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(q, h)-BERNSTEIN BASES AND BASIC HYPERGEOMETRIC SERIES

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Abstract

Quantum (q, h)-Bernstein bases and basic hypergeometric series are two seemingly unrelated mathematical entities. In this work, it is indicated that they are deeply interrelated theories. This new insight into two theories enables the provision of new proofs for two basic hypergeometric sums. The q-Chu-Vandermonde formula for basic hypergeometric series is proved by the partition of unity property for (q, h)-Bernstein bases, and the q-Pffaf-Saalschütz formula for basic hypergeometric series is proved by the Marsden identity for (q, h)-Bernstein bases.

Keywords: Quantum Bernstein bases, Marsden identity, basic hypergeometric series, q-Chu-Vandermonde formula, q-Pfaff-Saalschütz formula

1. Introduction

This work aims to investigate two different but related mathematical entities: quantum (q, h)-Bernstein bases and basic hypergeometric series. Quantum Bernstein bases were introduced first in Approximation Theory and later in Computer-Aided Geometric Design [11-18]. Quantum Bernstein bases, as an extension of the classical Bernstein bases, play a fundamental role in studying approximation methods for curves and surfaces. Quantum Bernstein bases come in two types: h-Bernstein bases and q-Bernstein bases. Quantum (q, h)-Bernstein bases are the most general form of the quantum Bernstein basis functions [6]. It generalizes the quantum h-Bernstein theory and the quantum q-Bernstein theory into a single quantum Bernstein theory. Hypergeometric series were initiated by Euler and Gauss in the early 19th century. First used as solutions to ordinary and partial differential equations, hypergeometric series later gained extensive application in number theory, combinatorics, orthogonal polynomials, approximation theory, and physics [1-3]. Hypergeometric series exist in two types: classical and q-versions (basic hypergeometric series).

There are numerous identities for the hypergeometric series. Researchers employ a variety of methods to prove these identities [5,7-10]. The Chu-Vandermonde formula and the Pfaff-Saalschütz formula are among the few examples of hypergeometric series that are widely used to establish many other identities. Recently, in [5], researchers propose alternative proofs for the Chu-Vandermonde formula, the Pfaff-Saalschütz formula, and their q versions using the identities of quantum Bernstein bases. This work will prove the q-Chu Vandermonde and q-Pfaff Saalschütz formulas using identities for the (q,h)-Bernstein bases.

This paper is organized as follows. In Section 2, we recall the basic definitions, notation, and results for q-shifted factorials and basic hypergeometric series. Also, we briefly review the (q,h)-Bernstein bases, the (q,h)-partition of unity property, and the (q,h)-Marsden identity. In Section 3, we prove the q-Chu Vandermonde and q-Pfaff-Saalschütz formulas using identities for the (q,h)-Bernstein bases. In Section 4, we conclude with a short summary of our results along with a brief discussion of a few open problems for future research.

2. Preliminaries

2.1. q-Shifted factorials

Throughout this paper we will adopt the following standard definitions and notation for the q-integers, q-factorials, q-shifted factorials, and the q-binomial coefficients [1-3].

q-Integers

$$[0]_q = 1, \qquad [n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases} \qquad n = 1,2,\dots$$

q-Factorials

$$[0]_q! = 1, \qquad [n]_q! = \prod_{k=1}^{n} [k]_q \, , n = 1,2,\dots$$

q-Shifted factorials

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k)\, , \quad n = 1,2,\dots$$

Multiple q-Shifted Factorials

$$(a_1,\dots,a_m;q)_n = \prod_{k=0}^{m} (a_k;q)_n\, , \quad n = 0,1,\dots$$

q-Binomial Coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad k = 0,\dots,n.$$

We will use the following useful identities for q-shifted factorials in Section 3.

$$(q^{-n}E;q)_n = (-E)^n q^{-\binom{n+1}{2}}(q/E;q)_n,$$
$$(E;q)_{n-k} = (E;q)_n(-E)^{-k}q^{\binom{n-k}{2}-\binom{n}{2}}/(q^{1-n}/E;q)_k.$$

2.2 Basic hypergeometric series

The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\middle|\, q,z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}(-q^{(k-1)/2})^{k(s+1-r)}z^k. \tag{1}$$

If $r = s+1$ then, (1) turns into the following series:

$${}_{s+1}\phi_s\left(\begin{matrix}a_1,a_2,\dots,a_{s+1}\\ b_1,\dots,b_s\end{matrix}\middle|\, q,z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_{s+1};q)_k}{(q,b_1,\dots,b_s;q)_k}z^k. \tag{2}$$

In addition, if $a_j = q^{-n}$ for some $n \in \mathbb{N}$, then $(a_j;q)_k = (q^{-n};q)_k = 0$ for every $k \geq n+1$. Therefore, the right-hand side of (2) reduces to a finite sum:

$${}_{s+1}\phi_s\left(\begin{matrix}q^{-n},a_2,\dots,a_{s+1}\\ b_1,\dots,b_s\end{matrix}\middle|\, q,z\right) = \sum_{k=0}^{n}\frac{(q^{-n},a_2,\dots,a_{s+1};q)_k}{(q,b_1,\dots,b_s;q)_k}z^k. \tag{3}$$

2.3. (q,h)-Bernstein basis functions

In [6], Goldman and Simeonov define the (q,h)-Bernstein basis functions. Let $g(t) = qt + h$ with $q \neq 0,-1$ and the m-fold composition of $g(t)$ with itself be

$$g^{[m]}(t) = (\underbrace{g\circ g\circ\cdots\circ g}_{m})(t) = q^m t + [m]_q h.$$

For $b \neq g^{[i]}(a)$, $i = 0,1,\cdots,n-1$, the (q,h)-Bernstein basis functions of degree n on the interval $[a,b]$ are given explicitly by

$$B_k^n(t;[a,b];q;h) = \begin{bmatrix}n\\k\end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(t-g^{[i]}(a))\prod_{i=0}^{n-k-1}(b-g^{[i]}(t))}{\prod_{i=0}^{n-1}(b-g^{[i]}(a))}, \qquad k = 0,\dots,n. \tag{4}$$

We shall focus on the following two identities for the (q,h)-Bernstein bases:

The (q,h)-Partition of Unity Property

$$\sum_{k=0}^{n} B_k^n(t;[a,b];q;h) = 1. \tag{5}$$

The (q,h)-Marsden Identity

$$\prod_{i=0}^{n-1}(x-g^{[i]}(t)) = \sum_{k=0}^{n}\left(\prod_{v=k}^{n-1}(x-g^{[v]}(a))\prod_{v=0}^{k-1}(x-g^{[v]}(b))\right)B_k^n(t;[a,b];q;h) \qquad (6)$$

3. Identities for basic hypergeometric series

We begin with an alternative representation for the (q,h)-Bernstein basis function $B_k^n\big(t\,;[a,b]\,;q\,;h\big)$ in terms of the q-shifted factorials.

Proposition 3.1

$$B_k^n(t;[a,b];q;h) = \frac{((t-g(t))/(b-g(b));q)_n}{((a-g(a))/(b-g(b));q)_n}\frac{(q^{-n},(a-g(a))/(t-g(t));q)_k}{(q,q^{1-n}(b-g(b))/(t-g(t));q)_k}q^k. \quad (7)$$

Proof. Using the definition of q-integers yields

$$g^{[m]}(t) = q^m t + [m]_q h = q^m t + \frac{1-q^m}{1-q}h = \frac{h}{1-q} + \left(t-\frac{h}{1-q}\right)q^m.$$

Then, we obtain

$$\begin{aligned} t-g^{[i]}(a) &= t-\frac{h}{1-q}-\left(a-\frac{h}{1-q}\right)q^i = \left(t-\frac{h}{1-q}\right)\left(1-\frac{\big((1-q)a-h\big)}{\big((1-q)t-h\big)}q^i\right) \\ &= \left(\frac{t-g(t)}{1-q}\right)\left(1-\frac{\big(a-g(a)\big)}{\big(t-g(t)\big)}q^i\right). \end{aligned}$$

Similarly,

$$\begin{aligned} b-g^{[i]}(t) &= \left(\frac{b-g(b)}{1-q}\right)\left(1-\frac{\big(t-g(t)\big)}{\big(b-g(b)\big)}q^i\right), \\ b-g^{[i]}(a) &= \left(\frac{b-g(b)}{1-q}\right)\left(1-\frac{\big(a-g(a)\big)}{\big(b-g(b)\big)}q^i\right). \end{aligned}$$

Substituting these equations into equation (4) and using q-shifted factorials, we can rewrite (4) as

$$B_k^n(t;[a,b];q;h)$$
$$= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\left(\frac{t-g(t)}{1-q}\right)^k\left(\frac{a-g(a)}{t-g(t)};q\right)_k\left(\frac{b-g(b)}{1-q}\right)^{n-k}\left(\frac{t-g(t)}{b-g(b)};q\right)_{n-k}\frac{\left(\frac{b-g(b)}{1-q}\right)^{-n}}{\left(\frac{a-g(a)}{b-g(b)};q\right)_n}.$$

Invoking the identity

$$(E;q)_{n-k} = (E;q)_n(-E)^{-k}q^{\binom{n-k}{2}-\binom{n}{2}}/(q^{1-n}/E;q)_k$$

with $E = \frac{t-g(t)}{b-g(b)}$ and $E=q$, we obtain (7).

Next we will show that the basic hypergeometric formulas can be derived from the identities (5) and (6) for the (q,h)-Bernstein basis functions.

Corollary 3.2 The basic hypergeometric form of the partition of unity property is

$$ {}_3\phi_1\left(\begin{matrix} q^{-n}, & (a-g(a))/(t-g(t)) \\ & q^{1-n}(b-g(b))/(t-g(t)) \end{matrix}\middle| q,q\right) = \frac{((a-g(a))/(b-g(b));q)_n}{((t-g(t))/(b-g(b));q)_n}. \tag{8} $$

Proof. Substituting (7) into the partition of unity property (5) yields

$$ \frac{((t-g(t))/(b-g(b));q)_n}{((a-g(a))/(b-g(b));q)_n}\sum_{k=0}^{n}\frac{(q^{-n},(a-g(a))/(t-g(t));q)_k}{(q,q^{1-n}(b-g(b))/(t-g(t));q)_k}q^k = 1. $$

Multiplying both sides by $\frac{((a-g(a))/(b-g(b));q)_n}{((t-g(t))/(b-g(b));q)_n}$ and invoking (3), we obtain (8).

Theorem 3.3 (q-Chu-Vandermonde summation formula)

$$ 2\phi_1\left(\begin{matrix} q^{-n}, A \\ C \end{matrix}\middle| q,q\right) = A^n\frac{(C/A;q)_n}{(C;q)_n}. \tag{9} $$

Proof. We prove this summation formula using the basic hypergeometric form (8) of the partition of unity property for the (q,h)-Bernstein basis functions. Set

$(a-g(a))/(t-g(t)) = A$ and $q^{1-n}(b-g(b))/(t-g(t)) = C$

in (8). Then the left-hand side of (8) becomes the left-hand side of (9). The right-hand side of (8) becomes $\frac{(q^{1-n}A/C;q)_n}{(q^{1-n}/C;q)_n}$, which reduces to the right-hand side of (9) after applying

$$ (q^{-n}E;q)_n = (-E)^n q^{-\binom{n+1}{2}}(q/E;q)_n $$

with $E = \frac{qA}{C}$ and with $E = \frac{q}{C}$.

Corollary 3.4 The basic hypergeometric form of the (q,h)-Marsden identity is

$$ \begin{aligned} &{}_3\phi_2\left(\begin{matrix} q^{-n},(b-g(b))/(x-g(x)),(a-g(a))/(t-g(t)) \\ (a-g(a))/(x-g(x)),q^{1-n}(b-g(b))/(t-g(t))q \end{matrix}\middle| q,q\right) \\ &\quad = \frac{((a-g(a))/(b-g(b)),(t-g(t))/(x-g(x));q)_n}{((a-g(a))/(x-g(x)),(t-g(t))/(b-g(b));q)_n}. \end{aligned} \tag{10} $$

Proof. Using q-shifted factorials, we get that

$$ \prod_{i=0}^{n-1}(x-g^{[i]}(t)) = \left(x-\frac{h}{1-q}\right)^n\left(\frac{((1-q)t-h)}{((1-q)x-h)};q\right)_n = \left(\frac{x-g(x)}{1-q}\right)^n\left(\frac{(t-g(t))}{(x-g(x))};q\right)_n, $$

$$ \prod_{v=0}^{k-1}(x-g^{[v]}(b)) = \left(\frac{x-g(x)}{1-q}\right)^k\left(\frac{(b-g(b))}{(x-g(x))};q\right)_k, $$

$$\prod_{v=k}^{n-1}(x-g^{[v]}(a)) = \left(\frac{x-g(x)}{1-q}\right)^{n-k}\frac{\left(\frac{(a-g(a))}{(x-g(x))};q\right)_n}{\left(\frac{(a-g(a))}{(x-g(x))};q\right)_k}$$

Substituting these equations and the hypergeometric form of the (q,h)-Bernstein basis functions (7) into equation (6), we obtain that

$$\left(\frac{x-g(x)}{1-q}\right)^n\left(\frac{t-g(t)}{x-g(x)};q\right)_n =$$

$$\sum_{k=0}^{n}\left(\frac{x-g(x)}{1-q}\right)^{n-k}\frac{\left(\frac{a-g(a)}{x-g(x)};q\right)_n}{\left(\frac{a-g(a)}{x-g(x)};q\right)_k}\frac{q^k\left(\frac{x-g(x)}{1-q}\right)^k\left(\frac{b-g(b)}{x-g(x)};q\right)_k\left(\frac{t-g(t)}{b-g(b)};q\right)_n\left(q^{-n},\frac{a-g(a)}{t-g(t)};q\right)_k}{\left(\frac{a-g(a)}{b-g(b)};q\right)_n\left(q,q^{1-n}\frac{b-g(b)}{t-g(t)};q\right)_k}q^k$$

After simplifying, this equation reduces to

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k((a-g(a))/(t-g(t));q)_k((b-g(b))/(x-g(x));q)_k}{(q;q)_k((a-g(a))/(x-g(x));q)_k(q^{1-n}(b-g(b))/(t-g(t));q)_k}q^k$$
$$=\frac{((t-g(t))/(x-g(x));q)_n((a-g(a))/(b-g(b));q)_n}{((t-g(t))/(b-g(b));q)_n((a-g(a))/(x-g(x));q)_n}.$$

By (3), the basic hypergeometric form of this sum is (10).

Theorem 3.5 (q-Pfaff-Saalschütz summation formula)

$${}_3\phi_2\left(\begin{matrix}q^{-n},A,B\\C,D\end{matrix}\middle|\,q,q\right)=\frac{(C/A,C/B;q)_n}{(C,C/AB;q)_n}, \qquad (11)$$

where $CD=q^{1-n}AB$.

Proof. We derive the q-Pfaff-Saalschütz summation formula (11) from the basic hypergeometric form (10) of the (q,h)-Marsden identity. In (10) set

$(b-g(b))/(x-g(x))=A, \qquad (a-g(a))/(t-g(t))=B$

$(a-g(a))/(x-g(x))=C$
and
$q^{1-n}(b-g(b))/(t-g(t))=D.$

With these substitutions $CD=q^{1-n}AB$, and (10) becomes (11).

4. Conclusion

This study presents alternative proofs of two basic hypergeometric sums using two identities for quantum (q,h)-Bernstein bases. These findings show that basic hypergeometric series and quantum Bernstein bases are linked, despite the fact that these two theories originated from two different mathematical entities. We expect that this new link will lead in the future to new results. There are also transformation formulas for basic hypergeometric series. Can one prove some transformation formulas using identities and properties of quantum (q, h)-Bernstein bases? Also, can one confirm the properties of quantum (q, h)-Bernstein bases using identities or transformation formulas for basic hypergeometric formulas? We hope to investigate these open problems for future work.

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