(t), A(t), u(t))dt + g(X(T))\right], \end{aligned} \tag{2}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ represents the running gain and $g : \mathbb{R} \to \mathbb{R}$ corresponds to the terminal gain of the control task. Here, we assume that f and g are $\mathcal{C}^1$-functions with respect to x, y, a, u such that for all $x_i = x, y, a, u$,

$$E\left[\int_0^T \left(|f(t, X(t), Y(t), A(t), u(t))| + \left|\frac{\partial f}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2\right) dt + |g(X(T))| + |g_x(X(T))|^2\right] < \infty.$$

Hence, in a classical *unconstrained* stochastic control problem, our goal is to find the optimal control $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{3}$$

On the other hand, in this work, we formulate the constraints inspired by Theorem 11.3.1 of [19] but with completely different constraints. In this theorem, the author presents an approach for the stochastic control tasks with a condition at the terminal time $T > 0$ for a diffusion process. Later, [4] gave an application of this theorem and [28] extended this theorem to the stochastic differential games with regimes.

Furthermore, [5] stated a version of Theorem 11.3.1 of [19] with constraint types (5) and (6) for a jump-diffusion process. These constraints describe deterministic and stochastic Lagrange multipliers, correspondingly and are different than the terminal conditions given in Theorem 11.3.1 of [19]. But the authors do not investigate the Lagrange multipliers however they claimed that their existence is a crucial condition to apply the proved theorems, see Theorem 5.2 and 5.4 of [5]. In our work, we study a stochastic control problem for a jump-diffusion process with the *memory* and the *constraints* defined with (5) and (6). Hence, our work extends the theorems of [5] to a *delayed model.* Moreover, we develop an application for which the corresponding Lagrange multiplier exists. In that sense, we should underline that our work is the first work that completes the desired task with the constraints (5)-(6) and also, by inserting a delay term, we study a larger model.

We do not prefer to define many technical conditions over b, σ, η in this section. In Section 2, we will develop two fundamental theorems to approach stochastic control problems with the constraints (5) and (6). These can be solved by both Stochastic Maximum Principle (**SMP**) and Dynamic Programming Principle (**DPP**). Thus, the technical assumptions have to be determined specifically depending on the preferred method. We will highlight them in Section 3, while we are studying an optimal consumption problem.

This article is organized as follows: In Section 2, we introduce the mathematical formulation of our constrained stochastic control problem and demonstrate the corresponding theorems in a Lagrangian environment. Section 3 is devoted to developing a financial application, which formulates the optimal consumption process of a company with memory. The final section gives a conclusion.

2. Reformulation of the Control Task within the Context of Constraints

In this section, we develop two theorems which describe the optimal control process and investigate the corresponding Lagrange multipliers for a time-delayed stochastic control system.

Firstly, let us state the value function of the constrained control problem:

$$\phi(x, y, a) = J(u^*) = \sup_{u \in \Theta} J(x, y, a, u). \tag{4}$$

Here, $J(\cdot)$ is defined by Equation (2) and the supremum is taken over Θ of all admissible controls $u : \mathbb{R} \to \mathcal{U} \subset \mathbb{R}$ such that

$$E\left[\int_0^T M(t, X(t), Y(t), A(t), u(t)) dt\right] = 0, \tag{5}$$

or

$$\int_0^T M(t, X(t), Y(t), A(t), u(t)) dt = 0 \ a.s.. \tag{6}$$

$M : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is a C^1 function with respect to x, y, and a such that for $x_i = x, y, a, u$:

$$E\left[\int_0^T \left(|M(t, X(t), Y(t), A(t), u(t))| + \left|\frac{\partial M}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2\right) dt\right] < \infty.$$

Here, we study two types of constraints: The constraint type (5) represents a real valued Lagrange multiplier and the type (6) discovers a stochastic one.

Thus, we should specify the set of stochastic Lagrange multipliers as in [27]:

$$\Delta = \Big\{ \lambda : \Omega \to \mathbb{R} | \lambda \text{ is } \mathcal{F}_T - \text{measurable and } E[|\lambda|] < \infty \Big\}.$$

Now, by observing the Equation (4) and the constraints (5) and (6), let us present the *unconstrained* stochastic control problem in the following way:

$$\begin{aligned} \phi^{\lambda}(x,y,a) = & \sup_{u\in\Theta} J(x,y,a,u) \\ = & \sup_{u\in\Theta} E^{x,y,a}\Big[\int_0^T f(t,X(t),Y(t),A(t),u(t))dt + g(X^u(T)) \\ & + \lambda \int_0^T M(t,X(t),Y(t),A(t),u(t))dt\Big], \end{aligned} \tag{7}$$

subject to the system (1).

First, we will prove the following theorem corresponding to the type (6):

Theorem 1. *Assume that for all $\lambda \in \Delta_1 \subset \Delta$, we can develop $\phi^{\lambda}(x,y,a)$ and the optimal control process $u^{*,\lambda}$, which solves the unconstrained stochastic control problem (7) subject to the system (1). Moreover, assume that there exists $\lambda_0 \in \Delta_1$, such that*

$$\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0})dt = 0, \qquad a.s. \tag{8}$$

Then, $\phi(x,y,a) = \phi^{\lambda_0}(x,y,a)$ is obtained and $u^* = u^{*,\lambda_0}$ solves the constrained stochastic control problem (3) subject to (1) and (6).

Proof. The first inequality appears by definition of the optimal value function as follows:

$$\begin{aligned} \phi^{\lambda}(x,y,a) = & \; J(x,y,a,u^{*,\lambda}) \\ & = E^{x,y,a}\Big[\int_0^T f(t, X_t^{u^{*,\lambda}}, Y_t^{u^{*,\lambda}}, A_t^{u^{*,\lambda}}, u^{*,\lambda})dt \\ & + \lambda \int_0^T M(t, X_t^{u^{*,\lambda}}, Y_t^{u^{*,\lambda}}, A_t^{u^{*,\lambda}}, u_t^{*,\lambda})dt + g(X_T^{u^{*,\lambda}})\Big] \\ & \geq J(x,y,a,u^{\lambda}) \\ & = E^{x,y,a}\Big[\int_0^T f(t, X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u^{\lambda})dt \\ & + \lambda \int_0^T M(X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u_t^{\lambda})dt + g(X_T^{u^{\lambda}})\Big]. \end{aligned} \tag{9}$$

In particular, if $\lambda = \lambda_0$ exists and since $u_1 \in \Theta$ is feasible in the constrained control problem (3), then by (8):

$$\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0})dt = \int_0^T M(X_t^{u^{\lambda}}, Y_t^{u^{\lambda}}, A_t^{u^{\lambda}}, u_t^{\lambda})dt = 0 \tag{10}$$

Therefore, by (9) and (10):

$$\phi^{\lambda_0}(x,y,a) = J(u^{*,\lambda_0}) = J(x,y,a,u^{*,\lambda_0}) \geq J(x,y,a,u) = J(u),$$

for all $u \in \Theta$. Note that $u^{*,\lambda_0} \in \Theta$ and this completes the proof. □

The following theorem can be proved similarly for the constraint type (5).

Theorem 2. *Assume that for all $\lambda \in K \subset \mathbb{R}$, we can determine $\phi^{\lambda}(x, y, a)$ and the optimal control process $u^{*,\lambda}$ solving the unconstrained stochastic control problem (7) subject to (1). Furthermore, assume that there exists $\lambda_0 \in K$ such that*

$$E\left[\int_0^T M(t, X_t^{u^{*,\lambda_0}}, Y_t^{u^{*,\lambda_0}}, A_t^{u^{*,\lambda_0}}, u_t^{*,\lambda_0})dt\right] = 0.$$

Then, $\phi(x, y, a) = \phi^{\lambda_0}(x, y, a)$ and $u^* = u^{*,\lambda_0}$ solves the constrained stochastic control problem (3) subject to the model (1) and the constraint (5).

Remark 1. *Theorem 1 and 2 can be applied to a wide range of stochastic control problems by both* ***SMP*** *and* ***DPP*** *as long as it is possible to determine the corresponding Lagrange multipliers. If we prefer to apply DPP, we should be careful about Markov property. SDDEs provide a more realistic environment to interact but we loose Markov property. Moreover, since we have an initial path instead of an initial value for the system (1), our problem creates the corresponding partial differential equations so-called Hamilton-Jacobi-Bellman equations in an infinite dimensional space. Hence, a direct application of DPP is not mathematically possible (more details to handle such problems by DPP in [7, 8, 14] and reference therein).*

Remark 2. *To utilize SMP, we do not need any Markovian assumption different than DPP. Hence, in this work, we will combine the method described in Theorem 2 of our paper with Theorem 3.1 and Theorem 4.1 of [20] to find the optimal consumption process by SMP.*

Remark 3. *Our work is inspired from Theorem 11.3.1 of [19], but we should highlight that the constraint of Theorem 11.3.1 of [19] is defined at terminal time T as:*

$$E[M(X_T^u)] = 0, \tag{11}$$

which is completely different than our constraints (5)-(6). We put a condition over running gain component rather than the terminal gain. Moreover, we can see similar constraints in [5] but both [19] and [5] do not include memory impact.

Remark 4. *In [30], we studied memory impact within the framework of Lagrange multipliers similar to Equation 11, which is a different type of constraint as we stated in Remark 3. Furthermore, in [30], we focused on a dividend policy application in a regime-switching environment with a different control formulation. Our present work and [30] share a similar philosophy with completely different constraints and financial formulations.*

Now, let us present an application of Theorem 2 in finance.

3. Application to Finance

In this section, we will develop the formulation of an optimal consumption process that corresponds to the wealth process of a company with memory. This process evolves according to a time-delayed jump-diffusion model. The dynamics of the model carry past values of the wealth process in the form of $Y(t) = X(t - \delta),\ t \in [0, T]$, where $\delta > 0$ is a constant. Our purpose is to develop a more realistic consumption policy, which depends on the information about the historical performance of the company as well.

$\mu(\cdot)$ is a deterministic function and represents the appreciation rate of the company. Furthermore, we suppose that $\sigma(t)$ and $\eta(t, z)$, $t \in [0, T]$, are given bounded, square integrable and adapted processes. $\mathcal{U}$ is a non-empty, closed and convex subset of $\mathbb{R}$. In this section, our problem formulation justifies the technical assumptions provided in [20] thus, we are allowed to apply Theorem 3.1 and Theorem 4.1 of that article.

The consumption process is a càdlàg, $\mathcal{F}_t$-adapted control process, which satisfies:

$$E\left[\int_0^T |c(t)|^2\, dt\right] < \infty.$$

Let us state the wealth process $X(t) = X^c(t)$, which is a special form of Equation (1) as follows:

$$
\begin{aligned}
dX(t) = & \Big(X(t-\delta)\mu(t) - c(t)\Big)dt + X(t-\delta)\Big(\sigma(t)dW(t) \\
& + \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz)\Big), \ t \in [0,T], \\
X(t) = & \ \theta(t), \quad t \in [-\delta, 0],
\end{aligned}
\tag{12}
$$

where $\theta(\cdot)$ is a given nonnegative, deterministic and continuous function.

We assume that the company wants to maximize its wealth despite a quadratic running loss by balancing it corresponding to a constraint of linear running gain, which is described in terms of the control process. Moreover, the company aims to reach a level of a constant K times the terminal time $T > 0$. So we assume that the company takes into account time restrictions as well. We will develop and highlight the conditions over K at the end of our computations. Hence, our goal is to find the optimal consumption process $c^*(\cdot)$ by solving:

$$
\begin{aligned}
J(c^*) &= \sup_{c \in \Theta} J(c) \\
&= \sup_{c \in \Theta} E\Big[\int_0^T \alpha(t)c^2(t)dt + \beta X(T)\Big]
\end{aligned}
$$

subject to the system (12) and to the constraint:

$$
E\Big[\int_0^T \gamma(t)c(t)dt\Big] = TK, \ K \in \mathbb{R}, \tag{13}
$$

where $\alpha(\cdot) < 0$ and $\gamma(\cdot)$ are deterministic functions and $\beta \in \mathbb{R}$.

Now we can develop the Lagrangian form of this stochastic control problem as follows:

$$
\begin{aligned}
J(c^*) = & \sup_{c \in \Theta} J(c) \\
= & \sup_{c \in \Theta} E\Big[\int_0^T \alpha(t)c^2(t)dt + \lambda \int_0^T (\gamma(t)c(t) - K)dt + \beta X(T)\Big],
\end{aligned}
\tag{14}
$$

for which we aim to find $c^* = c^{\lambda,*}$ and the real-valued Lagrange multiplier $\lambda = \lambda^0$ described in Theorem 2.

Since we apply SMP to solve the problem (14), first, we define the Hamiltonian corresponding to the wealth process (12):

$$
\begin{aligned}
H(t,x,y,a,c,p,q,r(\cdot)) = & \ \alpha(t)c^2 + \lambda(\gamma(t)c - K) + (\mu(t)y - c)p + y\sigma(t)q \\
& + y\int_{\mathbb{R}_0} \eta(t,z)r(t,z)\nu(dz).
\end{aligned}
\tag{15}
$$

Note that it is clearly seen that Hamiltonian H is a concave function of x, y, a and c, hence the concavity condition over H is satisfied, see Theorem 3.1 of [20] is justified.

Furthermore, we should present the corresponding Anticipated Backward Stochastic Differential Equation (**Anticipated BSDE**) and solve it for unknown $p(t),\ q(t)$, and $r(t,z)$.

For $t \in [0,T]$, let us introduce:

$$
\begin{aligned}
dp(t) = & -E\Big[\Big(\mu(t+\delta)p(t+\delta) + \sigma(t+\delta)q(t+\delta) \\
& + \int_{\mathbb{R}_0} \eta(t+\delta,z)r(t+\delta,z)\nu(dz)\Big)\mathbf{1}_{[0,T-\delta]}(t)|\mathcal{F}_t\Big]dt \\
& + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz)
\end{aligned}
\tag{16}
$$

$$
p(T) = \beta. \tag{17}
$$

We call *Anticipated* to this type of BSDEs since as seen in $\mu, \sigma, \eta, p(\cdot), q(\cdot)$, and $r(\cdot,\cdot)$, the terms involve time-advanced values in the form of $t+\delta$ for $t \in [0,T]$. This type of BSDEs was first introduced and developed by Peng and Yang, see [23]. For technical definitions of the Hamiltonian (15) and the System

(16)-(17), please see Apendix 4 or Section 2 in [20]. Furthermore, see [25, 28] for the formulation of Anticipated BSDEs and their relation with SDDEs via different models.

We follow the technique described in [20] to find the solution for $p(\cdot), q(\cdot)$, and $r(\cdot,\cdot)$, which will be computed inductively in the following way:

Step 1: For $t \in [T-\delta, T]$, the corresponding adjoint equation becomes:

$$\begin{aligned} dp(t) &= q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz), \\ p(T) &= \beta, \end{aligned}$$

for which we have the solution:

$$p(t) = E[p(T)|\mathcal{F}_t] = \beta, \qquad t \in [T-\delta, T].$$

By martingale representation theorem, since the Lagrange multiplier is a real value, we choose $q = r = w = 0$. Hence, the Anticipated BSDE gets the form:

$$\begin{aligned} dp(t) &= -\mu(t+\delta)p(t+\delta)\mathbf{1}_{[0,T-\delta]}(t)dt, \quad t \le T, \\ p(t) &= \beta, \qquad t \in [T-\delta, T]. \end{aligned}$$

Step 2: We define:

$$h(t) = p(T-t), \qquad t \in [0,T]. \tag{18}$$

That way, we get a deterministic delay equation:

$$\begin{aligned} dh(t) = -dp(T-t) &= \mu(T-t+\delta)p(T-t+\delta)dt \\ &= \mu(T-t+\delta)h(t-\delta)dt, \qquad t \in [\delta, T], \\ h(t) = p(T-t) &= \beta, \quad t \in [0,\delta]. \end{aligned}$$

For such equations, again, we have an approach of solving inductively. Since we can compute $h(t)$ on $[(j-1)\delta, j\delta]$, we obtain:

$$\begin{aligned} h(t) &= h(j\delta) + \int_{j\delta}^{t} h'(s)ds \\ &= h(j\delta) + \int_{j\delta}^{t} \mu(T-s+\delta)h(s-\delta)ds \end{aligned} \tag{19}$$

for $t \in [j\delta, (j+1)\delta]$, $j = 1, 2, ...$

Now, we should maximize the Hamiltonian (15) with respect to c to get:

$$c^*(t) = \frac{1}{2}\alpha(t)(p(t) - \lambda\gamma(t)), \; t \in [0,T]. \tag{20}$$

As a consequence of the nature of constrained stochastic control problems, we should compute the value of Lagrange multiplier λ^0 to use Theorem 2 properly.

Solving stochastic delay equations require special approaches different than usual stochastic differential equations. By the Equation (20), the wealth process becomes:

$$\begin{aligned} dX(t) &= (X(t-\delta)\mu(t) - \frac{1}{2}\alpha(t)(p(t) - \lambda\gamma(t)))dt + X(t-\delta)(\sigma(t)dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz)), \; t \in [0,T], \\ X(t) &= \theta(t), \quad t \in [-\delta, 0]. \end{aligned} \tag{21}$$

We know that the SDDE (21) can be solved by successive Itô integrations over steps of length δ (see Section 1, page 7 in [18]). Specifically, we assume that terminal time $T = 2\delta$. This assumption is just for the sake of simplicity and does not pretend to show the complete methodology of applying the technique. Thus, the total duration that we study is the interval of $[-\delta, 2\delta]$.

First, for $t \in [0,T]$, let us define:

$$dL(t) = \sigma(t)dW(t) + \int_{\mathbb{R}_0} \eta(t,z)\tilde{N}(dt,dz).$$

By also observing (18) and (19), we provide the following open form of the solution process:

$$X(t)=\theta(t),\ \textbf{if}\quad -\delta\le t\le 0,$$

$$X(t)=\theta(0)+\int_0^t\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^t\theta(s-\delta)dL(s)\qquad\textbf{if}\quad 0\le t\le\delta,$$

$$X(t)=X(\delta)+\int_\delta^t\Big(\Big\{\theta(0)+\int_0^{v-\delta}\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}\mu(v)-\frac{1}{2}\alpha(v)(h(T-v)-\lambda\gamma(v))\Big)dv$$
$$+\int_\delta^t\Big\{\theta(0)+\int_0^{v-\delta}\Big(\theta(s-\delta)\mu(s)-\frac{1}{2}\alpha(s)(h(T-s)-\lambda\gamma(s))\Big)ds$$
$$+\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}dL(v)\qquad\textbf{if}\quad\delta\le t\le 2\delta=T.$$

Now, the values of $h(T-t)$, $t\in[0,T]$ at the above integrals can be determined by following the boundary values of the integrals and their relation with t. Remember that $T=2\delta$.
Then, by (19)

$$\text{if},\quad 0\le s\le t\le\delta,\quad\text{then},\quad\delta\le T-s\le 2\delta,$$
$$h(2\delta-s)=h(\delta)+\int_\delta^{2\delta-s}\mu(3\delta-u)h(u-\delta)du,$$
$$h(2\delta-s)=\beta\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big).$$

Moreover,

$$\text{if},\quad 0\le s\le v-\delta\quad\text{and}\quad\delta\le v\le t\le 2\delta,\ \text{then},\quad 0\le v-\delta\le t-\delta\le\delta,$$
$$\text{so},\quad 0\le s\le\delta,\quad\text{then}\quad\delta\le 2\delta-s\le 2\delta,\quad\text{then, by (19)},$$
$$h(2\delta-s)=\beta\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big).$$

Finally,

$$\text{if},\quad\delta\le v\le t\le 2\delta,\quad\text{then},\quad 0\le 2\delta-v\le\delta,\quad\text{then, by 18}\quad h(2\delta-v)=\beta.$$

Firstly, we change the value of $h(\cdot)$ according to the relevant intervals in the above solution processes and integrate the Equation (21) from 0 to 2δ by following the above δ-length description of $X(\cdot)$. Then, we apply expectation to both sides of the Equation (21).

Now, let us introduce the following terms:

$$A=\int_0^{2\delta}\alpha(s)\gamma(s)ds+E\Big[\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\alpha(s)\gamma(s)ds\Big)\Big\{\mu(v)dv+dL(v)\Big\}\Big]$$

and

$$B=\theta(0)-E\Big[X(2\delta)\Big]+\int_0^\delta\theta(s-\delta)\mu(s)ds$$
$$-\frac{1}{2}\beta\int_0^\delta\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds+E\Big[\int_0^\delta\theta(s-\delta)dL(s)\Big]$$
$$+\int_\delta^{2\delta}\theta(0)\mu(v)dv+\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\theta(s-\delta)\mu(s)ds\Big)\mu(v)dv$$
$$-\beta\int_\delta^{2\delta}\Big(\int_0^{v-\delta}\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds\Big)\frac{1}{2}\mu(v)dv$$

$$
\begin{aligned}
&+E\Big[\int_\delta^{2\delta}\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big)\mu(v)dv\Big]\\
&-\frac{1}{2}\beta\int_\delta^{2\delta}\alpha(v)dv+E\Big[\int_\delta^{2\delta}\Big\{\theta(0)+\int_0^{v-\delta}\theta(s-\delta)\mu(s)ds\\
&-\frac{1}{2}\beta\int_0^{v-\delta}\alpha(s)\Big(1+\int_\delta^{2\delta-s}\mu(3\delta-u)du\Big)ds+\int_0^{v-\delta}\theta(s-\delta)dL(s)\Big\}dL(v)\Big].
\end{aligned}
$$

Then, we get:

$$
\lambda=\frac{2B}{A}\qquad\text{on condition that}\quad A\neq 0. \tag{22}
$$

Now, by (18) and (19), let us make some observations about the constraint (13):

$$
\begin{aligned}
E\Big[\int_0^T\gamma(t)c(t)dt\Big]=&\frac{1}{2}\int_0^\delta\gamma(t)\alpha(t)\Big[\beta\Big(1+\int_\delta^{2\delta-t}\mu(3\delta-u)du\Big)-\lambda\gamma(t)\Big]dt\\
&+\frac{1}{2}\int_\delta^{2\delta}\gamma(t)\alpha(t)(\beta-\lambda\gamma(t))dt\\
=&2\delta K.
\end{aligned}
$$

Then, let us utilize the above equality to clarify λ and define the following terms:

$$
D=\frac{\beta}{2}\Big[\int_0^{2\delta}\gamma(t)\alpha(t)dt+\int_0^\delta\gamma(t)\alpha(t)\Big(\int_\delta^{2\delta-t}\mu(3\delta-u)du\Big)dt\Big]-2\delta K,
$$

and

$$
C=\int_0^{2\delta}\gamma^2(t)\alpha(t)dt.
$$

Then, we obtain:

$$
\lambda=\frac{2D}{C}\qquad\text{on condition that}\quad C\neq 0. \tag{23}
$$

Finally, by observations (22)-(23), we conclude that in order to use Theorem 2, we have to specify the K value in Equation (5) carefully such that

$$
\frac{D}{C}=\frac{B}{A}.
$$

By this final result, we determined explicitly the control process $c^*(\cdot)$, the Lagrange multiplier λ^0 and consequently, the solution for $p(\cdot)$ corresponding to the Anticipated BSDE (16)-(17), and all the technical assumptions required.

4. Conclusion and Future Work

In this work, we studied a constrained stochastic control problem and investigated the impact of delay term on Lagrange multipliers. We proved two theorems for two different types of constraints and gave an application in finance for the case of a real-valued Lagrange multiplier. We focused on the wealth process of a company, which evolves according to a jump-diffusion model with historical values in its dynamics. We observed that however the Theorems 1 and 2 are applicable for a wide range of control tasks by both SMP and DPP, determining the Lagrange multipliers remains as a challenge. It is not always easy to compute these parameters. Furthermore, the step of formulating these multipliers can not be ignored because the provided theorems are enforceable on the condition that there exists a Lagrange multiplier for which the constraint is justified. Despite this challenge, to the best of our knowledge, our article presents the first results for a delayed system with constraints in running gain of the control task and computes the corresponding Lagrange multiplier exactly. In our financial application, we clearly present the technical differences for solving a delayed SDE and a usual one by applying Itô's formula recursively.

Furthermore, since stochastic control theory is a discipline of sequential decision-making, we may encounter some challenges from the side of model selection. The decision maker may believe that her model is perfect. But in reality, generally, this is not the case. Especially, in finance, model misidentification can cause high financial losses. At this point, robust control designs different control or decision rules performing fare well across alternative models [11, 34]. Especially, in stochastic games, we handle model uncertainty in a relative entropy context as a penalty term [2, 6, 9]. It is known that Hansen and

Sargent [10] used a Lagrange multiplier theorem to convert the entropy constraint onto a penalty on perturbations from the model. Therefore, we would like to underline the potential of our work towards robust stochastic control and stochastic games.

Risk minimization and worst-case scenarios have significant value in quantitative finance and insurance since each action with uncertainty carries a potential for loss that cannot be underestimated. Therefore, as a further study, we aim to focus on the relation between Lagrange multipliers and robust control. These structures can be approached from the side of relative entropy as well as from the sides of Var and CVar concepts, see [9, 15, 16]. Furthermore, within the wide scope of *risk management*, Lagrange multipliers can be handled via computational methods such as deep learning and deep reinforcement learning, see [24, 31].

On the other hand, we strongly believe that however delay systems are demanding and challenging, they will be highlighted within the context of other hot fields such as Deep Learning. However, the aim of our research article is to provide theoretical and technical approaches, in [29], we present a collection of novel aspects within the intersection of computer science and stochastic optimal control under the memory component.

Declaration of Competing Interests The author declares no conflict of interest.

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Appendix

In order to apply SMP, we have to define corresponding Hamiltonian for a delayed system as follows: $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$,

$$\begin{aligned} H(t,x,y,a,u,p,q,r) =& f(t,x,y,a,u) + b(t,x,y,a,u)p + \sigma(t,x,y,a,u)q \\ &+ \int_{\mathbb{R}_0} \eta(t,x,y,a,u,z) r(t,z)\nu(dz) \end{aligned} \tag{24}$$

where $\mathcal{R}$ denotes the set of all functions
$r : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$, for which the integral in (24) converges.

Associated to H, the adjoint, unknown and adapted processes $(p(t) \in \mathbb{R} : t \in [0,T])$, $(q(t) \in \mathbb{R} : t \in [0,T])$, and $(r(t,z) \in \mathcal{R} : t \in [0,T], z \in \mathbb{R}_0)$ are described by the following Anticipated BSDE with jumps:

$$\begin{aligned} dp(t) &= E[\mu(t)|\mathcal{F}_t]dt + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) \\ p(T) &= g_x(X(T)), \end{aligned}$$

where

$$\begin{aligned} \mu(t) :=& -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot)) \\ &- \frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot)) \\ &\times \mathbf{1}_{[0,T-\delta]}(t) - e^{\rho t}\Big(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot)) \end{aligned}$$

$$\times e^{-\rho s}\mathbf{1}_{[0,T]}(s)ds\Big). \tag{25}$$

As seen in $\mu(t)$, we have the future values of $X(s), u(s), p(s), q(s)$, and $r(s,\cdot)$ for $s \leq t+\delta$ in Equation (25), hence we call *Anticipated* to this type of BSDEs.