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On Some New Rhaly Sequence Spaces and Rhaly Sections in BK-Space

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Abstract
In this paper, we introduce some new sequence spaces and sectional subspaces related to the Rhaly matrix and BK spaces. Furthermore, we investigate their relations and identities among these subspaces and duals.

Keywords: AK-space, Distinguished subspaces, Matrix domain, Rhaly matrix, rK-space
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1. Background, Preliminaries and Notations

Let w be the linear space of all complex or real valued sequences with the topology τ_w of coordinatwise convergence. A linear subspace of w is called a sequence space. A sequence space λ with a locally convex topology τ is a K-space if the inclusion map: $(\lambda,\tau)\to(w,\tau_w)$ is continuous. If τ is complete metrizable and locally convex, (λ,τ) is called FK-space. An FK-space whose topology is normable is called a BK-space. The basic properties of FK(BK-)-spaces may be found in [1, 2].

By ℓ_∞, c, c_0 and ℓ_p, we denote the spaces of all bounded, convergent, null and absolutely p-summable complex sequences, respectively, where $1\le p<\infty$. The spaces ℓ_∞, c and c_0 are BK-space endowed with the sup norm $\|x\|_\infty=\sup_{k\in\mathbb{N}}|x_k|$, and ℓ_p $(1\le p<\infty)$ is a BK-space with the norm $\|x\|_p=\left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}$, where $\mathbb{N}=\{0,1,2,3,\ldots\}$.

Let X and Y be two sequence spaces, and $A=(a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k,n\in\mathbb{N}$. Then, we say that A defines a *matrix transformation* from X into Y and we denote it by writing $A:X\to Y$, if for every sequence $x=(x_k)\in X$ the A-transform $Ax=\{(Ax)_n\}$ of x is in Y, where

$$(Ax)_n=\sum_{k=0}^{\infty}a_{nk}x_k \text{ for each } n\in\mathbb{N}. \tag{1.1}$$

By $(X:Y)$, we denote the class of all matrices A such that $A:X\to Y$. Thus, $A\in(X:Y)$ if and only if the series on the right side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in X$, and we have $Ax\in Y$ for all $x\in X$. Also, we write $A_n=(a_{nk})_{k\in\mathbb{N}}$ for the sequence in the n^{th} row of A.

The *domain* X_A of an infinite matrix A in a sequence space X is defined by

$$X_A:=\left\{x=(x_k)\in w: Ax\in X\right\} \tag{1.2}$$

which is a sequence space. Depending on the choice of the matrix A, X_A may include or be included by the original space X. Indeed if we choose $A = \Delta$, the backward difference matrix, then $c_\Delta \supset c$ ($bv = (\ell_1)_\Delta \supset \ell_1$) but in the case $A = \Delta^{-1} = S$, the summation matrix, $c_S = cs \subset c$ ($bs = (\ell_\infty)_S \subset \ell_\infty$), where both of two inclusions are strict. However, if we define $X = c_0 \oplus span\{z\}$ with $z = \{(-1)^k\}$, i.e., $x \in X$ if and only if $x = s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n = (-1)^n e^n$ for all $n \in \mathbb{N}$, we have $Ae = z \in X$ but $Az = e \notin X$ which gives that $z \in X \setminus X_A$ and $e \in X_A \setminus X$ where e^k is a sequence whose only nonzero term is 1 in k^{th} place for each $k \in \mathbb{N}$. That is to say that the sequence spaces X_A and X are overlap but neither contains the other. In the literature, there are many studies on the matrix domain, see for instance [3]-[19].

The *continuous dual* of a normed space X is defined as the space of all bounded linear functionals on X and denoted by X'. If A is triangle, that is $a_{nk} = 0$ if $k > n$ and $a_{nn} \neq 0$, and X is a sequence space, then $f \in X'_A$ if and only if $f = g \circ A$, $g \in X'$.

Let (X, P) be a locally convex space. A set $S \subset X$ is called *fundamental* if the span of S is dense in X. The useful results concerning with the fundamental set which are applications of Hahn-Banach Theorem as follows:

Corollary 1.1. *(i) $S \subset X$ is fundamental if and only if $f(S) = 0$ implies $f = 0$ for each $f \in X'$.*
(ii) Let S_1 and S_2 be non-empty subsets of X. The inclusion $S_1 \subset \overline{span\{S_2\}}$ holds if and only if $f(S_2) = 0$ implies $f(S_1) = 0$ for each $f \in X'$.

For the sequence spaces X, Y and Z, the multiplier space X^Y (or $M(X,Y)$) is defined by

$$X^Y = \{a = (a_k) \in w : \forall x \in X, x \cdot a = (x_k a_k) \in Y\},$$

and $X^{YZ} = (X^Y)^Z$. The β-, γ- and f-duals X^β, X^γ and X^f of a sequence space X are defined by

$$X^\beta := X^{cs} = \left\{ a = (a_k) \in w : \left(\sum_{k=0}^{n} a_k x_k \right)_{n \in \mathbb{N}} \in c \text{ for all } x = (x_k) \in X \right\},$$

$$X^\gamma := X^{bs} = \left\{ a = (a_k) \in w : \left(\sum_{k=0}^{n} a_k x_k \right)_{n \in \mathbb{N}} \in \ell_\infty \text{ for all } x = (x_k) \in X \right\},$$

and

$$X^f := \left\{ a = (a_k) \in w : \exists f \in X', a = (f(e^k)) \right\}$$

respectively.

Lemma 1.2. *[2] Let X be an* FK *space containing $\phi = span\{e^k\}$, and let Y and Z any sequence spaces. Then, the following assertions hold.*
(i) If $\phi \subset Y \subset Z$ then $\phi \subset X^Y \subset X^Z$,
(ii) if $X \subset Y$ then $X^Z \supset Y^Z$ and $X^f \supset Y^f$
(iii) $X \subset X^{YY}$
(iv) $X^Y = X^{YYY}$
(v) $X^\beta \subset X^\gamma \subset X^f$
(vi) $X^f = (\overline{\phi})^f$

Zeller in [20] introduced the theory of FK-spaces and investigated the properties of sectional convergence in [21]. Sectional boundedness in BK-spaces was studied by Sargent [22]. Given a BK-space $X \supset \phi$, we denote the n^{th} section of a sequence $x \in X$ by $x^{[n]} = \sum_{k=0}^{n} x_k e^k$, and we say that x has

AK-property when $\lim_{n\to\infty} \|x - x^{[n]}\|_X = 0$,
AB-property when $\sup_{n\in\mathbb{N}} \|x^{[n]}\|_X < \infty$,
AD-property when $x \in \overline{\phi}$ (closure of $\phi \subset X$),
SAK-property when $\lim_{n\to\infty} |f(x) - f(x^{[n]})| = 0$ for all $f \in X'$,
FAK-property when $(f(x^{[n]}) \in c$ for all $f \in X'$.

If one of these properties holds for every $x \in X$, then we say that the space X has that property. It is trivial that AK implies AB and AD. For example, the spaces c_0, cs and ℓ_p are AK and c, bs and ℓ_∞ are AB but not AD-spaces, where $1 \leq p < \infty$.

The distinguished subsets of summability domains and arbitrary FK spaces have been studied by Wilansky [2], Bennett [23], and several others [24]-[38].

We denote by $\mathfrak{U}$ the set of all real sequences $u=(u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For a sequence $u=(u_k) \in \mathfrak{U}$, the Rhaly (or Terraced) matrix $R_u = (r_{nk}(u))$ is defined by

$$r_{nk}(u) = \begin{cases} u_n & , \quad k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

For the special case $u_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, the Rhaly matrix R_u reduces to the Cesàro matrix of order 1. For more details on this topic, see [39, 40].

In this paper, we introduce the new sequence spaces $c_0(R_u), c(R_u)$ and $\ell_\infty(R_u)$, which are the domains of the Rhaly matrix R_u in the spaces c_0, c, and ℓ_∞, respectively, and study some of their properties. We also define sectional subspaces related to the Rhaly matrix in an FK space and investigate their relationships, identities and duals.

2. The Sequence Spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$

In the present section, we introduce the sequence spaces $c_0(R_u)$, $c(R_u))$ and $\ell_\infty(R_u)$ as the domain of the matrix R_u in the classical sequence spaces c_0, c and ℓ_∞, respectively and examine some properties of these spaces.

Throughout the study, $y=(y_n)$ will be the R_u-transform of a sequence $x=(x_k)$; that is,

$$y_n = (R_u x)_n = u_n \sum_{k=0}^{n} x_k \tag{2.1}$$

for all $n \in \mathbb{N}$. Since the matrix R_u is a triangle, it has an inverse. Multiplying the equality (2.1) with $1/u_n$, we have

$$\frac{1}{u_n} y_n = \sum_{k=0}^{n} x_k \tag{2.2}$$

for all $n \in \mathbb{N}$. Therefore, by using the relation (2.2) we see that

$$x_n = \frac{1}{u_n} y_n - \frac{1}{u_{n-1}} y_{n-1} \tag{2.3}$$

holds for all $n \in \mathbb{N}$, where $y_{-1} = 0$.

Now, by the equation (2.3) we have the following lemma:

Lemma 2.1. *The matrix R_u is invertible and its inverse $(R_u)^{-1} = (r_{nk}^{-1}(u))$ defined for all $k, n \in \mathbb{N}$ by*

$$r_{nk}^{-1}(u) = \begin{cases} (-1)^{n-k} \dfrac{1}{u_n} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n. \end{cases}$$

Let us introduce the sequence spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$ as the set of all sequences whose R_u-transforms are in the classical spaces c_0, c and ℓ_∞, respectively; that is

$$\begin{aligned} c_0(R_u) &:= \left\{ x=(x_k) \in w : \lim_{n\to\infty} u_n \sum_{k=0}^{n} x_k = 0 \right\}, \\ c(R_u) &:= \left\{ x=(x_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} u_n \sum_{k=0}^{n} x_k = \alpha \right\}, \\ \ell_\infty(R_u) &:= \left\{ x=(x_k) \in w : \sup_{n\in\mathbb{N}} \left| u_n \sum_{k=0}^{n} x_k \right| < \infty \right\}. \end{aligned}$$

With the notation of (1.2), the spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$ can be redefined, as follows:

$$c_0(R_u) = (c_0)_{R_u}, \quad c(R_u) = (c)_{R_u} \quad \text{and} \quad \ell_\infty(R_u) = (\ell_\infty)_{R_u}.$$

It is known from [1]-[3] and [5, 16] that if T is a triangle, then the domain X_T of T in a normed sequence space X is normed with $\|x\|_{X_T} = \|Tx\|_X$, and is linearly norm isomorphic to X and X_T has a basis if and only if X has a basis.

As a direct consequence of these facts, we have:

Corollary 2.2. *Let $Z \in \{c_0, c, \ell_\infty\}$. Then, the following statements hold:*

(a) The space $Z(R_u)$ is a BK-space endowed with the norm

$$\|x\|_{Z(R_u)} = \sup_{n\in\mathbb{N}} \left| u_n \sum_{k=0}^{n} x_k \right|.$$

(b) The spaces $Z(R_u)$ is linearly norm isomorphic to the space Z.

Corollary 2.3. *Define the sequence $b^{(k)}(u) = \left(b_n^{(k)}(u)\right)_{n\in\mathbb{N}}$ by*

$$b_n^{(k)}(u) := \begin{cases} (-1)^{n-k}\dfrac{1}{u_n} & , \quad k \le n \le k+1 \\ 0 & , \quad n < k \text{ or } n > k+1 \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, the following statements hold:

(a) The sequence $b^{(k)}(u)$ is a basis for the spaces $c_0(R_u)$ and every sequence $x \in c_0(R_u)$ has a unique representation of the form $x = \sum_{k=0}^{\infty}(R_u x)_k b^{(k)}(u)$.

(b) The set $\left\{\tilde{e}, b^{(k)}(u)\right\}$ is a basis for the space $c(R_u)$ and every sequence $x \in c(R_u)$ has a unique representation of the form $x = l\tilde{e} + \sum_{k=0}^{\infty}\left[(R_u x)_k - l\right] b^{(k)}(u)$, where $\tilde{e} = \left(\dfrac{1}{u_k} - \dfrac{1}{u_{k-1}}\right)$ for all $k \in \mathbb{N}$ and $(R_u x)_k \to l$, as $k \to \infty$.

(c) The space $\ell_\infty(R_u)$ does not have a basis.

Since the inclusions $c_0 \subset c \subset \ell_\infty$ hold strictly, we have:

Theorem 2.4. *The inclusions $c_0(R_u) \subset c(R_u) \subset \ell_\infty(R_u)$ hold strictly.*

Lemma 2.5. *Let X and Y be sequence spaces, and let A and B be triangle matrices. Then, the inclusion $X_A \subset Y_B$ holds if and only if the matrix BA^{-1} belongs to (X, Y).*

Proof. Suppose that $X_A \subset Y_B$. Then, every $t \in X_A$ is in Y_B. By the definitions Y_B and X_A, we have $Bt \in Y$ and $x = At \in X$. Since A is a triangle matrix, it is invertible. From the equality $x = At$, we can obtain $t = A^{-1}x$. Hence, for each $x \in X$ the sequence $BA^{-1}x$ is in Y. This shows that $BA^{-1} \in (X, Y)$.

Conversely,suppose that $BA^{-1} \in (X, Y)$. Take any sequence $t \in X_A$. By the definition of X_A, we have $At \in X$. Since $BA^{-1} \in (X, Y)$, for $At \in X$, we have $BA^{-1}(At) \in Y$, and thus $Bt \in Y$. Therefore, $t \in Y_B$. This shows that the inclusion $X_A \subset Y_B$ holds. □

By using matrix transformations and Lemma 2.5, we can easily prove that:

Theorem 2.6. *The following assertions hold.*
(a) If $(ku_k) \in \ell_\infty$ then $c_0 \subset c_0(R_u)$ and $\ell_\infty \subset \ell_\infty(R_u)$ strictly holds.
(b) If $(ku_k) \in c$ then $c \subset c(R_u)$ strictly holds.
(c) If $(ku_k) \in c_0$ then $\ell_\infty \subset c_0(R_u)$ holds.
(d) If $(\frac{1}{u_k} - \frac{1}{u_{k-1}}) \in \ell_\infty$ then the inclusion $c_0(R_u) \subset c_0$ and $\ell_\infty(R_u) \subset \ell_\infty$ hold.
(e) If $(\frac{1}{u_k} - \frac{1}{u_{k-1}}) \in c$ then the inclusion $c(R_u) \subset c$ holds.

We shall begin with quoting the lemma due to Stieglitz and Tietz [41] which is needed in proving Theorem 2.8.

Lemma 2.7. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*
(a) $A \in (c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{2.4}$$

(b) $A \in (c_0 : c)$ if and only if (2.4) and

$$\lim_{n\to\infty} a_{nk} = \alpha_k (k \in \mathbb{N}), \tag{2.5}$$

(c) $A \in (c : c)$ if and only if (2.4), (2.5) and

$$\lim_{n\to\infty}\sum_k a_{nk} = \alpha.$$

(d) $A \in (\ell_\infty : c)$ if and only if (2.4), and

$$\lim_{n\to\infty}\sum_k |a_{nk}| = \sum_k |\lim_{n\to\infty} a_{nk}|$$

By [3, Theorem 3.1], we have:

Theorem 2.8. *For a sequence $u = (u_k) \in \mathfrak{U}$, let us define the sets $A_1(u)$, $A_2(u)$ and $A_3(u)$, as follows:*

$$\begin{aligned}
A_1(u) &:= \left\{ a = (a_k) \in w : \left(\frac{a_k - a_{k+1}}{u_k} \right) \in \ell_1 \right\}, \\
A_2(u) &:= \left\{ a = (a_k) \in w : \left(\frac{a_k}{u_k} \right) \in \ell_\infty \right\}, \\
A_3(u) &:= \left\{ a = (a_k) \in w : \left(\frac{a_k}{u_k} \right) \in c \right\}, \\
A_4(u) &:= \left\{ a = (a_k) \in w : \left(\frac{a_k}{u_k} \right) \in c_0 \right\}.
\end{aligned}$$

Then, the following statements hold:

(i) $[c_0(R_u)]^\beta = A_1 \cap A_2$, $[c(R_u)]^\beta = A_1 \cap A_3$, $[\ell_\infty(R_u)]^\beta = A_1 \cap A_4$.

(ii) $[c_0(R_u)]^\gamma = [c(R_u)]^\gamma = [\ell_\infty(R_u)]^\gamma = A_1 \cap A_2$.

Proof. For $a = (a_n) \in w$ and $x = (x_n) \in X(R_u)$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left[\sum_{j=k-1}^{k} \frac{(-1)^{k-j}}{u_k} y_j \right] \\
&= \sum_{k=0}^{n-1} \left(\frac{a_k - a_{k+1}}{u_k} \right) y_k + \frac{a_n}{u_n} y_n \\
&= (D_u y)_n
\end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$, where $D_u = (d_{nk}(u))$ is defined by

$$d_{nk}(u) = \begin{cases} \dfrac{a_k - a_{k+1}}{u_k} &, \quad k < n, \\ \dfrac{a_n}{u_n} &, \quad k = n, \\ 0 &, \quad k > n \end{cases}$$

The equation (2.6) implies that $ax = (a_n x_n) \in cs$ whenever $x \in X(R_u)$ if and only if $D_u y \in c$ whenever $y \in X$. Therefore, we conclude that $a \in [X(R_u)]^\beta$ if and only if $D_u \in (X : c)$.

(i) To show that $[c_0(R_u)]^\beta = A_1 \cap A_2$, let us take $X = c_0$. It follows that $D_u \in (c_0 : c)$, which means the conditions (2.4) and (2.5) of Lemma 2.7 (b) are satisfied by the matrix D_u. Thus, $a = (a_k) \in A_1 \cap A_2$. Therefore, we have:

$$[c_0(A^{ru})]^\beta = A_1 \cap A_2.$$

By using the conditions of Lemma 2.7(c) and (d), the equalities $[c(R_u)]^\beta = A_1 \cap A_3$, $[\ell_\infty(R_u)]^\beta = A_1 \cap A_4$ can be proved similarly.

(ii) This is similar to the proof of Part (i) of the present theorem by using Lemma 2.7(a). To avoid the repetition of the similar statements, we omit the details.

□

3. Some Rhaly Subspaces of FK spaces

In this section, using sectional properties we define some new subspaces of a BK-space and give some relations between these spaces and duals.

Given a BK-space $X \supset \phi$, we define the n^{th} Rhaly section of a sequence $x \in X$ as $r_x^{[n]} = u_n \sum_{k=0}^{n} x_k e^k$.

Definition 3.1. *Let X be a BK space contaning ϕ. Then, a sequence $x = (x_k) \in X$ has the following properties:*

$$\begin{aligned} rK \text{ when } & \lim_{n\to\infty} \|x - r_x^{[n]}\|_X = 0, \\ rB \text{ when } & \sup_{n\in\mathbb{N}} \|r_x^{[n]}\|_X < \infty, \\ SrK \text{ when } & \lim_{n\to\infty} |f(x) - f(r_x^{[n]})| = 0 \text{ for all } f \in X', \\ FrK \text{ when } & (f(r_x^{[n]}) \in c \text{ for all } f \in X'. \end{aligned}$$

In connection to Definition 3.1, we can define the following distinguished subset of X;

$$\begin{aligned} X_{RS} &= \{x \in X : x \text{ has rK in X}\}, \\ &= \{x \in X : x = \lim_n u_n \sum_{k=1}^{n} \sum_{i=1}^{k} x_i e^i\} \\ X_{RW} &= \{x \in X : x \text{ has SrK in X}\}, \\ &= \{x \in X : \forall f \in X', f(x) = \lim_n u_n \sum_{k=1}^{n} \sum_{i=1}^{k} x_i f(e^i)\} \\ X_{RF^+} &= \{x \in w : x \text{ has FrK in X}\}, \\ &= \{x \in w : \left(u_n \sum_{k=1}^{n} x^{(k)}\right) \text{ is weakly Cauchy in } X\} \\ &= \{x \in X : \forall f \in X', (u_n f(e^n)) \in (c(R_u))_S\}, \\ X_{RB^+} &= \{x \in w : x \text{ has rB in X}\}, \\ &= \{x \in w : \left(u_n \sum_{k=1}^{n} x^{(k)}\right) \text{ is bounded in } X\} \\ &= \{x \in X : \forall f \in X', (u_n f(e^n)) \in (\ell_\infty(R_u))_S\}, \end{aligned}$$

and

$$X_{RF} = X_{RF^+} \cap X \text{ and } X_{RB} = X_{RB^+} \cap X,$$

where the matrix $S = (s_{nk})$ is defined as

$$s_{nk} = \begin{cases} 1 & , \quad k \le n, \\ 0 & , \quad k > n \end{cases} .$$

By definitions of X_{RS}, X_{RW}, X_{RF^+} and X_{RB^+} we have:

Theorem 3.2. *Let X be an* FK*-space containing ϕ. Then the following inclusions hold.*

$$\phi \subset X_{RS} \subset X_{RW} \subset X_{RF} \subset X_{RB} \subset X$$

Theorem 3.3. *Let X be an* FK*-space containing ϕ. Then $X_{RW} \subset \overline{\phi}$.*

Proof. Let $f \in X'$ with $\phi \subset Kern f$. Since for every $x \in X$ and $n \in \mathbb{N}$, $z^n = \left(a_n \sum_{k=1}^{n} x^{(k)}\right) \in \phi$, then $f(z^n) = 0$. This shows that $X_{RW} \subset Kern f$. By Corollary 1.1 (ii), we obtain the inclusion $X_{RW} \subset \overline{\phi}$. □

By definition of X_{RF^+} (X_{RB^+}), $z \in X_{RF^+}(X_{RB^+})$ if and only if $z \cdot y \in (c(R_u))_S((\ell_\infty(R_u))_S)$ for each $y \in X^f$, we have the following theorems.

Theorem 3.4. *Let X be an* FK*-space containing ϕ. Then $X_{RF} = (X^f)^{(c(R_u))_S}$.*

Theorem 3.5. *Let X be an* FK*-space containing ϕ. Then $X_{RB} = (X^f)^{(\ell_\infty(R_u))_S}$*

In the study of FK-spaces, understanding the relationships between different sequence spaces and their properties is crucial. In this context, we investigate the inclusions and equalities among subspaces defined by various properties of sequences. The following results explore these relationships, focusing on the inclusion properties of different sequence spaces associated with the properties rK, SrK, FrK and rB when the inclusion $X \subset Y$ holds for FK-spaces X and Y. These results shed light on the structure of these spaces and the behavior of the sequence spaces under certain conditions.

Theorem 3.6. *If $X \subset Y$ then $X_\lambda \subset Y_\lambda$ for $\lambda \in \{RS, RW, RB^+, RF^+, RB, RF\}$.*

Proof. For $\lambda = RS(RW)$, the continuity(weak continuity) of inclusion map $i : X \to Y$ gives the desired result.
Let $\lambda \in \{RB^+, RF^+\}$. The results follows from Theorem 3.4, 3.5 and Lemma 1.2(ii). □

Theorem 3.7. *If $\overline{\phi} \subset Y \subset X$, then $Y_{RB^+} = X_{RB^+}$ and $Y_{RF^+} = X_{RF^+}$.*

Proof. By Theorem 3.6 we have

$$\overline{\phi}_{RB^+} \subset Y_{RB^+} \subset X_{RB^+}.$$

By Theorem 3.5 and Lemma 1.2(vi) the first and the last are equal. □

Theorem 3.8. *Let X be an* FK*-space containing ϕ and $X \subset X_{RB}$. Then $X_{RS} = X_{RW} = \overline{\phi}$.*

Proof. Since the sequence of functions (f_n) defined by $f_n : X \to X$, $f_n(x) = x - u_n \sum_{k=1}^{n} x^{(k)}$ is pointwise bounded, hence equicontinuous by (7.0.2) of [2]. Since $f_n \to 0$ on ϕ then also $f_n \to 0$ on $\overline{\phi}$ by (7.0.3) of [2]. This is the desired conclusion. □

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