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On the Frobenius norms of circulant matrices with Ducci sequences and Narayana and Gaussian Narayana numbers

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Abstract. In the present paper, we obtain identities for Narayana numbers, like the sum of terms with even and odd subscripts, the sum of products of consecutive terms and the sum of squares of terms. Then, we find images DN and D^2N of n-tuple $N = (N_1,\ N_2,\ N_3,\ \dots,\ N_n)$ of Narayana numbers under a map $D : \mathbb{C} \to \mathbb{C}$ defined as $D(z_1,\ z_2,\ \dots,\ z_n) = (|z_2 - z_1|,\ |z_3 - z_2|,\ \dots,\ |z_n - z_{n-1}|,\ |z_n - z_1|)$. We are then determined the circulant, skew-circulant, and semi-circulant matrices of these images. We have been discovered Frobenius norms of these circulant matrices and relations among these norms. In addition, we find DG and D^2G by taking the n-tuple $G = (GN_1,\ GN_2,\ \dots,\ GN_n)$ of Gaussian Narayana numbers. After that, we create circulant, semi-circulant, and skew-circulant matrices of $G,\ DG,\ D^2G$, determine their Frobenius norms, and derive relationships between them. Then, we obtain relations between norms of matrices of Narayana numbers and Gaussian Narayana numbers. Finally, coding and decoding methods with the use of circulant matrices of Narayana numbers and Gaussian Narayana numbers have been introduced.

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1. Introduction

If we consider the n-tuple $A = (a_1,\ a_2,\ \dots,\ a_n)$, where $a_i's$ s are integers, then the Ducci map $D_1 :\ \mathbb{Z}^n \to\ \mathbb{Z}^n$ is defined as

$$D_1(a_1,\ a_2,\ a_3,\ \dots,\ a_n) = (|a_2 - a_1|,\ |a_3 - a_2|,\ |a_4 - a_3|,\ \dots,\ |a_n - a_{n-1}|,\ |a_n - a_1|),$$

and the sequence $(A,\ D_1A,\ D_1^2A,\ D_1^3A,\ \dots)$ is known as the Ducci sequence. Ducci sequence [7] was introduced in 1937 but E. Ducci made some observations on this map in the $1800's$. Therefore, the name of this sequence was attributed to E. Ducci.

Circulant matrices have many connections to problems in physics, image processing, probability, statistics, numerical analysis, number theory and geometry. In literature, there are many papers in which authors use different kinds of number sequences in circulant matrices. In a circulant matrix [8], one basic row of numbers is repeated again and again with a shift in position. The inverse, conjugate transpose, sum and product of circulant matrices are also circulant. In [20, 21], Solak et al. used Fibonacci sequence to define the Ducci map and construct circulant matrices. They found the Frobenius norm, l_p norm and spectral norm of these sequences. The authors of [14, 15, 19] discovered bound estimation of spectral norms of circulant matrices composed of Padovan, Perrin, Fibonacci, and Lucas numbers. In [17], upper and lower bounds were found for the spectral norms of r-circulant matrices made up of Fibonacci and Lucas numbers. In [5, 6], the authors considered r-circulant matrices made up of generalized Fibonacci numbers and found determinants, bounds for the Euclidean and spectral norms of these circulant matrices. In [16], Shen et al. have studied spectral norms of r-circulant matrices made up of k-Fibonacci and

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k-lucas numbers. Then, found lower and upper bounds of those circulant matrices. In [11,12], Euclidean and spectral norms of circulant matrices of k-Fermat and k-Mersenne Numbers and generalized Fermat numbers have been obtained. Bala et al. [1] studied norms of circulant matrices of Gaussian Fibonacci numbers.

In [22], Stakhov gave an application of Fibonacci p-matrix defined in coding theory. Fibonacci p-numbers were defined in 1977 by the relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1), \ p = 0, 1, 2, 3, ... \ \text{with } n \geq p+1$$

and initial terms

$$F_p(1) = F_p(2) = F_p(3) = ... = F_p(p) = F_(p+1) = 1.$$

He used Fibonacci p-matrix denoted as Q_p, where

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{(p+1)\times(p+1)}.$$

He represented the message as a square matrix M of order $(p+1)\times(p+1)$, $p = 0, 1, 2, ...$ and used Q_p^n as the coding matrix, i.e., $M \times Q_p^n = E$ for Fibonacci coding and $E \times Q_p^n = M$ for Fibonacci decoding, here E is the coded matrix. Basu et al. [4] gave a coding theory based on Fibonacci p-numbers and their m extension. Tas et al. [23] gave a Fibonacci coding-decoding algorithm based on Fibonacci Q matrix and they divided the message matrix into block matrices of size 2×2 by adding zeroes between words and at the end for making the matrix even sized. Tas et al. and Ucar et al. gave similar algorithms. Ucar et al. [24] used Fibonacci Q matrix and $R = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ matrix in their coding-decoding algorithm. These matrices are related by Lucas numbers as

$$R_n = RQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} l_{n+1} & l_n \\ l_n & l_{n-1} \end{pmatrix}.$$

Shtayat et al. [18] introduced a cryptography model based on the Padovan Q-matrix and Perrin R-matrix. This method uses different encryption and decryption keys of the message matrix, which will increase the security of the methods.

Here, we define a map for complex numbers. For an n-tuple $X = (z_1, \ z_2, \ ... \ , \ z_n)$, here $z_i's$ are complex numbers, define a map $D : \mathbb{C} \to \mathbb{C}$ as

$$D(z_1, \ z_2, \ ... \ , \ z_n) = (|z_2 - z_1|, \ |z_3 - z_2|, \ ... \ , \ |z_n - z_{n-1}|, \ |z_n - z_1|)$$

or $D : \mathbb{C} \to \mathbb{R}$ as

$$D(z_1, \ z_2, \ ... \ , \ z_{n-1}, \ z_n) = (|z_2 - z_1|, \ |z_3 - z_2|, \ ... \ , \ |z_n - z_{n-1}|, \ |z_n - z_1|).$$

In the paper, we will consider map of the first kind.

The present paper is organized as follows. In section 2, basic definitions are given. In section 3, some lemmas that will be used in further sections are proved. In section 4, n-tuple of Narayana numbers are taken and its images under the Ducci map are obtained. Further, Frobenius norms of these matrices and relations of these norms are calculated. In section 5, results related to Gaussian Narayana numbers are given. In section 6, relations between norms of Narayana and Gaussian Narayana numbers are obtained. In section 7, coding/decoding algorithms are given using Narayana and Gaussian Narayana numbers.

2. Preliminaries

The Narayana sequence is defined by second order recurrence relation

$$\begin{aligned} N_{n+1} &= N_n + N_{n-2} \text{ for all } n \geq 2 \\ &\text{where } N_0 = 0, \ N_1 = 1 \text{ and } N_2 = 1. \end{aligned} \tag{1}$$

Ozkan et al. [13] defined Gaussian Narayana sequence as

$$GN_{n+1} = GN_n + GN_{n-2} \text{ for all } n \geq 2$$

where $GN_0 = i,\ GN_1 = 1$ and $GN_2 = 1$.

We note that $GN_n = N_n + iN_{n-2}$ for all $n \geq 2$ where N_n is n-th Narayana number.

Table 1. Some values of Gaussian Narayana numbers

n	0	1	2	3	4	5	6	7	8	9	10	...
GN_n	i	1	1	$1+i$	$2+i$	$3+i$	$4+2i$	$6+3i$	$9+4i$	$13+6i$	$19+9i$	...

Definition 1. *[10] The Frobenius (Euclidean) norm of an $n \times n$ matrix $A = (a_{ij})$ is defined as*

$$\|A\|_F = \sqrt{\left(\sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}|^2\right)}.$$

Definition 2. *[8] An $n \times n$ circulant matrix has the form*

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix},$$

and is denoted by $A = circ\ (c_0,\ c_1,\ c_2,\ ...\ ,\ c_{n-1})$.

Definition 3. *[8] For an n-tuple $(c_0,\ c_1,\ c_2,\ ...\ ,\ c_{n-1})$, a skew-circulant matrix M is defined as*

$$M = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ -c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ -c_{n-2} & -c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ -c_1 & -c_2 & -c_3 & \dots & c_0 \end{pmatrix},$$

and is denoted by $M = scirc\ (c_0,\ c_1,\ c_2,\ ...\ ,\ c_{n-1})$.

Definition 4. *[8] For an n-tuple $(c_0,\ c_1,\ c_2,\ ...\ ,\ c_{n-1})$, a semi-circulant matrix is defined as*

$$N = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ 0 & c_0 & c_1 & \dots & c_{n-2} \\ 0 & 0 & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & c_0 \end{pmatrix},$$

and is denoted by $N = circ_s\ (c_0,\ c_1,\ c_2,\ ...\ ,\ c_{n-1})$.

3. Some Identities of Narayana Numbers

In the current section, we obtained some identities for Narayana numbers. Then, we find the sum of n terms with even and odd subscripts, the sum of products of consecutive terms and the sum of squares of terms. These identities will be used in the next section.

We will need the following lemmas to prove our results

Lemma 1. *Sum of first n terms with odd subscripts*

$$N_1 + N_3 + N_5 + ... + N_{2n-1} = \frac{2N_{2n} + 2N_{2n-1} + N_{2n-2} - 1}{3}.$$

Proof. From equation (1), we have

$$N_{2n+1} = N_{2n+2} - N_{2n-1}.$$

If we put $n = 0,\ 1,\ 2,\ ...\ ,\ n-1$, we get

$$N_1 = N_2 - N_{-1} = N_2,$$

as $N_2 = N_1 + N_{-1}$, i.e., $N_{-1} = 1 - 1 = 0$

$$\begin{aligned} N_3 &= N_4 - N_1, \\ N_5 &= N_6 - N_3, \\ &\dots \\ N_{2n-3} &= N_{2n-2} - N_{2n-5}, \\ N_{2n-1} &= N_{2n} - N_{2n-3}, \end{aligned}$$

respectively. Adding all of the above equalities,

$$N_1 + N_3 + N_5 + \dots + N_{2n-1} = (N_2 + N_4 + N_6 + \dots + N_{2n}) - (N_1 + N_3 + N_5 + \dots + N_{2n-3})$$
$$2(N_1 + N_3 + N_5 + \dots + N_{2n-1}) - N_{2n-1} = N_2 + N_4 + N_6 + \dots + N_{2n}.$$

It is well known from [3] that

$$\begin{aligned} N_1 + N_2 + N_3 + \dots + N_{2n} &= N_{2n+3} - 1 \\ (N_1 + N_3 + N_5 + \dots + N_{2n-1}) + (N_2 + N_4 + N_6 + \dots + N_{2n}) &= N_{2n+3} - 1 \\ (N_1 + N_3 + N_5 + \dots + N_{2n-1}) + 2(N_1 + N_3 + N_5 + \dots + N_{2n-1}) - N_{2n-1} &= N_{2n+3} - 1 \\ N_1 + N_3 + N_5 + \dots + N_{2n-1} &= \frac{2N_{2n} + 2N_{2n-1} + N_{2n-2} - 1}{3}. \end{aligned}$$

□

Corollary 1. *Sum of first n terms with even subscripts*

$$N_2 + N_4 + \dots + N_{2n} = \frac{4N_{2n} + N_{2n-1} + 2N_{2n-2} - 2}{3}.$$

Proof.

$$\begin{aligned} N_1 + N_2 + \dots + N_{2n} &= N_{2n+3} - 1 \\ (N_1 + N_3 + \dots + N_{2n-1}) + (N_2 + N_4 + \dots + N_{2n}) &= N_{2n+3} - 1 \\ \frac{2N_{2n} + 2N_{2n-1} + N_{2n-2} - 1}{3} + N_2 + N_4 + \dots + N_{2n} &= N_{2n+3} - 1 \quad \text{(By using Lemma 1)} \\ N_2 + N_4 + \dots + N_{2n} &= N_{2n+3} - 1 - \frac{2N_{2n} + 2N_{2n-1} + N_{2n-2} - 1}{3} \end{aligned}$$

By using the equality $N_{2n+3} = 2N_n + N_{2n-2} + N_{2n-1}$, we get

$$N_2 + N_4 + \dots + N_{2n} = \frac{4N_{2n} + N_{2n-1} + 2N_{2n-2} - 2}{3}.$$

□

Lemma 2. *Sum of product of consecutive terms*

$$\sum_{k=1}^{n} N_k N_{k+1} = \frac{1}{3}\left(4N_n^2 + N_{n-1}^2 + N_{n-2}^2 + 4N_{n-2}N_n + 2N_{n-1}N_n + N_{n-2}N_{n-1} - 1\right).$$

Proof. From [2] Narayana matrix sequence

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_n^2 \implies \mathcal{N}_{n-1}\mathcal{N}_{n+1} = \mathcal{N}_n^2$$

and

$$\mathcal{N}_n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \\ N_{n-2} & N_{n-4} & N_{n-3} \end{pmatrix} \begin{pmatrix} N_{n+2} & N_n & N_{n+1} \\ N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \end{pmatrix} = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix} \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}$$

By using the equality of two matrices,

$$N_n N_{n+2} + N_{n-2}N_{n+1} + N_n N_{n-1} = N_{n+1}^2 + 2N_{n-1}N_n$$

$$\begin{aligned}
N_nN_{n+2}+N_{n-2}N_{n+1}-N_{n-1}N_n &= N_{n+1}^2\\
N_n(N_{n+1}+N_{n-1})+N_{n-2}N_{n+1}-N_{n-1}N_n &= N_{n+1}^2\\
N_nN_{n+1}+N_{n-2}N_{n+1} &= N_{n+1}^2\\
N_nN_{n+1}+N_{n-2}(N_n+N_{n-2}) &= N_{n+1}^2\\
N_nN_{n+1}+N_{n-2}N_n+N_{n-2}^2 &= N_{n+1}^2\\
N_nN_{n+1}+N_{n-2}(N_{n-1}+N_{n-3}) &= N_{n+1}^2-N_{n-2}^2\\
N_nN_{n+1}+N_{n-2}N_{n-1}+N_{n-3}N_{n-2} &= N_{n+1}^2-N_{n-2}^2\\
\sum_{k=3}^{n+3}N_kN_{k+1}+\sum_{k=3}^{n+3}N_{k-2}N_{k-1}+\sum_{k=3}^{n+3}N_{k-3}N_{k-2} &= \sum_{k=3}^{n+3}N_{k+1}^2-\sum_{k=3}^{n+3}N_{k-2}^2\\
\sum_{k=3}^{n+3}N_kN_{k+1}+\sum_{k=1}^{n+1}N_kN_{k+1}+\sum_{k=0}^{n}N_kN_{k+1} &= \sum_{k=3}^{n+3}N_{k+1}^2-\sum_{k=0}^{n}N_{k+1}^2
\end{aligned}$$

$$\begin{aligned}
3\sum_{k=1}^{n}N_kN_{k+1} &= N_{n+2}^2+N_{n+3}^2+N_{n+4}^2-2N_{n+1}N_{n+2}-N_{n+2}N_{n+3}-N_{n+3}N_{n+4}-1\\
\sum_{k=1}^{n}N_kN_{k+1} &= \frac{1}{3}\left(N_{n+2}^2+N_{n+3}^2+N_{n+4}^2-2N_{n+1}N_{n+2}-N_{n+2}N_{n+3}-N_{n+3}N_{n+4}-1\right)\\
\sum_{k=1}^{n}N_kN_{k+1} &= \frac{1}{3}\left(4N_n^2+N_{n-1}^2+N_{n-2}^2+4N_{n-2}N_n+2N_{n-1}N_n+N_{n-2}N_{n-1}-1\right).
\end{aligned}$$

□

Lemma 3. *Sum of squares of n terms*
$N_1^2+N_2^2+...+N_n^2$

$$=\frac{1}{3}\Big(2N_{n+1}^2+N_n^2+2N_{n-1}^2-N_{n-2}^2-N_{n-3}^2-2N_{n-4}^2+2N_nN_{n-1}-4N_{n-1}N_{n-2}-2N_{n-2}N_{n-3}+1\Big).$$

Proof. Bala and Mishra introduced Narayana matrix sequence [2] with he help of Narayana sequence. Using the identity from [2]

$$\mathcal{N}_n^2=\mathcal{N}_{2n}.$$

From here,

$$\begin{pmatrix} N_{n+1}^2+2N_{n-1}N_n & N_{n+1}N_{n-1}+N_{n-2}N_{n-1}+N_nN_{n-3} & N_nN_{n+1}+N_{n-1}^2+N_nN_{n-2}\\ N_nN_{n+1}+N_{n-2}N_n+N_{n-1}^2 & N_{n-1}N_n+N_{n-2}^2+N_{n-1}N_{n-3} & N_n^2+2N_{n-2}N_{n-1}\\ N_{n-1}N_{n+1}+N_nN_{n-3}+N_{n-2}N_{n-1} & N_{n-1}^2+2N_{n-2}N_{n-3} & N_{n-1}N_n+N_{n-1}N_{n-3}+N_{n-2}^2\end{pmatrix}$$
$$=\begin{pmatrix} N_{2n+1} & N_{2n-1} & N_{2n}\\ N_{2n} & N_{2n-2} & N_{2n-1}\\ N_{2n-1} & N_{2n-3} & N_{2n-2}\end{pmatrix}.$$

By using the equality of two matrices, we obtain

$$N_{n+1}^2+2N_{n-1}N_n=N_{2n+1},$$

that is,

$$N_{n+1}^2=N_{2n+1}-2N_{n-1}N_n.$$

For $n=0,1,2,3,...,n-1$, we have

$$\begin{aligned}
N_1^2 &= N_1-2N_{-1}N_0,\\
N_2^2 &= N_3-2N_0N_1,\\
N_3^2 &= N_5-2N_1N_2,\\
&\ \ ...\\
N_n^2 &= N_{2n-1}-2N_{n-2}N_{n-1}.
\end{aligned}$$

By using Lemma 2 and adding all above equalities, we obtain

$$N_1^2 + N_2^2 + ... + N_n^2 = (N_1 + N_3 + ... + N_{2n-1}) - 2(N_0N_1 + N_1N_2 + ... + N_{n-2}N_{n-1})$$

$$= \frac{2N_{2n} + 2N_{2n-1} + N_{2n-2} - 1}{3} - \frac{2}{3}\left(4N_n^2 + N_{n-1}^2 + N_{n-2}^2 + 4N_{n-2}N_n + 2N_{n-1}N_n + N_{n-2}N_{n-1} - 1\right)$$

$$= \frac{1}{3}\left(2N_{2n} + 2N_{2n-1} + N_{2n-2} - 8N_{n-2}^2 - 2N_{n-3}^2 - 2N_{n-4}^2 - 8N_{n-4}N_{n-2} - 4N_{n-3}N_{n-2} - 2N_{n-4}N_{n-3} + 1\right).$$

It is well known from [9] that

$$N_{n+m} = N_{n-1}N_{m+2} + N_{n-2}N_m + N_{n-3}N_{m+1}$$

and

$$N_{2n} = N_{n+1}^2 + N_{n-1}^2 - N_{n-2}^2.$$

Using these equalities, we obtain

$$N_{2n-1} = N_{n-1}N_{n+1} + N_{n-2}N_{n-1} + N_{n-3}N_n$$
$$N_{2n-2} = N_n^2 + N_{n-2}^2 - N_{n-3}^2.$$

Therefore,

$$\sum_{k=1}^{n} N_k^2 = \frac{1}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - 9N_{n-2}^2 - 3N_{n-3}^2 - 2N_{n-4}^2 + 2N_{n-1}N_{n+1} + 2N_{n-2}N_{n-1} + 2N_{n-3}N_n$$
$$- 8N_{n-4}N_{n-2} - 4N_{n-3}N_{n-2} - 2N_{n-4}N_{n-3} - 3\Big)$$
$$= \frac{1}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} - 2N_{n-2}N_{n-3} + 1\Big).$$

□

Lemma 4. *Sum of squares of n Narayana numbers multiplying by subscripts, i.e.,*

$$\sum_{k=1}^{n} kN_k^2 = \frac{n}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} - 2N_{n-2}N_{n-3}$$
$$+ 1\Big) - \frac{1}{9}\Big(2N_{n+1}^2 - 12N_n^2 - 4N_{n-1}^2 + 7N_{n-2}^2 + 8N_{n-3}^2 + 4N_{n-4}^2 + 6N_nN_{n-1} + 10N_{n-2}N_{n-1} + 4N_{n-2}N_{n-3}$$
$$- 16N_{n-2}N_n + 12N_nN_{n+1} + 3n - 13\Big).$$

Proof. Let

$$A_n = \sum_{k=1}^{n} N_k^2 = \frac{1}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} - 2N_{n-2}N_{n-3} + 1\Big)$$

$$\sum_{k=1}^{n} kN_k^2 = N_1^2 + 2N_2^2 + 3N_3^2 + ... + (n-1)N_{n-1}^2 + nN_n^2$$
$$= nA_n - \sum_{i=1}^{n-1} A_i$$
$$= nA_n - \frac{1}{3}\sum_{i=1}^{n-1}\Big(2N_{i+1}^2 + N_i^2 + 2N_{i-1}^2 - N_{i-2}^2 - N_{i-3}^2 - 2N_{i-4}^2 + 2N_iN_{i-1} - 4N_{i-1}N_{i-2} - 2N_{i-2}N_{i-3} - 3\Big)$$
$$= nA_n - \frac{1}{3}\Big(\sum_{i=1}^{n} N_i^2 - 4\sum_{i=1}^{n} N_iN_{i+1} + N_n^2 + 2N_{n-1}^2 + 4N_{n-2}^2 + 3N_{n-3}^2 + 2N_{n-4}^2 + 2N_{n-3}N_{n-2}$$
$$+ 6N_{n-2}N_{n-1} + 4N_{n-1}N_n + 4N_nN_{n+1} + n - 6\Big)$$
$$= \frac{n}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} - 2N_{n-2}N_{n-3} + 1\Big)$$
$$- \frac{1}{9}\Big(2N_{n+1}^2 - 12N_n^2 - 4N_{n-1}^2 + 7N_{n-2}^2 + 8N_{n-3}^2 + 4N_{n-4}^2 + 6N_nN_{n-1} + 10N_{n-2}N_{n-1} + 4N_{n-2}N_{n-3}$$

$$- 16N_{n-2}N_n + 12N_nN_{n+1} + 3n - 13\Big).$$

□

4. Circulant Matrices Made up of Narayana Numbers

In this section, images DN and D^2N of an n-tuple $N = (N_1,\ N_2,\ N_3,\ \dots,\ N_n)$ of Narayana numbers have been found under the Ducci map. Then, we establish circulant, skew-circulant and semi-circulant matrices of these images. Further, we find Frobenius norms of these circulant matrices and relations among these norms.

Let us consider $N = (N_1,\ N_2,\ N_3,\ \dots,\ N_{n-1},\ N_n)$, we get

$$\begin{aligned} DN &= (N_2 - N_1,\ N_3 - N_2,\ \dots,\ N_n - N_{n-1},\ N_n - N_1) \\ &= (0,\ 0,\ N_1,\ \dots,\ N_{n-3},\ N_n - 1), \\ D^2N &= (0,\ N_1,\ N_2 - N_1,\ N_3 - N_2,\ \dots,\ N_{n-3} - N_{n-4},\ N_n - 1 - N_{n-3},\ N_n - 1) \\ &= (0,\ 1,\ 0,\ 0,\ N_1,\ \dots,\ N_{n-6},\ N_{n-1} - 1,\ N_n - 1). \end{aligned}$$

Circulant($circ$), skew-circulant($scirc$) and semi-circulant($circ_s$) matrices are defined in section 2 (Preliminaries)

$$circ\ N = circ\ (N_1,\ N_2,\ N_3,\ \dots,\ N_{n-1},\ N_n), \tag{2}$$

$$circ\ DN = circ\ (0,\ 0,\ N_1,\ \dots,\ N_{n-3},\ N_n - 1), \tag{3}$$

$$circ\ D^2N = circ\ (0,\ 1,\ 0,\ 0,\ N_1,\ \dots,\ N_{n-6},\ N_{n-1} - 1,\ N_n - 1), \tag{4}$$

$$scirc\ N = scirc\ (N_1,\ N_2,\ N_3,\ \dots,\ N_{n-1},\ N_n), \tag{5}$$

$$scirc\ DN = scirc\ (0,\ 0,\ N_1,\ \dots,\ N_{n-3},\ N_n - 1), \tag{6}$$

$$scirc\ D^2N = scirc\ (0,\ 1,\ 0,\ 0,\ N_1,\ \dots,\ N_{n-6},\ N_{n-1} - 1,\ N_n - 1), \tag{7}$$

$$circ_s\ N = circ_s\ (N_1,\ N_2,\ N_3,\ \dots,\ N_{n-1},\ N_n), \tag{8}$$

$$circ_s\ DN = circ_s\ (0,\ 0,\ N_1,\ \dots,\ N_{n-3},\ N_n - 1), \tag{9}$$

$$circ_s\ D^2N = circ_s\ (0,\ 1,\ 0,\ 0,\ N_1,\ \dots,\ N_{n-6},\ N_{n-1} - 1,\ N_n - 1). \tag{10}$$

From equations (2) and (5),

$$\|circ\ N\|_F^2 = \|scirc\ N\|_F^2 = n\sum_{k=1}^{n} N_k^2. \tag{11}$$

From equations (3) and (6),

$$\|circ\ DN\|_F^2 = \|scirc\ DN\|_F^2 = n\left(\sum_{k=1}^{n-3} N_k^2 + (N_n - 1)^2\right). \tag{12}$$

From equations (4) and (7),

$$\left\|circ\ D^2N\right\|_F^2 = \left\|scirc\ D^2N\right\|_F^2 = n\left(1 + \sum_{k=1}^{n-6} N_k^2 + (N_{n-1} - 1)^2 + (N_n - 1)^2\right). \tag{13}$$

Theorem 1. *Relation between norms of circ N and circ DN*

$$\|circ\ N\|_F^2 - \|circ\ DN\|_F^2 = n\left(N_{n-1}^2 + N_{n-2}^2 + 2N_n - 1\right).$$

Proof. From equations (11) and (12),

$$\begin{aligned} \|circ\ N\|_F^2 - \|circ\ DN\|_F^2 &= n\left(\sum_{k=1}^{n} N_k^2 - \sum_{k=1}^{n-3} N_k^2 - (N_n - 1)^2\right) \\ &= n\left(N_n^2 + N_{n-1}^2 + N_{n-2}^2 - (N_n - 1)^2\right) \\ &= n\left(N_{n-1}^2 + N_{n-2}^2 + 2N_n - 1\right). \end{aligned}$$

□

Theorem 2. *Relation between norms of circ DN and circ D^2N*

$$\|circ\ DN\|_F^2 - \left\|circ\ D^2N\right\|_F^2 = n\left(N_{n-3}^2 + N_{n-4}^2 + N_{n-5}^2 - N_{n-1}^2 - 2 + 2N_{n-1}\right).$$

Proof. From equations (12) and (13),

$$\begin{aligned}\|circ\ DN\|_F^2 - \left\|circ\ D^2N\right\|_F^2 &= n\left(\sum_{k=1}^{n-3} N_k^2 + (N_n-1)^2 - 1 - \sum_{k=1}^{n-6} N_k^2 - (N_{n-1}-1)^2 - (N_n-1)^2\right)\\ &= n\left(N_{n-3}^2 + N_{n-4}^2 + N_{n-5}^2 - N_{n-1}^2 - 2 + 2N_{n-1}\right).\end{aligned}$$

□

Theorem 3. *Relation between norms of circ N and circ D^2N*

$$\|circ\ N\|_F^2 - \left\|circ\ D^2N\right\|_F^2 = n\left(N_{n-2}^2 + N_{n-3}^2 + N_{n-4}^2 + N_{n-5}^2 + 2N_n + 2N_{n-1} - 3\right).$$

Proof. From theorems 1 and 2, we get the required result. □

From equation (8),

$$\begin{aligned}\|circ_s\ N\|_F^2 &= nN_1^2 + (n-1)N_2^2 + \ \dots\ + 2N_{n-1}^2 + N_n^2\\ &= (n+1)(N_1^2 + N_2^2 + \dots + N_n^2) - (N_1^2 + 2N_2^2 + 3N_3^2 + \dots + nN_n^2)\end{aligned} \tag{14}$$

$$\begin{aligned}&= \frac{n+1}{3}\Bigg(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} - 2N_{n-2}N_{n-3}\\ &+ 1\Bigg) - \frac{n}{3}\Bigg(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2}\\ &- 2N_{n-2}N_{n-3} + 1\Bigg) + \frac{1}{9}\Bigg(2N_{n+1}^2 - 12N_n^2 - 4N_{n-1}^2 + 7N_{n-2}^2 + 8N_{n-3}^2 + 4N_{n-4}^2 + 6N_nN_{n-1}\\ &+ 10N_{n-2}N_{n-1} + 4N_{n-2}N_{n-3} - 16N_{n-2}N_n + 12N_nN_{n+1} + 3n - 13\Bigg)\\ &= \frac{1}{9}\Bigg(8N_{n+1}^2 - 9N_n^2 + 2N_{n-1}^2 + 4N_{n-2}^2 + 5N_{n-3}^2 - 2N_{n-4}^2 + 12N_nN_{n-1} + 2N_{n-1}N_{n-2}\\ &- 2N_{n-2}N_{n-3} - 16N_{n-2}N_n + 12N_nN_{n+1} + 3n - 10\Bigg).\end{aligned}$$

From equation (9),

$$\begin{aligned}&\|circ_s\ DN\|_F^2 = (n-2)N_1^2 + (n-3)N_2^2 + \ \dots\ + 2N_{n-3}^2 + (N_n-1)^2\\ &= (n-1)(N_1^2 + N_2^2 + \dots + N_{n-3}^2) - (N_1^2 + 2N_2^2 + \dots + (n-3)N_{n-3}^2) + (N_n-1)^2\end{aligned} \tag{15}$$

$$= \frac{n-1}{3}\bigg(2N_{n-2}^2 + N_{n-3}^2 + 2N_{n-4}^2 - N_{n-5}^2 - N_{n-6}^2 - 2N_{n-7}^2 + 2N_{n-3}N_{n-4} - 4N_{n-4}N_{n-5}$$

$$- 2N_{n-5}N_{n-6} + 1\bigg) - \frac{n-3}{3}\bigg(2N_{n-2}^2 + N_{n-3}^2 + 2N_{n-4}^2 - N_{n-5}^2 - N_{n-6}^2 - 2N_{n-7}^2 + 2N_{n-3}N_{n-4}$$

$$- 4N_{n-4}N_{n-5} - 2N_{n-5}N_{n-6} + 1\bigg) + \frac{1}{9}\bigg(2N_{n-2}^2 - 12N_{n-3}^2 + 4N_{n-4}^2 + 7N_{n-5}^2 + 8N_{n-6}^2 + 4N_{n-7}^2$$

$$+ 6N_{n-3}N_{n-4} + 10N_{n-4}N_{n-5} + 4N_{n-5}N_{n-6} - 16N_{n-5}N_{n-3} + 12N_{n-3}N_{n-2} + 3n - 22\bigg) + (N_n - 1)^2$$

$$= \frac{1}{9}\bigg(14N_{n-2}^2 - 6N_{n-3}^2 + 8N_{n-4}^2 + N_{n-5}^2 + 2N_{n-6}^2 - 8N_{n-7}^2 + 18N_{n-3}N_{n-4} - 14N_{n-4}N_{n-5}$$

$$- 8N_{n-5}N_{n-6} - 16N_{n-5}N_{n-3} + 12N_{n-3}N_{n-2} + 3n - 16\bigg) + (N_n - 1)^2.$$

From equation (10),

$$\left\|circ_s\ D^2N\right\|_F^2 = n - 1 + (n-4)N_1^2 + (n-5)N_2^2 + \ \dots\ + 3N_{n-6}^2 + 2(N_{n-1} - 1)^2 + (N_n - 1)^2$$

$$= n - 1 + (n-3)(N_1^2 + N_2^2 + \dots + N_{n-6}^2) - (N_1^2 + 2N_2^2 + \dots + (n-6)N_{n-6}^2) + 2(N_{n-1} - 1)^2 + (N_n - 1)^2 \tag{16}$$

$$= n - 1 + \frac{n-3}{3}\bigg(2N_{n-5}^2 + N_{n-6}^2 + 2N_{n-7}^2 - N_{n-8}^2 - N_{n-9}^2 - 2N_{n-10}^2 + 2N_{n-6}N_{n-7} - 4N_{n-7}N_{n-8}$$

$$- 2N_{n-8}N_{n-9} + 1\bigg) - \frac{n-6}{3}\bigg(2N_{n-5}^2 + N_{n-6}^2 + 2N_{n-7}^2 - N_{n-8}^2 - N_{n-9}^2 - 2N_{n-10}^2 + 2N_{n-6}N_{n-7}$$

$$- 4N_{n-7}N_{n-8} - 2N_{n-8}N_{n-9} + 1\bigg) + \frac{1}{9}\bigg(2N_{n-5}^2 - 12N_{n-6}^2 - 4N_{n-7}^2 + 7N_{n-8}^2 + 8N_{n-9}^2 + 4N_{n-10}^2$$

$$+ 6N_{n-6}N_{n-7} + 10N_{n-8}N_{n-7} + 4N_{n-8}N_{n-9} - 16N_{n-8}N_{n-6} + 12N_{n-6}N_{n-5} + 3n - 31\bigg) + 2(N_{n-1} - 1)^2$$

$$+ (N_n - 1)^2$$

$$= n - 1 + \frac{1}{9}\bigg(20N_{n-5}^2 - 3N_{n-6}^2 + 14N_{n-7}^2 - 2N_{n-8}^2 - N_{n-9}^2 - 14N_{n-10}^2 + 24N_{n-6}N_{n-7} - 26N_{n-8}N_{n-7}$$

$$- 14N_{n-8}N_{n-9} - 16N_{n-8}N_{n-6} + 12N_{n-6}N_{n-5} + 3n - 22\bigg) + 2(N_{n-1} - 1)^2 + (N_n - 1)^2.$$

We will use equations (14), (15) and (16) in the following theorems.

Theorem 4. *Relation between norms of $circ_s\ N$ and $circ_s\ DN$*

$$\|circ_s\ N\|_F^2 - \|circ_s\ DN\|_F^2 = \frac{2}{3}\Big(2N_n^2 + N_{n-1}^2 + 2N_{n-2}^2 - N_{n-3}^2 - N_{n-4}^2 - 2N_{n-5}^2 + 2N_{n-1}N_{n-2}$$

$$- 4N_{n-2}N_{n-3} - 2N_{n-3}N_{n-4} + 1\Big) + 3N_{n-2}^2 + (N_n - 1)^2.$$

Proof. From equations (14) and (15),

$$\|circ_s\ N\|_F^2 - \|circ_s\ DN\|_F^2$$

$$= nN_1^2 + (n-1)N_2^2 + \ \dots\ + 2N_{n-1}^2 + N_n^2 - \big((n-2)N_1^2 + (n-3)N_2^2 + \ \dots\ + 2N_{n-3}^2 + (N_n - 1)^2\big)$$

$$= 2\left(N_1^2 + N_2^2 + \ \dots\ + N_{n-3}^2 + N_{n-2}^2 + N_{n-1}^2\right) + 3N_{n-2}^2 - (N_n - 1)^2$$

$$= \frac{2}{3}\Big(2N_n^2 + N_{n-1}^2 + 2N_{n-2}^2 - N_{n-3}^2 - N_{n-4}^2 - 2N_{n-5}^2 + 2N_{n-1}N_{n-2} - 4N_{n-2}N_{n-3} - 2N_{n-3}N_{n-4} + 1\Big)$$

$+ 3N_{n-2}^2 + (N_n - 1)^2.$

□

Theorem 5. *Relation between norms of $circ_s\ DN$ and $circ_s\ D^2N$*

$$\|circ_s\ DN\|_F^2 - \left\|circ_s\ D^2N\right\|_F^2 = -n + 1 + \frac{2}{3}\Big(N_{2n+3} + N_{2n-1} - 2N_{n+2}^2 - 2N_{n+3}^2 - 2N_{n+4}^2 + 4N_{n+1}N_{n+2} + 2N_{n+3}N_{n+5} + 1\Big) + 2N_{n-5}^2 + N_{n-4}^2 + 2(N_{n-1} - 1)^2.$$

Proof. From equations (15) and (16),

$$\begin{aligned}
&\|circ_s\ DN\|_F^2 - \left\|circ_s\ D^2N\right\|_F^2 \\
&= (n-2)N_1^2 + (n-3)N_2^2 + \ \dots\ + 2N_{n-3}^2 + (N_n - 1)^2 \\
&- \big(n - 1 + (n-4)N_1^2 + (n-5)N_2^2 + \ \dots\ + 3N_{n-6}^2 + 2(N_{n-1} - 1)^2 + (N_n - 1)^2\big) \\
&= -n + 1 + 2\left(N_1^2 + N_2^2 + \ \dots\ + N_{n-5}^2 + N_{n-4}^2 + N_{n-3}^2\right) + 2N_{n-5}^2 + N_{n-4}^2 + 2(N_{n-1} - 1)^2 \\
&= -n + 1 + \frac{2}{3}\Big(2N_{n-2}^2 + N_{n-3}^2 + 2N_{n-4}^2 + 2N_{n-5}^2 - N_{n-6}^2 - 2N_{n-7}^2 + 2N_{n-3}N_{n-4} - 4N_{n-4}N_{n-5} \\
&- 2N_{n-5}N_{n-6} + 1\Big) + N_{n-4}^2 + 2(N_{n-1} - 1)^2.
\end{aligned}$$

□

From above two theorems, we will get relations between $\|circ_s\ N\|_F^2$ and $\left\|circ_s\ D^2N\right\|_F^2$.

5. Circulant Matrices of Gaussian Narayana Numbers

In the current section, we find images DG and D^2G of an n-tuple $G = (GN_1,\ GN_2,\ \dots,\ GN_n)$ of Gaussian Narayana numbers under the Ducci map. Then, we establish circulant, semi-circulant and skew-circulant matrices of $G,\ DG,\ D^2G$ and find Frobenius norms of these matrices and obtain relations among these norms.

Let us consider the n-tuple $G = (GN_1,\ GN_2,\ \dots,\ GN_{n-1},\ GN_n)$, we get

$$\begin{aligned}
DG &= (|GN_2 - GN_1|,\ |GN_3 - GN_2|,\ \dots,\ |GN_n - GN_{n-1}|,\ |GN_n - GN_1|) \\
&= (0,\ |i|,\ |GN_1|,\ \dots,\ |GN_{n-3}|,\ |GN_n - 1|), \\
D^2G &= (|i|,\ |GN_1 - i|,\ |GN_2 - GN_1|,\ \dots,\ |GN_{n-3} - GN_{n-4}|,\ |GN_n - 1 - GN_{n-3}|,\ |GN_n - 1|) \\
&= (|i|,\ |1 - i|,\ 0,\ |i|,\ |GN_1|,\ \dots,\ |GN_{n-6}|,\ |GN_{n-1} - 1|,\ |GN_n - 1|).
\end{aligned}$$

Circulant($circ$), skew-circulant($scirc$) and semi-circulant($circ_s$) matrices are defined in section 2 (Preliminaries)

$$circ\ G = circ\ (GN_1, GN_2, \dots, GN_{n-1}, GN_n), \tag{17}$$
$$circ\ DG = circ\ (0, |i|, |GN_1|, \dots, |GN_{n-3}|, |GN_n - 1|), \tag{18}$$
$$circ\ D^2G = circ\ (|i|, |1-i|, 0, |i|, |GN_1|, \dots, |GN_{n-6}|, |GN_{n-1} - 1|, |GN_n - 1|), \tag{19}$$

$$scirc\ G = scirc\ (GN_1, GN_2, \dots, GN_{n-1}, GN_n), \tag{20}$$
$$scirc\ DG = scirc\ (0, |i|, |GN_1|, \dots, |GN_{n-3}|, |GN_n - 1|), \tag{21}$$
$$scirc\ D^2G = scirc\ (|i|, |1-i|, 0, |i|, |GN_1|, \dots, |GN_{n-6}|, |GN_{n-1} - 1|, |GN_n - 1|), \tag{22}$$

$$circ_s\ G = circ_s\ (GN_1, GN_2, \dots, GN_{n-1}, GN_n), \tag{23}$$
$$circ_s\ DG = circ_s\ (0, |i|, |GN_1|, \dots, |GN_{n-3}|, |GN_n - 1|), \tag{24}$$
$$circ_s\ D^2G = circ_s\ (|i|, |1-i|, 0, |i|, |GN_1|, \dots, |GN_{n-6}|, |GN_{n-1} - 1|, |GN_n - 1|). \tag{25}$$

Further, we will find norms of these circulant, skew-circulant, semi-circulant matrices.

From equations (17) and (20),

$$\begin{aligned}
\|circ\ G\|_F^2 &= n\sum_{k=1}^{n} |GN_k|^2 \\
&= n\sum_{k=1}^{n} |N_k + iN_{k-2}|^2 \\
&= n\left(\sum_{k=1}^{n} N_k^2 + \sum_{k=1}^{n} N_{k-2}^2\right) \\
&= n\left(2\sum_{k=1}^{n} N_k^2 - N_{n-1}^2 - N_n^2\right) \\
&= n\Big(\frac{2}{3}\Big(2N_{n+1}^2 + N_n^2 + 2N_{n-1}^2 - N_{n-2}^2 - N_{n-3}^2 - 2N_{n-4}^2 + 2N_nN_{n-1} - 4N_{n-1}N_{n-2} \\
&\quad - 2N_{n-2}N_{n-3} + 1\Big) - N_{n-1}^2 - N_n^2\Big).
\end{aligned} \tag{26}$$

From equations (18) and (21),

$$\begin{aligned}
\|circ\ DG\|_F^2 &= n\left(\sum_{k=1}^{n-3} |GN_k|^2 + |i| + |GN_n - 1|\right) \\
&= n\left(\sum_{k=1}^{n-3} |N_k + iN_{k-2}|^2 + 1 + |N_n + iN_{n-2} - 1|\right) \\
&= n\left(\sum_{k=1}^{n-3} N_k^2 + \sum_{k=1}^{n-3} N_{k-2}^2 + 1 + (N_n - 1)^2 + N_{n-2}^2\right) \\
&= n\left(2\sum_{k=1}^{n-3} N_k^2 - N_{n-4}^2 - N_{n-3}^2 + 1 + (N_n - 1)^2 + N_{n-2}^2\right) \\
&= \frac{2n}{3}\Big(2N_{n-2}^2 + N_{n-3}^2 + 2N_{n-4}^2 - N_{n-5}^2 - N_{n-6}^2 - 2N_{n-7}^2 + 2N_{n-3}N_{n-4} - 4N_{n-4}N_{n-5} \\
&\quad - 2N_{n-5}N_{n-6} + 1\Big) + n\left(-N_{n-4}^2 - N_{n-3}^2 + 1 + (N_n - 1)^2 + N_{n-2}^2\right).
\end{aligned} \tag{27}$$

From equations (19) and (22),

$$\begin{aligned}
\left\|circ\ D^2G\right\|_F^2 &= n\left(\sum_{k=1}^{n-6} |GN_k|^2 + 2\,|i| + |GN_{n-1} - 1| + |GN_n - 1| + |1 - i|\right) \\
&= n\left(\sum_{k=1}^{n-6} |N_k + iN_{k-2}|^2 + 2 + |N_{n-1} + iN_{n-3} - 1| + |N_n + iN_{n-2} - 1| + 2\right) \\
&= n\left(\sum_{k=1}^{n-6} N_k^2 + \sum_{k=1}^{n-6} N_{k-2}^2 + 4 + (N_{n-1} - 1)^2 + N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2\right) \\
&= n\left(2\sum_{k=1}^{n-6} N_k^2 - N_{n-7}^2 - N_{n-6}^2 + 4 + (N_{n-1} - 1)^2 + N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2\right)
\end{aligned} \tag{28}$$

$$= \frac{2n}{3}\Big(2N_{n-5}^2 + N_{n-6}^2 + 2N_{n-7}^2 - N_{n-8}^2 - N_{n-9}^2 - 2N_{n-10}^2 + 2N_{n-6}N_{n-7} - 4N_{n-7}N_{n-8} - 2N_{n-8}N_{n-9} + 1\Big) + n\left(-N_{n-7}^2 - N_{n-6}^2 + 4 + (N_{n-1} - 1)^2 + N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2\right).$$

Theorem 6. *Relation between norms of circ G and circ DG*

$$\|circ\ G\|_F^2 - \|circ\ DG\|_F^2 = n\left(N_{n-1}^2 + N_{n-2}^2 + N_{n-3}^2 + N_{n-4}^2 + 2N_n - 2\right).$$

Proof. From equations (26) and (27),

$$\begin{aligned}\|circ\ G\|_F^2-\|circ\ DG\|_F^2 &= n\left(\sum_{k=1}^{n}N_k^2+\sum_{k=1}^{n}N_{k-2}^2\right)-n\left(\sum_{k=1}^{n-3}N_k^2+\sum_{k=1}^{n-3}N_{k-2}^2+1+(N_n-1)^2+N_{n-2}^2\right)\\ &= n\left(N_{n-2}^2+N_{n-1}^2+N_n^2+N_{n-2}^2+N_{n-3}^2+N_{n-4}^2-1-(N_n-1)^2-N_{n-2}^2\right)\\ &= n\left(N_{n-1}^2+N_{n-2}^2+N_{n-3}^2+N_{n-4}^2+2N_n-2\right).\end{aligned}$$

□

Theorem 7. *Relation between norms of circ DG and circ D^2G*

$$\|circ\ DG\|_F^2-\left\|circ\ D^2G\right\|_F^2=n\left(N_{n-4}^2+2N_{n-5}^2+N_{n-6}^2+N_{n-7}^2-3-(N_{n-1}-1)^2\right).$$

Proof. From equations (27) and (28),

$$\begin{aligned}&\|circ\ DG\|_F^2-\left\|circ\ D^2G\right\|_F^2=n\left(\sum_{k=1}^{n-3}N_k^2+\sum_{k=1}^{n-3}N_{k-2}^2+1+(N_n-1)^2+N_{n-2}^2\right)\\ &-n\left(\sum_{k=1}^{n-6}N_k^2+\sum_{k=1}^{n-6}N_{k-2}^2+4+(N_{n-1}-1)^2+N_{n-3}^2+(N_n-1)^2+N_{n-2}^2\right)\\ &=n(N_{n-5}^2+N_{n-4}^2+N_{n-3}^2+N_{n-7}^2+N_{n-6}^2+N_{n-5}^2-3-(N_{n-1}-1)^2-N_{n-3}^2)\\ &=n\left(N_{n-4}^2+2N_{n-5}^2+N_{n-6}^2+N_{n-7}^2-3-(N_{n-1}-1)^2\right).\end{aligned}$$

□

From above two theorems we will get the relations between $\|circ\ G\|_F^2$ and $\|circ\ DG\|_F^2$.

From equation (23),

$$\begin{aligned}&\|circ_s\ G\|_F^2=n\left|GN_1\right|^2+(n-1)\left|GN_2\right|^2+\ \dots\ +2\left|GN_{n-1}\right|^2+\left|GN_n\right|^2\\ &=n\left|N_1+iN_{-1}\right|^2+(n-1)\left|N_2+iN_0\right|^2+\ \dots\ +2\left|N_{n-1}+iN_{n-3}\right|^2+\left|N_n+iN_{n-2}\right|^2\\ &=nN_1^2+(n-1)N_2^2+(n-2)N_3^2+(n-2)N_1^2+\ \dots\ +3N_{n-2}^2+3N_{n-4}^2+2N_{n-1}^2+2N_{n-3}^2+N_n^2+N_{n-2}^2\\ &=2n(N_1^2+N_2^2+...+N_{n-1}^2)-2(N_1^2+2N_2^2+...+(n-1)N_{n-1}^2)+N_n^2\end{aligned}\tag{29}$$

$$\begin{aligned}&=\frac{2}{9}\Bigg(8N_n^2-15N_{n-1}^2+2N_{n-2}^2+4N_{n-3}^2+5N_{n-4}^2+N_{n-5}^2+12N_{n-1}N_{n-2}-2N_{n-3}N_{n-2}-2N_{n-3}N_{n-4}\\ &-16N_{n-3}N_{n-1}+12N_{n-1}N_n+3n-13\Bigg)+N_n^2.\end{aligned}$$

From equation (24),

$$\begin{aligned}&\|circ_s\ DG\|_F^2=(n-1)\left|i\right|^2+(n-2)\left|GN_1\right|^2+\ \dots\ +3\left|GN_{n-4}\right|^2+2\left|GN_{n-3}\right|^2+\left|GN_n-1\right|^2\\ &=n-1+(n-2)\left|N_1+iN_{-1}\right|^2+(n-3)\left|N_2+iN_0\right|^2+\ \dots\ +2\left|N_{n-3}+iN_{n-5}\right|^2+\left|N_n+iN_{n-2}-1\right|^2\\ &=n-1+(n-2)N_1^2+(n-3)N_2^2+(n-4)N_3^2+(n-4)N_1^2+\ \dots\ +5N_{n-6}^2+5N_{n-8}^2+4N_{n-5}^2+4N_{n-7}^2\\ &+3N_{n-4}^2+3N_{n-6}^2+2N_{n-3}^2+2N_{n-5}^2+(N_n-1)^2+N_{n-2}^2\\ &=n-1+(2n-4)(N_1^2+N_2^2+...+N_{n-3}^2)-2(N_1^2+2N_2^2+...+(n-3)N_{n-3}^2)+(N_n-1)^2-N_{n-4}^2\\ &+N_{n-2}^2\end{aligned}\tag{30}$$

$$\begin{aligned}&=n-1+(N_n-1)^2-N_{n-4}^2+N_{n-2}^2+\frac{2}{9}\Bigg(8N_{n-2}^2-9N_{n-3}^2+2N_{n-4}^2+4N_{n-5}^2+5N_{n-6}^2+N_{n-7}^2\\ &+12N_{n-3}N_{n-4}-2N_{n-4}N_{n-5}-2N_{n-5}N_{n-6}-16N_{n-5}N_{n-3}+12N_{n-3}N_{n-2}+3n-19\Bigg).\end{aligned}$$

From equation (25),

$$\left\|circ_s\ D^2G\right\|_F^2 = n\,|i|^2 + (n-1)\,|1-i|^2 + (n-3)\,|i|^2 + \ \dots\ + 3\,|GN_{n-6}|^2 + 2\,|GN_{n-1}-1|^2 + |GN_n - 1|^2$$
$$= n + 2(n-1) + n - 3 + (n-4)\,|N_1 + iN_{-1}|^2 + (n-5)\,|N_2 + iN_0|^2 + \ \dots\ + 5\,|N_{n-8} + iN_{n-10}|^2$$
$$+ 4\,|N_{n-7} + iN_{n-9}|^2 + 3\,|N_{n-6} + iN_{n-8}|^2 + 2\,|N_{n-1} + iN_{n-3} - 1|^2 + |N_n + iN_{n-2} - 1|^2$$
$$= 4n - 5 + (2n-8)(N_1^2 + N_2^2 + \dots + N_{n-7}^2 + N_{n-6}^2) - 2(N_1^2 + 2N_2^2 + \dots + (n-7)N_{n-7}^2 + (n-6)N_{n-6}^2)$$
$$- 2N_{n-7}^2 - N_{n-6}^2 + 2(N_{n-1}-1)^2 + 2N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2 \tag{31}$$
$$= 4n - 5 + \frac{2}{9}\Bigg(14N_{n-5}^2 - 6N_{n-6}^2 + 8N_{n-7}^2 + N_{n-8}^2 + 2N_{n-9}^2 - 2N_{n-10}^2 + 18N_{n-6}N_{n-7}$$
$$- 14N_{n-8}N_{n-7} - 8N_{n-8}N_{n-9} - 16N_{n-8}N_{n-6} + 12N_{n-6}N_{n-5} + 3n - 25\Bigg) - 2N_{n-7}^2 - N_{n-6}^2$$
$$+ 2(N_{n-1}-1)^2 + 2N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2.$$

Now, with the help of these norms we will prove the following theorems.

Theorem 8. *Relation between norms of $circ_s\ G$ and $circ_s\ DG$*

$$\|circ_s\ G\|_F^2 - \|circ_s\ DG\|_F^2 = \frac{4}{3}\Big(2N_{n-1}^2 + N_{n-2}^2 + 2N_{n-3}^2 - N_{n-4}^2 - N_{n-5}^2 - 2N_{n-6}^2 + 2N_{n-2}N_{n-3}$$
$$- 4N_{n-3}N_{n-4} - 2N_{n-4}N_{n-5} + 1\Big) + N_{n-4}^2 + 2N_{n-1}^2 - n + 2N_n - N_{n-2}^2.$$

Proof. From equations (29) and (30),

$$\|circ_s\ G\|_F^2 - \|circ_s\ DG\|_F^2$$
$$= (2n-2)N_1^2 + (2n-4)N_2^2 + \ \dots\ + 6N_{n-3}^2 + 4N_{n-2}^2 + 2N_{n-1}^2 + N_n^2 - n + 1 - (2n-6)N_1^2$$
$$- (2n-8)N_2^2 - (2n-10)N_3^2 - \ \dots\ - 8N_{n-6}^2 - 6N_{n-5}^2 - 3N_{n-4}^2 - 2N_{n-3}^2 - (N_n - 1)^2 - N_{n-2}^2$$
$$= -n + 1 + 4\left(N_1^2 + N_2^2 + \ \dots\ + N_{n-4}^2 + N_{n-3}^2\right) + N_{n-4}^2 + 3N_{n-2}^2 + 2N_{n-1}^2 + N_n^2 - (N_n - 1)^2$$
$$= 4(N_1^2 + N_2^2 + \dots + N_{n-3}^2 + N_{n-2}^2) + N_{n-4}^2 + 2N_{n-1}^2 - n + 2N_n - N_{n-2}^2$$
$$= \frac{4}{3}\Big(2N_{n-1}^2 + N_{n-2}^2 + 2N_{n-3}^2 - N_{n-4}^2 - N_{n-5}^2 - 2N_{n-6}^2 + 2N_{n-2}N_{n-3} - 4N_{n-3}N_{n-4} - 2N_{n-4}N_{n-5}$$
$$+ 1\Big) + N_{n-4}^2 + 2N_{n-1}^2 - n + 2N_n - N_{n-2}^2.$$

□

Theorem 9. *Relation between norms of $circ_s\ DG$ and $circ_s\ D^2G$*

$$\|circ_s\ DG\|_F^2 - \left\|circ_s\ D^2G\right\|_F^2 = \frac{4}{3}\Big(2N_{n-3}^2 + N_{n-4}^2 + 2N_{n-5}^2 - N_{n-6}^2 - N_{n-7}^2 - 2N_{n-8}^2 + 2N_{n-4}N_{n-5}$$
$$- 4N_{n-5}N_{n-6} - 2N_{n-6}N_{n-7} + 1\Big) - 3n + 4 + 2N_{n-5}^2 - N_{n-4}^2 + 2N_{n-7}^2 + N_{n-6}^2 - 2(N_{n-1} - 1)^2.$$

Proof. From equations (30) and (31),

$$\|circ_s\ DG\|_F^2 - \left\|circ_s\ D^2G\right\|_F^2$$
$$= n - 1 + (2n-6)N_1^2 + (2n-8)N_2^2 + (2n-10)N_3^2 + \ \dots\ + 8N_{n-6}^2 + 6N_{n-5}^2 + 3N_{n-4}^2$$
$$+ 2N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2 - 4n + 4 - (2n-10)N_1^2 - (2n-12)N_2^2 - \ \dots\ - 10N_{n-9}^2$$
$$- 8N_{n-8}^2 - 4N_{n-7}^2 - 3N_{n-6}^2 - 2(N_{n-1} - 1)^2 - 2N_{n-3}^2 - (N_n - 1)^2 - N_{n-2}^2$$
$$= -3n + 4 + 4\left(N_1^2 + N_2^2 + \ \dots\ + N_{n-4}^2\right) + 2N_{n-5}^2 - N_{n-4}^2 + 2N_{n-7}^2 + N_{n-6}^2 - 2(N_{n-1} - 1)^2$$
$$= \frac{4}{3}\Big(2N_{n-3}^2 + N_{n-4}^2 + 2N_{n-5}^2 - N_{n-6}^2 - N_{n-7}^2 - 2N_{n-8}^2 + 2N_{n-4}N_{n-5} - 4N_{n-5}N_{n-6} - 2N_{n-6}N_{n-7}$$
$$+ 1\Big) - 3n + 4 + 2N_{n-5}^2 - N_{n-4}^2 + 2N_{n-7}^2 + N_{n-6}^2 - 2(N_{n-1} - 1)^2.$$

□

From the above two theorems, we will get relation between $\|circ_s\ G\|_F^2$ and $\left\|circ_s\ D^2G\right\|_F^2$.

6. Relations Between Frobenius Norms of Narayana and Gauss Narayana Sequences

In this section, relations among norms of matrices of Narayana numbers and Gaussian Narayana numbers have been established.

From equation (26),

$$\begin{aligned}\|circ\ G\|_F^2 &= n\left(\sum_{k=1}^{n} N_k^2 + \sum_{k=1}^{n} N_{k-2}^2\right)\\ &= n\left(\sum_{k=1}^{n} N_k^2 + \sum_{k=-1}^{n-2} N_k^2\right)\\ &= n\left(2\sum_{k=1}^{n} N_k^2 - N_{n-1}^2 - N_n^2 + N_{-1}^2 + N_0^2\right)\\ &= 2\|circ\ N\|_F^2 - n(N_n^2 + N_{n-1}^2).\end{aligned}$$

From equation (27),

$$\begin{aligned}\|circ\ DG\|_F^2 &= n\left(\sum_{k=1}^{n-3} N_k^2 + \sum_{k=1}^{n-3} N_{k-2}^2 + (N_n-1)^2 + N_{n-2}^2 + 1\right)\\ &= n\left(\sum_{k=1}^{n-3} N_k^2 + \sum_{k=-1}^{n-5} N_k^2 + (N_n-1)^2 + N_{n-2}^2 + 1\right)\\ &= n\left(2\sum_{k=1}^{n-3} N_k^2 - N_{n-4}^2 - N_{n-3}^2 + (N_n-1)^2 + N_{n-2}^2 + 1\right)\\ &= 2\|circ\ DN\|_F^2 + n(2N_n + N_{n-2}^2 - N_{n-4}^2 - N_{n-3}^2 - N_n^2).\end{aligned}$$

From equation (28),

$$\begin{aligned}\left\|circ\ D^2G\right\|_F^2 &= n\left(\sum_{k=1}^{n-6} N_k^2 + \sum_{k=1}^{n-6} N_{k-2}^2 + 4 + (N_{n-1}-1)^2 + N_{n-3}^2 + (N_n-1)^2 + N_{n-2}^2\right)\\ &= n\left(\sum_{k=1}^{n-6} N_k^2 + \sum_{k=-1}^{n-8} N_{k-2}^2 + 4 + (N_{n-1}-1)^2 + N_{n-3}^2 + (N_n-1)^2 + N_{n-2}^2\right)\\ &= n\left(2\sum_{k=1}^{n-6} N_k^2 - N_{n-7}^2 - N_{n-6}^2 + 4 + (N_{n-1}-1)^2 + N_{n-3}^2 + (N_n-1)^2 + N_{n-2}^2\right)\\ &= 2\left\|circ\ D^2N\right\|_F^2 + n\left(-N_{n-7}^2 - N_{n-6}^2 - (N_{n-1}-1)^2 - (N_n-1)^2 + N_{n-3}^2 + N_{n-2}^2 + 2\right).\end{aligned}$$

Theorem 10. *Relation between norms of circ G and circ N*

$$\begin{aligned}\|circ_s\ G\|_F^2 - \|circ_s\ N\|_F^2 = \frac{1}{9}\Big(&2N_n^2 - 12N_{n-1}^2 - 4N_{n-2}^2 + 7N_{n-3}^2 + 8N_{n-4}^2 + 4N_{n-5}^2 + 6N_{n-1}N_{n-2}\\ &+ 10N_{n-3}N_{n-2} + 4N_{n-3}N_{n-4} - 16N_{n-3}N_{n-1} + 12N_{n-1}N_n + 3n - 16\Big).\end{aligned}$$

Proof. From equations (29) and (14),

$$\|circ_sG\|_F^2 = 2n(N_1^2 + N_2^2 + ... + N_{n-1}^2) - 2(N_1^2 + 2N_2^2 + ... + (n-1)N_{n-1}^2) + N_n^2.$$

and

$$\begin{aligned}\|circ_s\ N\|_F^2 &= (n+1)(N_1^2 + N_2^2 + ... + N_{n-1}^2 + N_n^2) - (N_1^2 + 2N_2^2 + ... + nN_n^2)\\ &= (n+1)(N_1^2 + N_2^2 + ... + N_{n-1}^2) - (N_1^2 + 2N_2^2 + ... + (n-1)N_{n-1}^2) + N_n^2.\end{aligned}$$

From here,

$$\|circ_s\ G\|_F^2 - \|circ_s\ N\|_F^2 = (n-1)(N_1^2 + N_2^2 + ... + N_{n-1}^2) - (N_1^2 + 2N_2^2 + ... + (n-1)N_{n-1}^2).$$

By using Lemma 3 and 4,

$$\|circ_s\ G\|_F^2 - \|circ_s\ N\|_F^2 = \frac{1}{9}\Big(2N_n^2 - 12N_{n-1}^2 - 4N_{n-2}^2 + 7N_{n-3}^2 + 8N_{n-4}^2 + 4N_{n-5}^2 + 6N_{n-1}N_{n-2} + 10N_{n-3}N_{n-2} + 4N_{n-3}N_{n-4} - 16N_{n-3}N_{n-1} + 12N_{n-1}N_n + 3n - 16\Big).$$

□

Theorem 11. *Relation between norms of circ DG and circ DN*

$$\|circ_s\ DG\|_F^2 - \|circ_s\ DN\|_F^2 = n - 1 - N_{n-4}^2 + \frac{1}{9}\Big(2N_{n-2}^2 - 12N_{n-3}^2 - 4N_{n-4}^2 + 7N_{n-5}^2 + 8N_{n-6}^2 + 4N_{n-7}^2 + 6N_{n-3}N_{n-4} + 10N_{n-5}N_{n-4} + 4N_{n-5}N_{n-6} - 16N_{n-5}N_{n-3} + 12N_{n-3}N_{n-2} + 3n - 22\Big).$$

Proof. From equations (30) and (15),

$$\|circ_s\ DG\|_F^2 = n - 1 + (2n-4)(N_1^2 + N_2^2 + ... + N_{n-3}^2) - 2(N_1^2 + 2N_2^2 + ... + (n-3)N_{n-3}^2) + (N_n - 1)^2 - N_{n-4}^2.$$

and

$$\|circ_s\ DN\|_F^2 = (n-1)(N_1^2 + N_2^2 + ... + N_{n-3}^2) - (N_1^2 + 2N_2^2 + ... + (n-3)N_{n-3}^2) + (N_n - 1)^2.$$

From here,

$$\|circ_s\ DG\|_F^2 - \|circ_s\ DN\|_F^2 = n - 1 + (n-3)(N_1^2 + N_2^2 + ... + N_{n-3}^2) - (N_1^2 + 2N_2^2 + ... + (n-3)N_{n-3}^2) - N_{n-4}^2.$$

By using Lemma 3 and 4

$$\|circ_s\ DG\|_F^2 - \|circ_s\ DN\|_F^2 = n - 1 - N_{n-4}^2 + \frac{1}{9}\Big(2N_{n-2}^2 - 12N_{n-3}^2 - 4N_{n-4}^2 + 7N_{n-5}^2 + 8N_{n-6}^2 + 4N_{n-7}^2 + 6N_{n-3}N_{n-4} + 10N_{n-5}N_{n-4} + 4N_{n-5}N_{n-6} - 16N_{n-5}N_{n-3} + 12N_{n-3}N_{n-2} + 3n - 22\Big).$$

□

Theorem 12. *Relation between norms of circ D^2G and circ D^2N*

$$\|circ_s\ D^2G\|_F^2 - \|circ_s\ D^2N\|_F^2 = 3n - 4 - 2N_{n-7}^2 - N_{n-6}^2 + 2N_{n-3}^2 + N_{n-2}^2 + \frac{1}{9}\left(8N_{n-5}^2 - 9N_{n-6}^2 + 2N_{n-7}^2 + 4N_{n-8}^2 + 5N_{n-9}^2 + N_{n-10}^2 + 12N_{n-6}N_{n-7} + 2N_{n-8}N_{n-7} - 2N_{n-8}N_{n-9} - 16N_{n-8}N_{n-6} + 12N_{n-6}N_{n-5} + 3n - 28\right).$$

Proof. From equations (31) and (16),

$$\|circ_s\ D^2G\|_F^2 = 4n - 5 + (2n-8)(N_1^2 + N_2^2 + ... + N_{n-6}^2) - 2(N_1^2 + 2N_2^2 + ... + (n-6)N_{n-6}^2) - 2N_{n-7}^2 - N_{n-6}^2 + 2(N_{n-1} - 1)^2 + 2N_{n-3}^2 + (N_n - 1)^2 + N_{n-2}^2.$$

and

$$\|circ_s\ D^2N\|_F^2 = n - 1 + (n-3)(N_1^2 + N_2^2 + ... + N_{n-6}^2) - (N_1^2 + 2N_2^2 + ... + (n-6)N_{n-6}^2) + 2(N_{n-1} - 1)^2 + (N_n - 1)^2.$$

From here,

$$\|circ_s\ D^2G\|_F^2 - \|circ_s\ D^2N\|_F^2 = 3n - 4 + (n-5)(N_1^2 + N_2^2 + ... + N_{n-6}^2) - (N_1^2 + 2N_2^2 + ... + (n-6)N_{n-6}^2) - 2N_{n-7}^2 - N_{n-6}^2 + 2N_{n-3}^2 + N_{n-2}^2.$$

By using Lemma 3 and 4,

$$\|circ_s\ D^2G\|_F^2 - \|circ_s\ D^2N\|_F^2 = 3n - 4 - 2N_{n-7}^2 - N_{n-6}^2 + 2N_{n-3}^2 + N_{n-2}^2 + \frac{1}{9}\Bigg(8N_{n-5}^2 - 9N_{n-6}^2 + 2N_{n-7}^2 + 4N_{n-8}^2 + 5N_{n-9}^2 + N_{n-10}^2 + 12N_{n-6}N_{n-7} + 2N_{n-8}N_{n-7} - 2N_{n-8}N_{n-9} - 16N_{n-8}N_{n-6} + 12N_{n-6}N_{n-5} + 3n - 28\Bigg).$$

□

7. Applications in Coding/Decoding

In this section, we define a coding and decoding algorithm using matrices of Narayana and Gaussian Narayana numbers. We represent the message in the square matrix M of order n and use the circulant matrix G of order n as the coding matrix, its inverse as the decoding matrix. The following transformations are well known.

- Coding transformation: $E = M \times G$
- Decoding transformation: $M = E \times G^{-1}$, here E is coded matrix.

7.1. Example for Coding/Decoding of Circulant Matrices of Narayana Numbers.

Example 1. *Let us represent the initial message into a square matrix of order* 4 *as*

$$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix}.$$

Coding: Consider circulant matrix of order 4 *as coding matrix, i.e.,*

$$G = \begin{pmatrix} N_1 & N_2 & N_3 & N_4 \\ N_4 & N_1 & N_2 & N_3 \\ N_3 & N_4 & N_1 & N_2 \\ N_2 & N_3 & N_4 & N_! \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

Coded matrix will be

$$\begin{aligned} E &= M \times G \\ &= \begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 13 & 14 & 12 \\ 17 & 18 & 14 & 16 \\ 12 & 14 & 13 & 11 \\ 13 & 14 & 18 & 15 \end{pmatrix}. \end{aligned}$$

Decoding: G is the circulant matrix of order 4 *with Narayana numbers. Then, its inverse matrix is obtained as*

$$G^{-1} = \frac{1}{-5}\begin{pmatrix} 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \\ -4 & 1 & 1 & 1 \end{pmatrix}.$$

Hence the decoded matrix, i.e., original message is as follows;

$$\begin{aligned} M &= E \times G^{-1} \\ &= \begin{pmatrix} 11 & 13 & 14 & 12 \\ 17 & 18 & 14 & 16 \\ 12 & 14 & 13 & 11 \\ 13 & 14 & 18 & 15 \end{pmatrix} \frac{1}{-5}\begin{pmatrix} 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \\ -4 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

$$= \frac{1}{-5}\begin{pmatrix} 11 & 13 & 14 & 12 \\ 17 & 18 & 14 & 16 \\ 12 & 14 & 13 & 11 \\ 13 & 14 & 18 & 15 \end{pmatrix}\begin{pmatrix} 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \\ -4 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix}.$$

7.2. **Example for Coding/Decoding of Circulant Matrices of Gaussian Narayana Numbers.**

Example 2. *Let us consider the initial message into a matrix of order* 3 *as*

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 4 \end{pmatrix}.$$

Conding: Take circulant matrix of order 3 *as coding matrix, i.e.,*

$$G' = \begin{pmatrix} GN_1 & GN_2 & GN_3 \\ GN_3 & GN_1 & GN_2 \\ GN_2 & GN_3 & GN_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1+i \\ 1+i & 1 & 1 \\ 1 & 1+i & 1 \end{pmatrix}.$$

So, coded matrix is

$$\begin{aligned} E &= M \times G' \\ &= \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1+i \\ 1+i & 1 & 1 \\ 1 & 1+i & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6+i & 6+3i & 6+2i \\ 12+4i & 12+5i & 12+3i \\ 7+2i & 7+4i & 7+i \end{pmatrix}, \text{ this is coded matrix.} \end{aligned}$$

Decoding: G' is the circulant matrix of order 3 *with Gaussian Narayana numbers. Then, its inverse matrix is obtained as*

$$G'^{-1} = \frac{1}{-3-i}\begin{pmatrix} -i & -1+2i & -i \\ -i & -i & -1+2i \\ -1+2i & -i & -i \end{pmatrix}.$$

Hence, the decoded matrix, i.e., original message is as follows;

$$\begin{aligned} M &= E \times G'^{-1} \\ &= \begin{pmatrix} 6+i & 6+3i & 6+2i \\ 12+4i & 12+5i & 12+3i \\ 7+2i & 7+4i & 7+i \end{pmatrix}\frac{1}{-3-i}\begin{pmatrix} -i & -1+2i & -i \\ -i & -i & -1+2i \\ -1+2i & -i & -i \end{pmatrix} \\ &= \frac{1}{-3-i}\begin{pmatrix} 6+i & 6+3i & 6+2i \\ 12+4i & 12+5i & 12+3i \\ 7+2i & 7+4i & 7+i \end{pmatrix}\begin{pmatrix} -i & -1+2i & -i \\ -i & -i & -1+2i \\ -1+2i & -i & -i \end{pmatrix} \\ &= \frac{1}{-3-i}\begin{pmatrix} -6-2i & -3-i & -9-3i \\ -9-3i & -12-4i & -15-5i \\ -3-i & -6-2i & -12-4i \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 4 \end{pmatrix}. \end{aligned}$$

8. Conclusions

Various identities for Narayana numbers have been discovered. Norms of circulant matrices made up of Narayana and Gaussian Narayana numbers are found and have perceived relations among these norms. We conclude that in message encryption and decryption various algorithms can be developed using Narayana and Gaussian Narayana numbers.

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References

[1] Bala, R., Mishra, V., On the circulant matrices with Ducci sequence and Gaussian Fibonacci numbers, American Institute of Physics Conference Series, Vol 2352, No 1 (2021), 030012.

[2] Bala, R., Mishra, V., Narayana matrix sequence, *Proc. Jangjeon Math. Soc.*, 25(4) (2022), 427-434.

[3] Bala, R., Mishra, V., Generalised (k,t)-Narayana sequence, *J. Indones. Math. Soc.*, 30(1) (2024), 121-138.

[4] Basu, M., Prasad, B., Coding theory on the m-extension of the Fibonacci p-numbers, *Chaos Soliton Fract*, 42(4) (2009), 2522-2530.

[5] Bozkurt, D., Tam, T. Y., Determinants and inverses of r-circulant matrices associated with a number sequence, *Linear Multilinear A*, 63(10) (2015), 2079-2088.

[6] Chandoul, A., On the norms of r-circulant matrices with generalized Fibonacci numbers, *J. Algebra Comb. Discrete Struct. Appl.*, 4(1) (2017), 13-21.

[7] Ciamberlini, C., Marengoni, A., On an interesting numerical curiosity, *Mathematics Periodicals*, 17(4) (1937), 25-30.

[8] Davis, P. J., Circulant matrices, American Mathematical Society, A Wiley-Interscience Publication, 2013.

[9] Flaut, C., Shpakivskyi, V., On generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions, *Adv. Appl. Clifford Algebras*, 23(3) (2013), 673-688.

[10] Horn, R. A., Johnson, C.R., Matrix Analysis, Cambridge University Press, 2012.

[11] Kuloglu, B., Eser, E., Ozkan, E., The r-circulant matrices associated with k-Fermat and k-Mersenne numbers, *WSEAS Trans. Math.*, 22(1) (2023), 531-543.

[12] Kuloglu, B., Engin, E., Ozkan, E., On the properties of r-circulant matrices involving generalized Fermat numbers, *Sak. Univ. J. Sci*, 27(5) (2023), 956-965.

[13] Ozkan, E., Kuloglu, B., On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials, *Asian-Eur. J. Math.*, 14(6) (2020), 1-16.

[14] Pacheenburawana, A., Sintunavarat, W., On the spectral norms of r-circulant matrices with the Padovan and Perrin sequences, *J. Math. Anal.*, 9(3) (2018), 110-122.

[15] Radicic, B., On k-circulant matrices with the Lucas numbers, *Filomat*, 32(11) (2018), 4037-4046.

[16] Shen, S. Q., Cen, J. M., On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sci.*, 5(12) (2010), 569-578.

[17] Shen, S., Cen, J., On the bounds for the norms of r-circulant matrices with the Fibonacci and Lucas numbers, *Appl. Math. Comput.*, 216(10) (2010), 2891-2897.

[18] Shtayat, J., Al-Kateeb, A. A., The Perrin R-matrix and more properties with an application, *J. Discrete Math. Sci. Cryptogr.*, 25(1) (2022), 41-52.

[19] Solak, S., On the norms of circulant matrices with the Fibonacci and Lucas numbers, *Appl. Math. Comput.*, 160(1) (2005), 125-132.

[20] Solak, S., Bahsi, M., Kan, O., On the circulant matrices with Ducci sequences and Fibonacci numbers, *Filomat*, 32(15) (2018), 5501-5508.

[21] Solak, S., Bahşi, M., Some properties of circulant matrices with Ducci sequences, *Linear Algebra Appl.*, 542(01) (2018), 557-568.

[22] Stakhov, A. P., Fibonacci matrices, a generalization of the "Cassini formula", and a new coding theory, *Chaos Solit. Fractals*, 30(1) (2006), 56-66.

[23] Tas, N., Ucar, S., Ozgur, N. Y., Kaymak, O. O., A new coding/decoding algorithm using Fibonacci numbers, *Discrete Math. Algorithms Appl.*, 10(02) (2018), 1-12.

[24] Ucar, S., Tas, N., Ozgur, N. Y., A new application to coding theory via Fibonacci and Lucas numbers, *Math. Sci. Appl. E-Notes*, 7(1) (2019), 62-70.