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State Feedback Control of Multiagent Singular Linear Systems Representing Brain Neural Networks

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Abstract

A multi-agent singular system is an extension of a traditional multi-agent system. The behavior of neural networks within the brain is crucial for cognitive functions, making it essential to understand the learning processes and the development of potential disorders. This study utilizes the analysis of singular linear systems representing brain neural networks to delve into the complexities of the human brain. In this context, the digraph approach is a powerful method for unraveling the intricate neural interconnections. Directed graphs, or digraphs, provide an intuitive visual representation of the causal and influential relationships among different neural units, facilitating a detailed analysis of network dynamics. This work explores the use of digraphs in analyzing singular linear multi-agent systems that model brain neural networks, emphasizing their significance and potential in enhancing our understanding of cognition and brain function.

1. Introduction

The brain's architecture constitutes a complex recurrent neuronal network that can depicted through a digraph representation (see Figure 1.1). In this depiction, nodes symbolize distinct brain regions, while edges signify the intensity of connections formed between these regions during specific task execution.

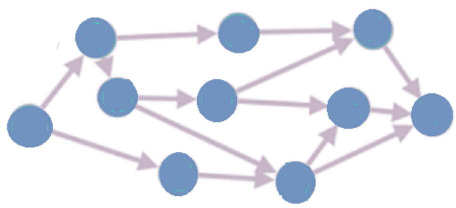

Figure 1.1: Recurrent Neuronal Network.

The term "neuronal network" refers to a specific framework for understanding brain function, wherein neurons serve as the fundamental computational entities and computational processes are interpreted through the lens of network interactions.
It has been shown ([1]) that cognitive control and the ability to control brain dynamics powerfully suggest improving cognitive functions and reversing the possible disorder in learning processes. The human brain can travel between diverse cognitive states. Its most significant function is in linking multiple sources of information in large-scale networks that are required to solve complex cognitive problems and strengthen memory.
Kriegeskorte in [2], states that neuronal network models fix a starting point of a new time for computational neuroscience, in which participants bear a part in real-world labors that require broad knowledge and elaborate calculations.
Neural networks that control their functions have been managed using dynamic linear control systems. In this study, neural networks are considered multi-agent systems, meaning they are systems of linear dynamic systems interconnected through a predetermined topology.

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Digraphs offer an intuitive visual representation of the causal and influential relationships between different neural units, allowing a detailed analysis of the network dynamics, [3].
Various fields of engineering employ multi-agent systems to solve synchronization problems and address consensus problems of the systems (see, for example, [4, 5]). On the other hand, it should be said that neural networks are also being studied as non-linear dynamic systems (see, for example, [6]).
In many instances, it is challenging and costly to convert a description of the brain into a multi-agent linear dynamic system. This process involves a combination of differential equations and purely algebraic constraints, naturally transforming them into state-space equations such as:

$$\left.\begin{array}{ll}\dot{x}^1(t) & = A_1x^1(t)+B_1u^1(t)\\ \vdots & \\ \dot{x}^k(t) & = A_kx^k(t)+B_ku^k(t)\end{array}\right\}, \tag{1.1}$$

where $A_i \in M_n(\mathbb{R})$, $B_i \in M_{n\times m}(\mathbb{R})$, $x^i(t) \in \mathbb{R}^n$, $u^i(t) \in \mathbb{R}^m$, $1 \le i \le k$. García-Planas in [7] showed that a description of the system using equations of the type

$$\left.\begin{array}{ll}E_1\dot{x}^1(t) & = A_1x^1(t)+B_1u^1(t)\\ \vdots & \\ E_k\dot{x}^k(t) & = A_kx^k(t)+B_ku^k(t)\end{array}\right\}, \tag{1.2}$$

where $E_i \in M_n(\mathbb{R})$ are allowed to be singular matrices, may be much more suitable.
A block diagram is plotted in Figure 1.2.

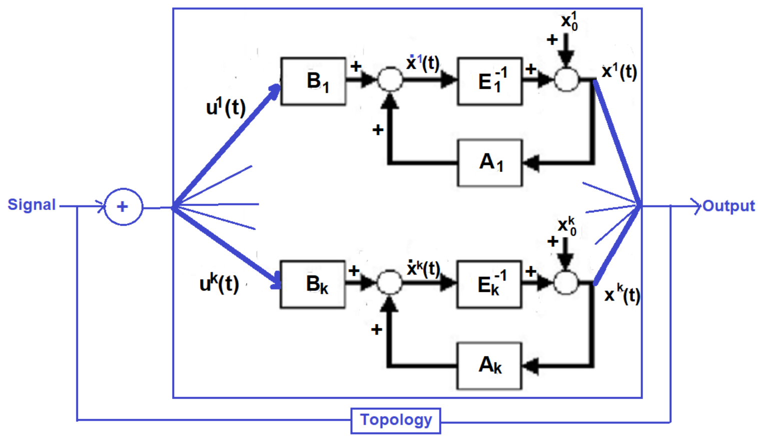

Figure 1.2: Diagram of a multi-agent singular system.

Systems and control theory can provide insights into the theoretical control of the human brain. Research on brain interfaces and neuromodulation indicates that changes in regional activity, measured by evoked potentials or other methods, can lead to alterations in brain function dynamics, [8].
Although fully understanding the relationship between mathematical control measures and the limited knowledge of cognitive control in neuroscience is challenging, even slight advances in this field can encourage further research and efforts to address learning difficulties such as dyscalculia and other issues like the phenomenon of forgetting, [8].
Structural controllability theory can be a powerful method for managing structured linear systems. In [7], Garcia-Planas demonstrated that structural controllability is a mathematical tool applicable to multi-agent singular systems, where each agent follows a predetermined structure.

2. Preliminaries

To investigate the proposed control problems, addressing the complexity inherent in the brain's structure is essential. This complexity necessitates dividing the global model into several local submodels, each representing distinct brain regions with intricate and interconnected network structures. By breaking down the global model into these local submodels, we can better understand and manage each region's specific dynamics and interactions. This approach enables us to conceptualize the brain as a collection of neuronal subnetworks, each contributing to the overall function and working together towards common cognitive and physiological objectives. By focusing on these localized models, we can develop more targeted and effective control strategies, ultimately enhancing our ability to understand and influence brain function as a cohesive multi-network system.
Let's consider a group of k agents as described in (1.2).
In our specific setup, the agents communicate according to the topology defined by the graph $\mathscr{G}$ with

i) Set of Vertices: $V = \{1,\dots,k\}$,
ii) Set of Edges: $\mathscr{E} = \{(i,j) \mid i,j \in V\} \subset V \times V$.

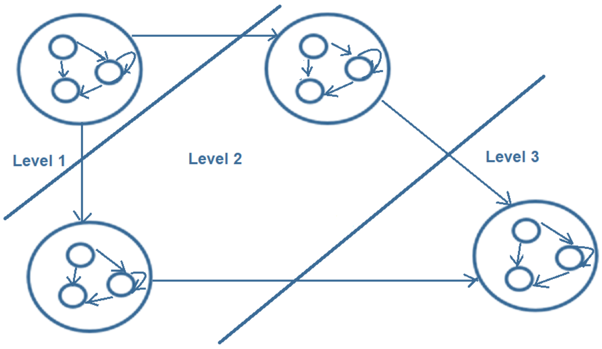

Figure 2.1: Multiagent graph.

The Figure (2.1) shows the graph that defines the topology on the participating agents in the system.
It is well known that each graph has an associated matrix called Laplacian; this matrix is defined as

$$\mathscr{L} = (l_{ij}) = \begin{cases} |\mathscr{N}_i| & \text{if } i = j, \\ -1 & \text{if } j \in \mathscr{N}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Writing:

$$\mathscr{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathscr{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix}, \quad \mathscr{U}(t) = \begin{pmatrix} u^1(t) \\ \vdots \\ u^k(t) \end{pmatrix},$$

$$\mathscr{E} = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_k \end{pmatrix}, \mathscr{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \mathscr{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}.$$

With these notations, it is possible to describe the multisystem as a system:

$$\mathscr{E}\dot{\mathscr{X}}(t) = \mathscr{A}\mathscr{X}(t) + \mathscr{B}\mathscr{U}(t).$$

The description of the local interrelation between systems defined by the considered topology is given by the control:

$$u^i(t) = F_i \sum_{j \in \mathscr{N}_i} (x^i(t) - x^j(t)), \ 1 \le i \le k. \tag{2.1}$$

That in a matrix description is

$$\mathscr{F}\mathscr{U}(t) = \mathscr{F}(\mathscr{L} \otimes I_n)\mathscr{X}(t),$$

where $\mathscr{F} = \begin{pmatrix} F_1 & & \\ & \ddots & \\ & & F_k \end{pmatrix}$.
Then, the multisystem with interrelation control is described as:

$$\mathscr{E}\dot{\mathscr{X}}(t) = \mathscr{A}\mathscr{X}(t) + \mathscr{B}\mathscr{F}\mathscr{U}(t) = (\mathscr{A} + \mathscr{B}\mathscr{F}(\mathscr{L} \otimes I_n))\mathscr{X}(t). \tag{2.2}$$

3. Controllability Character of a Singular Linear System

Controllability is a crucial property of dynamical systems, which is why there is an extensive body of literature addressing this concept ([7, 9, 10], among others).
Typical features of singular systems $E\dot{x}(t) = Ax(t) + Bu(t)$, which are unknown in the realm of state-space systems $\dot{x}(t) = Ax(t) + Bu(t)$ are possible impulsive responses to nonimpulsive excitations as well as provision for the consistency of initial conditions, complicating the analysis and design of control strategies. Understanding these features is essential for determining the controllability of such systems.
The controllability character can be defined using the following rank conditions which are generalizations of Hautus [9], and Kalman [10], conditions.

Proposition 3.1. *The dynamical Singular system $E\dot{x}(t) = Ax(t) + Bu(t)$ is controllable if and only if:*

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n,\\ &\operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix} = n,\ \forall s \in \mathbb{C}.\end{aligned}$$

Proposition 3.2 ([11]). *The dynamical system $E\dot{x}(t) = Ax(t) + Bu(t)$ is controllable if and only if the rank of the following matrix in $M_{n^2\times((n-1)n+nm)}(\mathbb{C})$ is n^2*

$$\operatorname{rank}\begin{pmatrix} E & 0 & 0 & \dots & 0 & B & 0 & 0 & \dots & 0 & 0\\ A & E & 0 & \dots & 0 & 0 & B & 0 & \dots & 0 & 0\\ 0 & A & E & \dots & 0 & 0 & 0 & B & \dots & 0 & 0\\ & & \dots & \dots & & & & \dots & & & \\ 0 & 0 & 0 & \dots & E & 0 & 0 & 0 & \dots & B & 0\\ 0 & 0 & 0 & \dots & A & 0 & 0 & 0 & \dots & 0 & B \end{pmatrix} = n^2.$$

We give evidence of the work applying it to simple example of a graph represented in Figure 3.1.

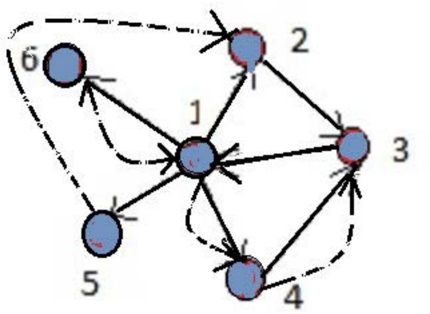

Figure 3.1: Example of a graph.

The system is

$$E = \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix},\quad A = \begin{pmatrix} 0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&1&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0 \end{pmatrix},\quad B = \begin{pmatrix} 1&0&0\\2&0&0\\1&0&1\\0&1&0\\3&0&1\\1&0&0 \end{pmatrix}.$$

We have that

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = 6,\\ &\operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix} = 6,\ \forall s \in \mathbb{C}.\end{aligned}$$

Then, the singular system is controllable. So, there exist matrices F and G in such away that$(E+BF)$ is invertible and the standard system $\dot{x}(t) = (E+BF)^{-1}(A+BG)x(t) + Bu(t)$ is controllable. For example, we can consider $F = \begin{pmatrix} 0&0&1&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0 \end{pmatrix}$ and $G = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}$.

The fact that

$$\begin{aligned} \operatorname{rank} B &= 3,\\ \operatorname{rank}\begin{pmatrix} B & (E+BF)^{-1}(A+BG)B \end{pmatrix} &= 5,\\ \operatorname{rank}\begin{pmatrix} B & (E+BF)^{-1}(A+BG)B & ((E+BF)^{-1}(A+BG))^2B \end{pmatrix} &= 6, \end{aligned}$$

ensures that taking suitable controls u_1, u_2, u_3 it is possible to reach a desired state from a fixed initial state in a finite time.
For example, taking $u_1 = (1,0,-2.5)$, $u_2 = (-0.5,0,3)$ and $u_3 = (-1.5,0,3.5)$, it is possible to reach the node 5 from the node 6.
Writing $\mathbf{A} = (E+BF)^{-1}(A+BG)$, we have:

$$\mathbf{A}^3x_6 + \mathbf{A}^2Bu_1 + \mathbf{A}Bu_2 + Bu_3 = x_5.$$

Determining which B matrices ensure system controllability is challenging, especially when these matrices require the minimum number of inputs. Liu et al. [12] suggest “the maximum coincidence algorithm” based on the network representation of the matrix to select the control

nodes to ensure that systems are controllable. Yuan et al. in [13] exhibit a general framework based on the maximum multiplicity theory to investigate the exact controllability of multiplex interrelated networks, focussing the study on the controllability amount defined by the minimum set of drivers that are needed to control steering the whole system toward any desired state but the authors do not construct the possible drivers. García-Planas in [14] builds the matrices (drivers) based on the matrix A eigenvalues and its geometric multiplicity.
Given a linear dynamical system as $E\dot{x}(t) = Ax(t) + Bu(t)$ that for plainness, from now on, we can write as the triple of matrices (E,A,B). It is well known that the system has many possible control matrices B that can assure the controllability condition; for that, it suffices to consider invertible matrices $B \in Gl(n;\mathbb{R})$.
The objective is to identify the set of all possible matrices B with the minimum number of columns, corresponding to the minimum number $n_B(E,A)$ of independent controllers necessary to manage the entire network.
Controllability with the minimum number of inputs is referred to as *exact controllability*.

Definition 3.3. *Let (E,A) be a pair of matrices defining the homogeneous singular system $E\dot{x}(t) = Ax(t)$. The exact controllability $n_B(E,A)$ is the minimum of the rank of all possible matrices B making the system $E\dot{x}(t) = Ax(t) + Bu(t)$ controllable.*

$$n_B(E,A) = \min\{\operatorname{rank} B, \forall B \in M_{n\times i}\, 1 \leq i \leq n, (E,A,B) \text{ controllable}\}.$$

If confusion is not possible we will write simply n_B.
To want to know n_B makes it attractive to be able to use systems with the same properties, but due to the simplicity of their expression, the computation is immediate. In this sense, we consider the following equivalence relation:

Definition 3.4. *We say that two systems (E,A,B) and $(\bar{E},\bar{A},\bar{B})$ are equivalent if and only if, there exist invertible matrices P and Q such that $(\bar{E},\bar{A},\bar{B}) = (QEP,QAP,QB)$.*

This equivalence relation corresponds with strict equivalence of the pencil $\begin{pmatrix} sE-A & B \end{pmatrix}$. So, the collection of invariants of the pencil are the invariants for the system.
Besides, it is easy to prove that n_B is invariant under this equivalence relation,

Proposition 3.5. *The exact controllability n_B is invariant under equivalence relation considered, that is to say: for any couple of invertible matrices (Q,P),*

$$n_B(E,A) = n_{QB}(QEP,QAP).$$

Proof.

$$\operatorname{rank}\begin{pmatrix} QEP & QB \end{pmatrix} = \operatorname{rank} Q\begin{pmatrix} E & B \end{pmatrix}\begin{pmatrix} P & \\ & I \end{pmatrix} = \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix},$$

$$\operatorname{rank}\begin{pmatrix} sQEP-QAP & QB \end{pmatrix} = \operatorname{rank} Q\begin{pmatrix} sE-A & B \end{pmatrix}\begin{pmatrix} P & \\ & I \end{pmatrix} = \operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix}.$$

□

As a consequence, if necessary we can consider (E,A,B) in a simpler form. In particular, the triple of matrices (E,A,B) can be reduced to a weaker form called "Quasi-Weierstraß form" (see [15]) in the following manner:
Let $P = \begin{pmatrix} V & W \end{pmatrix}$ and $Q = \begin{pmatrix} EV & AW \end{pmatrix}^{-1}$. Matrices $V \in M_{n\times r}(C)$ and $W \in M_{n\times(n-r)}(C)$ are in such a way that $\begin{pmatrix} V & W \end{pmatrix}$ and $\begin{pmatrix} EV & AW \end{pmatrix}$ are invertible.

$$(QEP,QAP,QB) = \left(\begin{pmatrix} I_r & \\ & N \end{pmatrix}, \begin{pmatrix} A_r & \\ & I_{n-r} \end{pmatrix}, \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}\right) = (\tilde{E},\tilde{A},\tilde{B}),$$

where A_r is some matrix and N is nilpotent.
The vector spaces $\operatorname{Im} V$ and $\operatorname{Im} W$ are spanned by the generalized eigenvector at the finite and infinite eigenvalues respectively, and they are derived by the following recursive subspace iteration with a limited number of steps called Wong sequences [16].

$$\begin{array}{ll} V_0 = C^n, & V_{i+1} = \{v \in C^n \mid Av \in E(V_i)\}, \\ W_0 = \{0\}, & W_{i+1} = \{v \in C^n \mid Ev \in A(W_i)\}. \end{array}$$

verifying

$$\begin{array}{l} V_0 \supseteq V_1 \supseteq \ldots \supseteq V_\ell = V_{\ell+1} = \ldots V_{\ell+q} = V^* \supseteq \operatorname{Ker} A, \\ W_0 \subseteq W_1 \subseteq \ldots \subseteq w_m = W_{m+1} = \ldots W_{m+q} = W^*. \end{array}$$

It is easy to prove that $\ell = m$ and satisfy $AV^* \subseteq EV^*$ and $EW^* \subseteq AW^*$.
Matrices V and W are defined in such away that $V^* = \operatorname{Im} V$ and $W^* = \operatorname{Im} W$.

Remark 3.6. *Not every matrix B having n_B columns is valid to make the system controllable. For example if $E = I$, $A = \operatorname{diag}(1,2,3)$ and $B = (1,0,0)^t$, the system (A,B) is not controllable,* (rank $\begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$, *or equivalently* rank $\begin{pmatrix} A-\lambda I & B \end{pmatrix} = 2$ *for* $\lambda = 2,3$.

For standard systems we have the following result.

Proposition 3.7 ([17]). *Let $\mu(\lambda_i) = \dim \operatorname{Ker}(A - \lambda_i I)$ be the geometric multiplicity of the eigenvalue λ_i. Then,*

$$n_B = \max_i \{\mu(\lambda_i)\}$$

In [14] a manner to obtain a set of minimal number of controls is presented.
We now extend the theorem to the case of singular systems.

Theorem 3.8. *Let (E,A) the fixed homogeneous part of a singular system. The exact controllability n_B is computed in the following manner.*

$$n_B = \max\{n_E, \mu(\lambda_i)\}$$

where $n_E = \operatorname{rank}(E,B)$, $\mu(\lambda_i) = \dim \operatorname{Ker}(\lambda_i E - A)$ and λ_i (for each i) is the eigenvalue of pencil $sE - A$.

Proof. Proposition 3.5 permit us to consider the system in its canonical reduced form

$$\operatorname{rank}(E,B) = \operatorname{rank}\left(\begin{pmatrix} I & \\ & N \end{pmatrix}\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}\right) = n_1 + \operatorname{rank}(N, \bar{B}_2),$$

$$N = \operatorname{diag}(N_1, \dots, N_{n_E}) \text{ with } N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

Taking

$$\bar{B}_2 = \begin{pmatrix} 0 & & \\ \vdots & & \\ 1 & & \\ & \ddots & \\ & & 0 \\ & & \vdots \\ & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} = \begin{pmatrix} w_1^{\infty} & \dots & w_{n_E}^{\infty} \end{pmatrix},$$

we have that $\operatorname{rank}(N, \bar{B}_2) = n_2$.

$$\operatorname{rank}\begin{pmatrix} \lambda_i E - A & B \end{pmatrix} = \operatorname{rank}\left(\lambda_i \begin{pmatrix} I & \\ & N \end{pmatrix} - \begin{pmatrix} J & \\ & I \end{pmatrix}\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}\right) = n_2 + \operatorname{rank}\begin{pmatrix} \lambda_i I - J & \bar{B}_1 \end{pmatrix}.$$

$$J = \operatorname{diag}(J_1(\lambda_1), \dots, J_r(\lambda_r)), J_i(\lambda_i) = \operatorname{diag}(J_{i_1}(\lambda_i), \dots, J_{i_{r_i}}(\lambda_i)),$$

and

$$J_{i_j}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}.$$

Taking

$$\bar{B}_{1_i} = \begin{pmatrix} 0 & & \\ \vdots & & \\ 1 & & \\ & \ddots & \\ & & 0 \\ & & \vdots \\ & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} = \begin{pmatrix} w_1^{\lambda_i} & \dots & w_{\mu_i(\lambda_i)}^{\lambda_i} \end{pmatrix},$$

we have that rank $(\lambda_i I - J, \bar{B}_{i_i}) = \mu(\lambda_i)$.
Consider now the following collection of vectors

$$\begin{array}{l} w_1^{\lambda_1}, \dots, w_\ell^{\lambda_1}, \\ \vdots \\ w_1^{\lambda_r}, \dots, w_\ell^{\lambda_r}, \\ w_1^{\infty}, \dots, w_\ell^{\infty}, \end{array}$$

where $\ell = \max\left(\mu(\lambda_1), \dots, \mu(\lambda_r), n_E\right)$ and we complete each series of vectors with the zero vectors in case its length is less than ℓ.
Finally we construct the family

$$w_1 = w_1^{\lambda_1} + \dots + w_1^{\lambda_r} + w_1^{\infty}, \dots, w_\ell = w_\ell^{\lambda_1} + \dots + w_\ell^{\lambda_r} + w_\ell^{\infty}.$$

Clearly,

$$\begin{array}{l} \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n, \\ \operatorname{rank}\begin{pmatrix} \lambda E - A & B \end{pmatrix} = n, \text{ for all } \lambda \in \mathbb{C}. \end{array}$$

Now, it suffices to remark that if we consider $B = (b_{ij}) \in M_{n\times m}(\mathbb{C})$ with $m < \ell$, we have

i) if $\ell = n_E$ then rank $\begin{pmatrix} E & B \end{pmatrix} < n$,
ii) if $\ell = \mu(\lambda_i)$ then rank $\begin{pmatrix} \lambda_i E - A & B \end{pmatrix} < n$.

□

4. Controllability of Multiagent Singular Neural Networks

We are concerned about bringing the output of the system (1.1) to a reference value and keeping it there, we can ensure that it is possible when the system is controllable. If topology relating systems is not considered, unquestionably, the system (1.1) is controllable if and only if each subsystem is controllable, and, in this case, there is feedback in which we obtain the requested solution.
We can be interested with the control (2.1) and ask for the stability of the system (2.2)
If having considered this control the resulting system (2.2) has not the desired eigenvalues, we can try to consider different feedback F_i so that, with the new control)with feedback $= K_i$),

$$u^i(t) = K_i \sum_{j\in\mathcal{N}_i} (x^i(t) - x^j(t)),\ 1 \le i \le k, \tag{4.1}$$

the system has appointed, eigenvalues to take a requested output of the system.
In some cases could be attentive in a solution such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0,\ 1 \le i, j \le k.$$

Namely, finding solutions for each subsystem reaching all, the same point.

Proposition 4.1. *Considering the control* $u^i(t) = K_i \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ *the closed-loop system can be detailed as*

$$\mathscr{E}\dot{\mathscr{X}}(t) = (\mathscr{A} + \mathscr{B}\mathscr{K}(\mathscr{L} \otimes I_n))\mathscr{X}(t).$$

where $\mathscr{K}$ is the diagonal matrix $\begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_k \end{pmatrix}$.
Computing the matrix $\mathscr{A} + \mathscr{B}\mathscr{K}(\mathscr{L} \otimes I_n)$ we obtain

$$\begin{pmatrix} A_1 + l_{11}B_1K_1 & l_{12}B_1K_1 & \dots & l_{1k}B_1K_1 \\ l_{21}B_2K_2 & A_2 + l_{22}B_2K_2 & \dots & l_{2k}B_2K_2 \\ \vdots & \vdots & \ddots & \vdots \\ l_{k1}B_kK_k & l_{k2}B_kK_k & \dots & A_k + l_{kk}B_kK_k \end{pmatrix}.$$

In the special case where all systems in the multi-system have the same dynamics, this is $E_i = E$, $A_i = A$, $B_i = B$ and $K_i = K$ Proposition 4.1 can be rewritten as follows:

Proposition 4.2. *Considering the control* $u^i(t) = K\sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ *the closed-loop system for a multiagent with identical linear dynamical mode, is detailed as*

$$(I_k \otimes E)\dot{\mathscr{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathscr{L} \otimes I_n))\mathscr{X}.$$

It is also interesting to study the case that we can also consider external controls that allow us to obtain the desired eigenvalues.
It is of interest to recognize the minimum set of driver nodes needed to achieve full control of networks having arbitrary structures and link-weight distributions.
In our particular setup, the objective is to find the collection of all possible matrices E, having the minimum number of columns corresponding to the minimum number $n_D((\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n)))$ of independent drivers that are necessary to control the whole network.
Given the protocol as (4.1) with K the feedback gain matrix, and defining

$$\mathscr{U}_{\text{ext}}(t)=\begin{pmatrix}u^1_{\text{ext}}(t)\\ \vdots\\ u^k_{\text{ext}}(t)\end{pmatrix},\ \mathscr{D}=\begin{pmatrix}D_1&&\\&\ddots&\\&&D_k\end{pmatrix}.$$

Proposition 4.3. *With these notations the system can be described as*

$$\mathscr{E}\dot{\mathscr{X}}(t)=(\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n))\mathscr{X}(t)+\mathscr{D}\mathscr{U}_{ext}(t).$$

And the minimum number of controls $\mathscr{D}$ necessary to ensure the controllability of the system is

$$n_{\mathscr{D}}=\max_i(n_{\mathscr{E}},\mu(\lambda_i))$$

where $n_{\mathscr{E}}=nk-\operatorname{rank}\mathbb{E}$ and $\mu(\lambda_i)=\dim\operatorname{Ker}(s\mathscr{E}-(\mathscr{A}+\mathbb{B}\mathbb{K}(\mathscr{L}\otimes I_n)),\mathscr{D})$.

Example 4.4. *Consider the case where* $E_i=E=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $A_i=A=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $B_i=B=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $K_i=K=\begin{pmatrix}1&1\end{pmatrix}$ *corresponding to the Figure 4.1.*

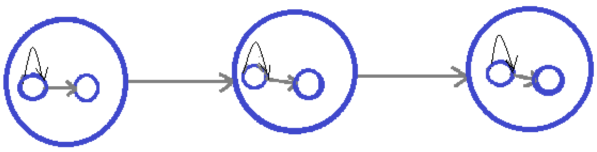

Figure 4.1: Neural network

The Laplacian matrix is

$$\mathscr{L}=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&0\end{pmatrix}.$$

The matrix $\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n)$ *is*

$$\begin{pmatrix}1&0&-1&0&0&0\\1&0&0&0&0&0\\0&0&0&1&-1&0\\0&0&1&0&0&0\\0&0&0&0&1&-1\\0&0&0&0&1&0\end{pmatrix}.$$

In this case $n_{\mathscr{D}}=3$.
Taking

$$\mathscr{D}=\begin{pmatrix}D_1&&\\&D_2&\\&&D_3\end{pmatrix}=\begin{pmatrix}1&0&0\\2&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix},$$

we have

$$\operatorname{rank}\begin{pmatrix}E&0&0&0&0&B&0&0&0&0&0\\A&E&0&0&0&0&B&0&0&0&0\\0&A&E&0&0&0&0&B&0&0&0\\0&0&A&E&0&0&0&0&B&0&0\\0&0&0&A&E&0&0&0&0&B&0\\0&0&0&0&A&0&0&0&0&0&B\end{pmatrix}=36.$$

Then, the multisystem is controllable.

5. Discussion and Conclusion

The concept of "control" implies taking action and represents the human endeavor to intervene in the environment to ensure survival and continually enhance the quality of life. Many control problems can be addressed through a mathematical model that describes the physical system in question with equations representing the system's state.

Although control is a fundamental issue in numerous network systems, more studies are still needed to quantitatively explore the control of directed networks, which are the most common configuration in real-world systems.

The primary issue is the network's size. Liu et al. [12] have developed tools to investigate the controllability of networks with arbitrary sizes and topologies using the controllability matrix, focusing on a few driver nodes within the network.

In [8], Gu et al. define different types of controllability (global, regional, average, modal, and boundary) from various perspectives to apply to neural systems. These different viewpoints can help analyze the various roles in controlling the dynamic trajectories of brain network functions.

This paper considers the brain network as a linear, discrete-time, time-invariant multisystem, which allows for considering a more significant number of nodes. In 2018, Abiodun et al. ([18]) conducted a survey on the state of the art in the applications of artificial neural networks.

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