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HERMITE-HADAMARD TYPE INEQUALITIES FOR $LOG-$CONVEX STOCHASTIC PROCESSES

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Abstract – The main aim of the present note is to introduce log-convex stochastic processes and to contact correlation between convex stochastic processes and log-convex stochastic processes. We also prove some Hadamard-type inequalities for log-convex stochastic processes with the help of the special means.

Keywords – *Hermite Hadamard Inequality, log-convex functions, convex stochastic process, log-convex stochastic process*

1 Introduction

In 1980, Nikodem [13] introduced the convex stochastic processes in his article. Later in 1995, Skowronski [9] presented some further results on convex stochastic processes. Moreover, in 2011, Kotrys [7] derived some Hermite-Hadamard type inequalities for convex stochastic processes. In 2014, Maden *et.al.* [24] introduced the convex stochastic processes in the first sense and proved Hermite-Hadamard type inequalities to these processes. Also in 2014, Set *et.al.* [25] presented the convex stochastic processes in the second sense and they investigated Hermite-Hadamard type inequalities for these processes. Moreover, in recent papers [22, 23], strongly λ-GA-convex stochastic processes and preinvex stochastic processes has been introduced.

A function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1) holds, then f is concave. For some recent results related to this classic result, see the books [2, 4, 5, 6] and the papers [14, 15, 16, 17, 18, 19, 20, 21] where further references are given.

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

Recently, log-convex functions have gained much interest in mathematics and its sub-areas such as optimization theory. Let $f : I \to \mathbb{R}$ be a function where I is an interval of real numbers. f is said to be convex on I if the following inequality holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{3}$$

A function $f : I \to [0, \infty)$ is said to be *log*-convex (or multiplicatively convex) if $\log(f)$ is convex or namely the following inequality

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{(1-\lambda)} \tag{4}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Moreover, any *log*-convex function is a convex function since the inequality

$$[f(x)]^{\lambda}[f(y)]^{(1-\lambda)} \leq \lambda f(x) + (1-\lambda)f(y) \tag{5}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. [1, p.7]

Let $f : I \subseteq \mathbb{R} \to [0, \infty)$ be a log-convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f(\frac{a+b}{2}) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)dx\right] \leq \sqrt{f(a)f(b)} \tag{6}$$

is well known in the literature as Hermite-Hadamard inequality for log-convex functions. Both inequalities hold in the reversed direction if f is concave.[18]

Furtermore, in [16], Dragomir and Mond proved that the following inequalities of Hermite–Hadamard type hold for log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(t)dt\right] \\ &\leq \frac{1}{b-a}\int_a^b G\left(f\left(t\right) + fa + b - t\right)dt \\ &\leq \frac{1}{b-a}\int_a^b f\left(t\right)dt \\ &\leq L\left(f\left(a\right), f(b)\right) \end{aligned} \tag{7}$$

More information about *log*-convex functions and their properties can be found in [1, 10, 11, 12].

In this paper we propose the generalization of convexity of this kind for stochastic processes.

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{F}-measurable$. Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space and let $T \subset \mathbb{R}$ be time. A collection of random variables $X(t, w)$, $t \in T$ with values in $\mathbb{R}$ is called a *stochastic process*. If $X(t, w)$ takes values in $S = \mathbb{R}^d$, it is called a $vector - valued$ stochastic process. If the time T can be a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process. If time is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a *stochastic process with continuous time*. For any fixed $\omega \in \Omega$, one can regard $X(t, w)$ as a function of t. It is called a *sample function* of the stochastic process. In the case of a $vector - valued$ process, it is a *sample path*, a curve in $\mathbb{R}^d$. Throughout the paper, we restrict our attention stochastic processes with continuous time, i.e. , index set $T = [0, \infty)$.

Definition 1.1. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : T \times \Omega \to \mathbb{R}$ is

i. convex if

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(v, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in [0,1]$. . This class of stochastic process are denoted by C.

ii. $\lambda-$*convex (where* λ *is a fixed number in* $(0,1)$ *if*

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(u, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in (0,1)$. This class of stochastic process is denoted by C_λ.

iii. Wright-convex if

$$X(\lambda u + (1-\lambda) v, \cdot) + X((1-\lambda)u + \lambda v, \cdot) \leq X(u, \cdot) + X(v, \cdot)$$

for all $u, v \in T$ *and* $\lambda \in [0,1]$. This class of stochastic process is denoted by W.

iv. Jensen-convex if

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}$$

[7, 8, 9, 13]

Clearly, $C \subseteq C_\lambda \subset W$ and $C_{\frac{1}{2}} \subseteq C_\lambda$, for all $\lambda \in (0,1)$. [9]

Definition 1.2. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : T \times \Omega \to \mathbb{R}$ is called

i. continuous in probability in interval T *if for all* $t_0 \in T$

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$$

where $P - \lim$ *denotes the limit in probability;*

ii. mean-square continuous *in the interval* T *if for all* $t_0 \in T$

$$P - \lim_{t \to t_0} E[X(t, \cdot) - X(t_0, \cdot)] = 0$$

where $E[X(t, \cdot)]$ *denotes the expectation value of the random variable* $X(t, \cdot)$*;*

iii. increasing (decreasing) if for all $u, v \in T$ *such that* $t < s$,

$$X(u, \cdot) \leq X(v, \cdot), (X(u, \cdot) \geq X(v, \cdot))$$

iv. monotonic if it is increasing or decreasing;

v. differantiable at a point $t \in T$ *if there is a random variable* $X'(t, \cdot) : T \times \Omega \to \mathbb{R}$

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}$$

We say that a stochastic process $X : T \times \Omega \to \mathbb{R}$ is continuous (differantiable) if it is continuous (differantiable) at every point of the interval T. [7, 8, 9, 13]

Definition 1.3. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval with $E[X(t)^2] < \infty$ for all $t \in T$. Let $[a, b] \subset T$, $a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds:

$$\lim_{n\to\infty} E[(X(\Theta_k(t_k - t_{k-1}) - Y)^2] = 0.$$

Then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot) \ (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \le \int_a^b Z(t,\cdot)dt \ (a.e.),$$

where $X(t,\cdot) \le Z(t,\cdot)$ *(a.e.)* in $[a,b]$ [3].

In throughout the paper, we will consider the stochastic processes that is *with continuous time* and *mean-square continuous*.

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes:

If $X : T \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have [7]

$$X\left(\frac{u+v}{2},\cdot\right) \le \frac{1}{v-u}\int_u^v X(t,\cdot)dt \le \frac{X(u,\cdot)+X(v,\cdot)}{2} \quad (8)$$

The main subject of this paper is to extend some well-known results concerning log-convex functions to log-convex stochastic processes. Also, we investigate the relationship between log-convex stochastic processes and convex stochastic processes. Moreover, we propose well-known Hermite-Hadamard type inequalities for $\log-$convex stochastic processes by the help of aritmetic and geometric means.

2 Hermite-Hadamard Inequality For $\log-$Convex Stochastic Process

Definition 2.1. Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : T \times \Omega \to [0,\infty)$ is *log−convex if*

$$X(\lambda s + (1-\lambda)t,\cdot) \le [X(s,\cdot)]^{\lambda}[X(t,\cdot)]^{1-\lambda} \quad (9)$$

for all $s,t \in T$ and $\lambda \in [0,1]$.

This class of stochastic process is denoted by C_l.

Proposition 2.2. If $X : T\times\Omega \to [0,\infty)$ is a log-convex stochastic process, then X is convex stochastic process. That is, $C_l \subseteq C$ for all $\lambda \in [0,1]$.

Proof. The proof is obvious from (9) and the arithmetic-geometric mean inequality which is known as the inequality

$$[X(s,\cdot)]^{\lambda}[X(t,\cdot)]^{1-\lambda} \le \lambda X(s,\cdot) + (1-\lambda)X(t,\cdot) \quad (10)$$

f*or all* $s,t \in T$and $\lambda \in [0,1]$. □

Proposition 2.3. Let $f : T \to [0,\infty)$ and $X : T\times\Omega \to [0,\infty)$ be a function and a stochastic process, respectively. If f and X are convex and f is increasing, then $f \circ X$ is convex.

Proof. Since f and X are convex and f is increasing

$$\begin{aligned}(f\circ X)(\lambda s+(1-\lambda)t,\cdot) &= f(X(\lambda s+(1-\lambda)t,\cdot))\\ &\le f(\lambda X(s,\cdot)+(1-\lambda)X(t,\cdot))\\ &\le \lambda f(X(s,\cdot))+(1-\lambda)f(X(t,\cdot))\\ &= \lambda(f\circ X)(s,\cdot)+(1-\lambda)(f\circ X)(X(t,\cdot))\end{aligned}$$

for all $s,t\in T$ and $\lambda\in[0,1]$.

Let us recall the Hermite-Hadamard inequality

$$X\left(\frac{u+v}{2},\cdot\right)\le\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\le\frac{X(u,\cdot)+X(v,\cdot)}{2}$$

where $X:T\times\Omega\to\mathbb{R}$ is a convex stochastic process on the interval $T\times\Omega$, $u,v\in T$ and $u<v$. ☐

Note that if we apply the above inequality for the $log-$convex stochastic process $X:T\times\Omega\to(0,\infty)$, we have that

$$\ln\left[X\left(\frac{u+v}{2},\cdot\right)\right]\le\frac{1}{v-u}\int_u^v \ln[X(t,\cdot)]\,dt\le\frac{\ln[X(u,\cdot)]+\ln[X(v,\cdot)]}{2} \tag{11}$$

from which we get

$$X\left(\frac{u+v}{2},\cdot\right)\le\exp\left[\frac{1}{v-u}\int_u^v \ln[X(t,\cdot)]\,dt\right]\le\sqrt{X(u,\cdot)X(v,\cdot)} \tag{12}$$

which is an inequality of Hadamard's type for log-convex stochastic process.

Let us denote by $A(u,v)$ the arithmetic mean of the nonnegative real numbers, and by $G(u,v)$ the geometric mean of the same numbers.

Note that, by the use of these notations, Hadamard's inequality (8) can be written in the form:

$$X(A(u,v),\cdot)\le\frac{1}{v-u}\int_u^v A(X(t,\cdot)+X(u+v-t,\cdot))dt\le A(X(u,\cdot)+X(v,\cdot))$$

It is easy to see this as

$$\int_u^v X(t,\cdot)\,dt=\int_u^v X(u+v-t,\cdot)\,dt$$

We now prove a similar result for log-convex stochastic process and geometric means.

Theorem 2.4. Let $X:T\times\Omega\to[0,\infty)$ be a log-convex stochastic process on $T\times\Omega$ and $u,v\in T$ with $u<v$. Then one has the inequality:

$$X(A(u,v),\cdot)\le\frac{1}{v-u}\int_u^v G(X(t,\cdot),X(u+v-t,\cdot))dt\le G(X(u,\cdot),X(v,\cdot)) \tag{13}$$

Proof. Since X is log-convex, we have that

$$X(\lambda s+(1-\lambda)t,\cdot)\le[X(s,\cdot)]^{\lambda}[X(t,\cdot)]^{1-\lambda}$$

for all $\lambda\in[0,1]$ and

$$X((1-\lambda)s+\lambda t,\cdot)\le[X(s,\cdot)]^{1-\lambda}[X(t,\cdot)]^{\lambda}$$

for all $\lambda\in[0,1]$.

If we multiply the above inequalities and take square roots, we obtain

$$G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))\leq G(X(u,\cdot),X(v,\cdot))$$

Integrating this inequality on $[0,1]$ over λ, we get

$$\int_0^1 G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))d\lambda\leq G(X(u,\cdot),X(v,\cdot))$$

If we change the variable $t:=\lambda u+(1-\lambda)v$, $\lambda\in[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 G(X(\lambda s+(1-\lambda)t,\cdot),X((1-\lambda)s+\lambda t,\cdot))d\lambda\\ =&\ \frac{1}{v-u}\int_u^v G(X(t,\cdot),X(u+v-t,\cdot))dt\end{aligned}$$

and the second inequality in (13) is proved.

Now, by (9) , for $\lambda=\frac{1}{2}$, we have that

$$X\left(\frac{s+t}{2},\cdot\right)\leq G(X(s,\cdot),X(t,\cdot))$$

for all $u,v\in T$.

If we choose $s:=\lambda u+(1-\lambda)v$, $t:=(1-\lambda)u+\lambda v$, we get the inequality

$$X\left(\frac{u+v}{2},\cdot\right)\leq G(X(\lambda u+(1-\lambda)v,\cdot),X((1-\lambda)u+\lambda v,\cdot)) \tag{14}$$

for all $\lambda\in[0,1]$. Integrating this inequality on $[0,1]$ over λ, the first inequality in (13) is proved. □

Corollary 2.5. With the above assumptions, $u\geq 0$ and X nondecreasing on $T\times\Omega$, we have the inequality:

$$X(G(u,v),\cdot)\leq\frac{1}{v-u}\int_u^v G(X(t,\cdot),X(u+v-t,\cdot))dt\leq G(X(u,\cdot),X(v,\cdot)) \tag{15}$$

The following result offers another inequality of Hadamard type for convex stochastic process.

Corollary 2.6. Let $X:T\times\Omega\to[0,\infty)$ be a convex stochastic process on $T\times\Omega$ and $u,v\in T$ with $u<v$. Then one has the inequalities:

$$\begin{aligned}&X\left(\frac{u+v}{2},\cdot\right)\\ \leq&\ \ln\left[\frac{1}{b-a}\int_u^v\exp\left[X(t,\cdot)+X(u+v-t,\cdot)\right]dt\right]\\ \leq&\ \frac{X(u,\cdot)+X(v,\cdot)}{2}\end{aligned} \tag{16}$$

Proof. Define the mapping $g:T\to(0,\infty)$, $g(t)=\exp(X(t,\cdot))$, which is clearly log-convex on I.

Now, if we apply Theorem 2.4, we obtain

$$\exp X(\frac{u+v}{2},\cdot)\leq\frac{1}{b-a}\int_u^v\sqrt{\exp X(t,\cdot)X(u+v-t,\cdot)}dt\leq\sqrt{\exp X(u,\cdot)X(v,\cdot)},$$

which implies (16). □

The following theorem for log-convex stochastic process also holds.

Theorem 2.7. Let $X : T \times \Omega \to (0, \infty)$ be a log-convex stochastic process on $T \times \Omega$ and $u, v \in T$ with $u < v$. Then, one has the inequalities:

$$\begin{aligned} X\left(\frac{u+v}{2}, \cdot\right) &\leq \exp\left[\frac{1}{v-u}\int_u^v \ln\left[X(t,\cdot)\right]dt\right] \quad (17) \\ &\leq \frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right)dt \\ &\leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \\ &\leq L\left(X(u,\cdot), X(v,\cdot)\right), \end{aligned}$$

where $L(p,q) := \frac{p-q}{\ln p - \ln q}$ if $p \neq q$ and $L(p,p) := p$.

Proof. The first inequality in (17) was proved before. We now have that

$$G\left(X(t,\cdot) + X(u+v-t,\cdot)\right) = \exp\left[\ln G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right]$$

for all $t \in [u, v]$.

Integrating this equality on $[u, v]$ and using the well-known Jensen's integral inequality for the convex mapping $\exp(\cdot)$, we have that

$$\begin{aligned} & \frac{1}{v-u}\int_u^v G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)dt \quad (18) \\ = \ & \frac{1}{v-u}\int_u^v \exp\left[\ln\left(G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right)\right]dt \\ \geq \ & \exp\left[\frac{1}{v-u}\int_u^v \ln\left(G\left(X(t,\cdot) + X(u+v-t,\cdot)\right)\right)dt\right] \\ = \ & \exp\left[\frac{1}{v-u}\int_u^v \frac{\ln X(t,\cdot) + \ln X(u+v-t,\cdot)}{2}dt\right] \\ = \ & \exp\left[\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\,dt\right]. \end{aligned}$$

It is clear that

$$\int_u^v \ln X(t,\cdot)\,dt = \int_u^v \ln X(u+v-t,\cdot)\,dt.$$

By the aritmetic mean -geometric mean inequality we have that

$$G\left(X(t,\cdot), X(u+v-t,\cdot)\right) \leq \frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}, t \in [u,v]$$

from which, by integration, we get

$$\frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right)dt \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt$$

and the third inequality in (18) is proved.

To prove the last inequality, we observe, by the log-convexity of X, that

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \left[X(u,\cdot)\right]^{\lambda}\left[X(v,\cdot)\right]^{1-\lambda} \quad (19)$$

for all $u, v \in T$.

Integrating (19) over λ in $[0, 1]$, we have

$$\int_0^1 X(\lambda u + (1-\lambda)v, \cdot)\, d\lambda \le \int_0^1 [X(u,\cdot)]^{\lambda} [X(v,\cdot)]^{1-\lambda}\, d\lambda.$$

As

$$\int_0^1 X(\lambda u + (1-\lambda)v, \cdot)\, d\lambda = \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt$$

and

$$\int_0^1 [X(u,\cdot)]^{\lambda} [X(v,\cdot)]^{1-\lambda}\, d\lambda = L[X(u,\cdot), X(v,\cdot)],$$

the theorem is proved. □

Corollary 2.8. Let $X : T \times \Omega \to \mathbb{R}$ be a convex stochastic process on $T \times \Omega$ and $u, v \in T$ with $u < v$. Then one has the inequalities:

$$\begin{aligned} \exp\left[X\left(\frac{u+v}{2}, \cdot\right)\right] &\le \exp\left[\frac{1}{v-u}\int_u^v X(t,\cdot)\, dt\right] \\ &\le \frac{1}{v-u}\int_u^v \exp\left[\frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}\right] dt \\ &\le \frac{1}{v-u}\int_u^v \exp[X(t,\cdot)]\, dt \\ &\le E(X(u,\cdot), X(v,\cdot)), \end{aligned} \tag{20}$$

where E is the exponential mean, i.e.,

$$E(p,q) := \frac{\exp p - \exp q}{p-q} \text{ for } p \ne q \text{ and } E(p,p) = p.$$

Remark 2.9. Note that the inequality

$$\exp\left(\frac{1}{v-u}\int_u^v \ln[X(t,\cdot)]\, dt\right) \tag{21}$$

$$\le \frac{1}{v-u}\int_u^v G(X(t,\cdot), X(u+v-t,\cdot))\, dt \tag{22}$$

$$\le \frac{1}{v-u}\int_u^v X(t,\cdot)\, dt$$

holds for every strictly positive and integrable stochastic process $X : I \times \Omega \to \mathbb{R}$ and the inequality

$$\exp\left[\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\, dt\right] \tag{23}$$

$$\le \frac{1}{v-u}\int_u^v \exp\left(\frac{X(t,\cdot) + X(u+v-t,\cdot)}{2}\right) dt \tag{24}$$

$$\le \frac{1}{v-u}\int_u^v \exp X(t,\cdot)\, dt$$

holds for every $X : T \times \Omega \to \mathbb{R}$ an integrable stochastic on $[u, v]$.

Taking into account that the above two inequalities hold, we can assert that for every $X : T \times \Omega \to (0, \infty)$ an integrable stochastic process on $[u, v]$ we have the inequalities:

$$
\begin{aligned}
& \exp\left(\frac{1}{v-u}\int_u^v \ln X(t,\cdot)\,dt\right) \qquad (25)\\
\leq\ & \frac{1}{v-u}\int_u^v G\left(X(t,\cdot), X(u+v-t,\cdot)\right)dt\\
\leq\ & \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\\
\leq\ & \ln\left[\frac{1}{v-u}\int_u^v \exp A\left(X(t,\cdot), X(u+v-t,\cdot)\right)\right]dt\\
\leq\ & \ln\left[\frac{1}{v-u}\int_u^v \exp X(t,\cdot)\,dt\right],
\end{aligned}
$$

which is of interest in itself.

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