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# NUMERICAL METHODS FOR DISCONTINUOUS STURM-LIOUVILLE PROBLEMS

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**Abstract** – This study is devoted to determining the eigenvalues and eigenfunctions of a discontinuous Sturm-Liouville Problem. By modifying the finite difference method, we have developed a numerical approximation to the eigenvalues and eigenfunctions.

**Keywords** – Sturm-Liouville, Discontinuous, Numerical Solution, Eigenvalues, Eigenfunctions, Transmission Conditions

## 1 Introduction

Many physical systems are connected to a Sturm-Liouville problem. The computation of eigenvalues of Sturm-Liouville problems is therefore important to many problems in Mathematical Physics. Sturm-Liouville systems arise from vibration problems in continuous media with non-uniform properties, such as the propagation of sonar in water and the seismic waves in the Earth [4]. In the classical sense, the Sturm-Liouville problem is replaced by a first order differential equation that is solved using Shooting methods. Added in this class are the Prüfer phase methods ([5] and [8]). In the recent years, pursued with considerable success by a number of researchers including Pruess ([9], [10]) and Paine and de Hoog [7], a simpler problem is constructed by replacing the coefficients in the Sturm-Liouville problem by piecewise constants. Anderssen and de Hoog [6] extend the results to the Liouville normal form with general boundary conditions. Moreover, some important results in this field have also been obtained for discontinuous Sturm-Liouville systems. It should be mentioned that O. Sh. Mukhtarov and his colleagues [1, 2, 3] have constructed boundary value problems with discontinuities where an eigenparameter appears not only in the differential equation, but also in the boundary and transmission conditions.

The main goal of this study is to extend the finite difference method to a particular discontinuous Sturm-Liouville problem.

Specifically, we shall compute approximations to the eigenvalues and eigenfunctions of a problem with the following transmission and boundary conditions:

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$$y'' + \lambda y = 0 \quad x \in [0, c) \cup (c, \pi] \tag{1}$$

$$y(0) = y(\pi) = 0 \tag{2}$$

$$\gamma_1 y(c_+) = \gamma_2 y(c_-) \tag{3}$$

### 1.1 Finite Difference Method

Under certain conditions, we can use Taylor's formulae:

$$y(x + h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2 + \dots \tag{4}$$

and

$$y(x - h) = y(x) - y'(x)h + \frac{y''(x)}{2!}h^2 + \dots \tag{5}$$

Combining the two, we get

$$y(x + h) + y(x - h) \approx 2y(x) + y''(x)h^2, \tag{6}$$

discarding all higher-order terms. Thus, to approximate  $y''(x)$ , we can use the difference equation

$$y''(x) \approx \frac{y(x - h) - 2y(x) + y(x + h)}{h^2}. \tag{7}$$

Substituting this approximation into the Sturm-Liouville equation (1.1), we get

$$-\frac{y(x - h) - 2y(x) + y(x + h)}{h^2} \approx \lambda y(x). \tag{8}$$

We now partition the interval  $[0, \pi]$  into  $0 = x_0 < x_1 \dots < x_N < x_{N+1} = \pi$  nodes, where  $x_j = jh$  and  $h = \frac{\pi}{N+1}$ . We seek a solution to the Sturm-Liouville problem on these nodes. Naturally, we let

$$y(x_0) = y(0) = 0$$

and

$$y(x_{N+1}) = y(\pi) = 0$$

However, the solution is unknown at the interior points  $x_j, j = 1, 2, \dots, N$ .

By evaluating the difference equation 8 at  $x_j$ , we obtain

$$-h^{-2}[y(x_j - h) - 2y(x_j) + y(x_j + h)] \approx \lambda y(x_j).$$

Because  $h$  is the increment between consecutive points on our partition,  $x_j - h = x_{j-1}$  and  $x_j + h = x_{j+1}$ . Hence,

$$-h^{-2}[y(x_{j-1}) - 2y(x_j) + y(x_{j+1})] \approx \lambda y(x_j).$$

We now replace  $y(x_j)$  with  $y_j$  and iterate the equation

$$-h^{-2}[y_{j-1} - 2y_j + y_{j+1}] = \lambda y_j \tag{9}$$

in order to find an approximate solution to the Sturm-Liouville problem.

It must be noted that  $y(x_j)$  represents the *exact* value of the Sturm-Liouville solution evaluated at  $x_j$ , while the variable  $y_j$  represents the approximation to  $y(x_j)$  at  $x_j$ .

For  $j = 1$ , equation 9 becomes

$$-h^{-2}[y_0 - 2y_1 + y_2] = \lambda y_1. \tag{10}$$

Similarly, for  $j = 2, 3$ , and  $4$ , we get

$$\begin{aligned} -h^{-2}[y_1 - 2y_2 + y_3] &= \lambda y_2, \\ -h^{-2}[y_2 - 2y_3 + y_4] &= \lambda y_3, \\ -h^{-2}[y_3 - 2y_4 + y_5] &= \lambda y_4. \end{aligned}$$

Finally, for  $j = N$ , we get

$$-h^{-2}[y_{N-1} - 2y_N + y_{N+1}] = \lambda y_N. \tag{11}$$

However, the boundary conditions require

$$\begin{aligned} y_0 = y(x_0) = y(0) &= 0, \\ y_{N+1} = y(x_{N+1}) = y(\pi) &= 0, \end{aligned}$$

so equation 10 becomes  $-h^{-2}[-2y_1 + y_2] = \lambda y_1$ . Similarly, equation 11 becomes  $-h^{-2}[y_{N-1} - 2y_N] = \lambda y_N$ . The full set is a system of  $N$  equations in  $N$  unknowns.

$$\begin{aligned} 2h^{-2}y_1 - h^{-2}y_2 &= \lambda y_1, \\ -h^{-2}y_1 + 2h^{-2}y_2 - h^{-2}y_3 &= \lambda y_2, \\ -h^{-2}y_2 + 2h^{-2}y_3 - h^{-2}y_4 &= \lambda y_3, \\ -h^{-2}y_3 + 2h^{-2}y_4 - h^{-2}y_5 &= \lambda y_4, \\ &\vdots \\ -h^{-2}y_{N-1} + 2h^{-2}y_N &= \lambda y_N. \end{aligned} \tag{12}$$

This system can be written in matrix form as

$$\begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & 0 & \dots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & 0 & \dots & 0 \\ 0 & -h^{-2} & 2h^{-2} & -h^{-2} & \dots & 0 \\ 0 & 0 & -h^{-2} & 2h^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2h^{-2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix}. \tag{13}$$

By denoting

$$M = \begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & 0 & \dots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & 0 & \dots & 0 \\ 0 & -h^{-2} & 2h^{-2} & -h^{-2} & \dots & 0 \\ 0 & 0 & -h^{-2} & 2h^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2h^{-2} \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix}$$

the system (13) can be written in the form

$$M\vec{y} = \lambda\vec{y}.$$

## 1.2 Neumann Conditions

Let us consider the same Sturm-Liouville equation, but this time attach a Neumann condition at the right endpoint.

$$-y''(x) = \lambda y(x), \quad y(0) = 0, \quad y'(\pi) = 0 \tag{14}$$

Since we still have  $-y''(x) = \lambda y(x)$ , we can re-use equation (9), repeated here for convenience.

$$-h^{-2}[y_{j-1} - 2y_j + y_{j+1}] = \lambda y_j$$

Because of the Neumann condition  $y'(\pi) = 0$ , we do not know the value of the solution at the right endpoint (i.e., we do not know  $y(\pi)$ ). Therefore, if we partition the interval as above, we will need to compute  $y_1, y_2, \dots, y_{N+1}$  instead of  $y_1, y_2, \dots, y_N$ . Solving this system requires the addition of an

extra equation for the variable  $y_{N+1}$ . The resulting system looks like this:

$$\begin{aligned}
 -h^{-2}[y_0 - 2y_1 + y_2] &= \lambda y_1, \\
 -h^{-2}[y_1 - 2y_2 + y_3] &= \lambda y_2, \\
 -h^{-2}[y_2 - 2y_3 + y_4] &= \lambda y_3, \\
 -h^{-2}[y_3 - 2y_4 + y_5] &= \lambda y_4, \\
 -h^{-2}[y_4 - 2y_5 + y_6] &= \lambda y_5, \\
 &\vdots \\
 -h^{-2}[y_{N-2} - 2y_{N+1} + y_{N+2}] &= \lambda y_{N+1}.
 \end{aligned} \tag{15}$$

However, this gives us  $N + 1$  equations in  $N + 3$  unknowns. Because of the Dirichlet condition at the left endpoint, we know that

$$y_0 = y(x_0) = y(0) = 0.$$

This condition eliminates the unknown  $y_0$ . However, we will need to obtain an estimate for  $y_{N+2}$ .

We can use a forward difference equation to approximate the first derivative:

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}.$$

Using  $h$  as the step size in our partition of  $[0, \pi]$ , we have

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{h}.$$

Using  $y_j$  as an approximation of  $y(x_j)$ , we can write

$$y'_j = \frac{y_{j+1} - y_j}{h}$$

or equivalently

$$y_{j+1} = y_j + h y'_j.$$

For  $j = N + 1$ , this gives us the equation  $y_{N+2} = y_{N+1} + h y'_{N+1}$ . However, the Neumann condition on the right endpoint of  $[0, \pi]$  is

$$y'_{N+1} = y'(x_{N+1}) = y'(\pi) = 0.$$

Thus, we get  $y_{N+2} = y_{N+1}$ .

Substituting  $y_0 = 0$  and  $y_{N+2} = y_{N+1}$  into the first and last equations respectively of system (15) gives us  $N + 1$  equations in  $N + 1$  unknowns:

$$\begin{aligned}
 -h^{-2}[-2y_1 + y_2] &= \lambda y_1, \\
 -h^{-2}[y_1 - 2y_2 + y_3] &= \lambda y_2, \\
 -h^{-2}[y_2 - 2y_3 + y_4] &= \lambda y_3, \\
 -h^{-2}[y_3 - 2y_4 + y_5] &= \lambda y_4, \\
 -h^{-2}[y_4 - 2y_5 + y_6] &= \lambda y_5, \\
 &\vdots \\
 -h^{-2}[y_N - y_{N+1}] &= \lambda y_{N+1}.
 \end{aligned}$$

This system can be written in matrix form.

$$\begin{bmatrix}
 2h^{-2} & -h^{-2} & 0 & \cdots & 0 \\
 -h^{-2} & 2h^{-2} & h^{-2} & \cdots & 0 \\
 0 & -h^{-2} & 2h^{-2} & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & h^{-2}
 \end{bmatrix}
 \begin{bmatrix}
 y_1 \\
 y_2 \\
 y_3 \\
 \vdots \\
 y_{N+1}
 \end{bmatrix}
 = \lambda
 \begin{bmatrix}
 y_1 \\
 y_2 \\
 y_3 \\
 \vdots \\
 y_{N+1}
 \end{bmatrix}. \tag{16}$$

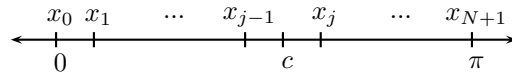


Figure 1: Partitioning the solution interval  $[0; \pi]$

By denoting

$$N = \begin{bmatrix} 2h^{-2} & -h^{-2} & 0 & \dots & 0 \\ -h^{-2} & 2h^{-2} & -h^{-2} & \dots & 0 \\ 0 & -h^{-2} & 2h^{-2} & \dots & 0 \\ 0 & 0 & -h^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h^{-2} \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N+1} \end{bmatrix}$$

the system (16) reduced to the form

$$N\vec{y} = \lambda\vec{y}.$$

## 2 A Numerical Method for Solving Discontinuous Sturm-Liouville Problems

We now consider the Sturm-Liouville equation described in (1)- (3), which holds over the finite interval  $[0, \pi]$  except at one inner point  $c \in [0, \pi]$ .

First, we partition the interval  $[0, \pi]$  such that  $x_{j-1} < c < x_j$  as in Figure 1. We then use equation (9) to find an approximate solution at those points to the left of  $c$ :  $y_0, y_1, \dots, y_{j-1}$  at  $x_0, x_1, \dots, x_{j-1}$  by . Once  $y_{j-1}$  is known, the transmission condition (3) gives us a value for  $y_j$ :

$$y_j = \frac{\gamma_2}{\gamma_1} y_{j-1}$$

at  $x_j$ , on the right-hand side of  $c$ . Finally, with  $y_j$  known, the values  $y_{j+1}, \dots, y_n$  at  $x_{j+1}, \dots, x_n$  can be calculated from equation (9).

To generalize this solution, consider the new variable  $Y_j$ :

$$Y_j = \begin{cases} y_j, & j \neq j^* \\ \frac{\gamma_2}{\gamma_1} y_{j-1}, & j = j^* \end{cases} \tag{17}$$

where

$$j^* = \min\{k | x_k > c\}.$$

System (12) then becomes

$$\begin{aligned} -h^{-2}[Y_0 - 2Y_1 + Y_2] &= \lambda Y_1 \\ -h^{-2}[Y_1 - 2Y_2 + \frac{\gamma_1}{\gamma_2} Y_3] &= \lambda Y_2 \\ -h^{-2}[\frac{\gamma_2}{\gamma_1} Y_2 - 2Y_3 + \frac{\gamma_2}{\gamma_1} Y_4] &= \lambda Y_3 \\ -h^{-2}[\frac{\gamma_1}{\gamma_2} Y_3 - 2Y_4 + Y_5] &= \lambda Y_4 \\ -h^{-2}[Y_4 - 2Y_5 + Y_6] &= \lambda Y_5 \\ &\vdots \\ -h^{-2}[Y_{N-1} - 2Y_N + Y_{N+1}] &= \lambda Y_N. \end{aligned}$$

for  $j = 1, \dots, N$ . The matrix form of this system is written as follows:

$$h^{-2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_N \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_N \end{bmatrix}. \tag{18}$$

### 3 Numerical Illustration

**Example 3.1.** For  $N = 30$ ,  $\frac{\gamma_1}{\gamma_2} = \frac{1}{2}$  and  $x_2 < c < x_3$ , we consider the discontinuous Sturm-Liouville problem:

$$\begin{aligned} -y''(x) &= \lambda y(x) \quad x \in [0, c) \cup (c, \pi] \\ y(0) &= y(\pi) = 0 \\ \gamma_2 y(c_+) &= \gamma_1 y(c_-) \end{aligned} \tag{19}$$

By applying the transformation (17) we obtain the following system of linear equations:

$$\begin{aligned} -h^{-2}[Y_0 - 2Y_1 + Y_2] &= \lambda Y_1 \\ -h^{-2}[Y_1 - 2Y_2 + \frac{\gamma_1}{\gamma_2} Y_3] &= \lambda Y_2 \\ -h^{-2}[\frac{\gamma_2}{\gamma_1} Y_2 - 2Y_3 + \frac{\gamma_2}{\gamma_1} Y_4] &= \lambda Y_3 \\ -h^{-2}[\frac{\gamma_1}{\gamma_2} Y_3 - 2Y_4 + Y_5] &= \lambda Y_4 \\ -h^{-2}[Y_4 - 2Y_5 + Y_6] &= \lambda Y_5 \\ &\vdots \\ -h^{-2}[Y_{28} - 2Y_{29} + Y_{31}] &= \lambda Y_{30} \end{aligned}$$

This system can be written as the following matrix equation:

$$h^{-2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_{30} \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ \vdots \\ Y_{30} \end{bmatrix}. \tag{20}$$

For this system, the following MatLAB commands generate approximations to the eigenvalue and the eigenfunctions, as shown in Figure 2.

v =

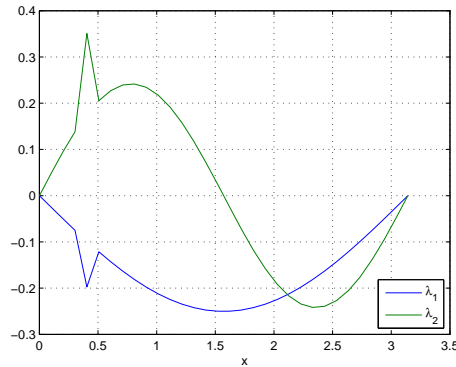


Figure 2: The first two eigenfunctions

Columns 1	through 5					Columns 29	through 30
-0.0253	0.0487	0.0703	-0.0917	0.1146	...	-0.2098	-0.2036
-0.0504	0.0954	0.1341	-0.1685	0.2005	...	0.1848	0.2154
-0.0749	0.1382	0.1857	-0.2181	0.2359	...	0.0470	-0.0243
-0.1974	0.3508	0.4404	-0.4645	0.4242	...	-0.4525	-0.3794
-0.1215	0.2054	0.2345	-0.2088	0.1350	...	0.1522	0.2250
-0.1430	0.2269	0.2273	-0.1515	0.0239	...	0.0922	-0.0483
-0.1630	0.2392	0.1993	-0.0696	-0.0932	...	-0.2334	-0.1739
-0.1814	0.2417	0.1529	0.0235	-0.1868	...	0.1134	0.2322
-0.1979	0.2343	0.0926	0.1129	-0.2335	...	0.1335	-0.0718
-0.2124	0.2173	0.0238	0.1839	-0.2215	...	-0.2310	-0.1563
-0.2247	0.1914	-0.0473	0.2251	-0.1539	...	0.0700	0.2371
-0.2347	0.1576	-0.1140	0.2299	-0.0476	...	0.1694	-0.0946
-0.2423	0.1174	-0.1702	0.1973	6 0.0707	...	-0.2191	-0.1370
-0.2474	0.0724	-0.2108	0.1328	0.1712	...	0.0236	0.2396
-0.2499	0.0245	-0.2321	0.0468	0.2287	...	0.1983	-0.1164
-0.2499	-0.0245	-0.2321	-0.0468	0.2287	...	-0.1983	-0.1164
-0.2474	-0.0724	-0.2108	-0.1328	0.1712	...	-0.0236	0.2396
-0.2423	-0.1174	-0.1702	-0.1973	0.0707	...	0.2191	-0.1370
-0.2347	-0.1576	-0.1140	-0.2299	-0.0476	...	-0.1694	-0.0946
-0.2247	-0.1914	-0.0473	-0.2251	-0.1539	...	-0.0700	0.2371
-0.2124	-0.2173	0.0238	-0.1839	-0.2215	...	0.2310	-0.1563
-0.1979	-0.2343	0.0926	-0.1129	-0.2335	...	-0.1335	-0.0718
-0.1814	-0.2417	0.1529	-0.0235	-0.1868	...	-0.1134	0.2322
-0.1630	-0.2392	0.1993	0.0696	-0.0932	...	0.2334	-0.1739
-0.1430	-0.2269	0.2273	0.1515	0.0239	...	-0.0922	-0.0483
-0.1215	-0.2054	0.2345	0.2088	0.1350	...	-0.1522	0.2250
-0.0987	-0.1754	0.2202	0.2322	0.2121	...	0.2262	-0.1897
-0.0749	-0.1382	0.1857	0.2181	0.2359	...	-0.0470	-0.0243
-0.0504	-0.0954	0.1341	0.1685	0.2005	...	-0.1848	0.2154
-0.0253	-0.0487	0.0703	0.0917	0.1146	...	0.2098	-0.2036

The eigenvalues are

n	1	2	3	.....	29	30
$\lambda_n$	0.9991	3.9863	8.9309	....	385.4923	388.4795

The first two eigenfunctions are shown in Figure 2.

**Example 3.2.** For  $N = 30$ ,  $\frac{\gamma_1}{\gamma_2} = \frac{1}{2}$  and  $x_2 < c < x_3$ , we consider the same discontinuous Sturm-Liouville problem with a Neumann boundary condition:

$$-y''(x) = \lambda y(x) \quad x \in [0, c) \cup (c, \pi] \tag{21}$$

$$y(0) = y'(\pi) = 0$$

$$\gamma_2 y(c_+) = \gamma_1 y(c_-)$$

By applying the transformation (17) we obtain the following system of linear equations:

$$\begin{aligned} -h^{-2}[-2Y_1 + Y_2] &= \lambda Y_1 \\ -h^{-2}[Y_1 - 2Y_2 + \frac{\gamma_1}{\gamma_2} Y_3] &= \lambda Y_2 \\ -h^{-2}[\frac{\gamma_2}{\gamma_1} Y_2 - 2Y_3 + \frac{\gamma_2}{\gamma_1} Y_4] &= \lambda Y_3 \\ -h^{-2}[\frac{\gamma_1}{\gamma_2} Y_3 - 2Y_4 + Y_5] &= \lambda Y_4 \\ -h^{-2}[Y_4 - 2Y_5 + Y_6] &= \lambda Y_5 \\ &\vdots \\ -h^{-2}[Y_{30} + Y_{31}] &= \lambda Y_{31} \end{aligned}$$

The matrix form is

$$h^{-2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -\frac{\gamma_1}{\gamma_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{\gamma_2}{\gamma_1} & 2 & -\frac{\gamma_2}{\gamma_1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\gamma_1}{\gamma_2} & 2 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ \vdots \\ Y_{31} \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ \vdots \\ Y_{31} \end{bmatrix} \tag{22}$$

The following commands were used to generate the eigenvalues and eigenfunctions.

v =



Columns 1	through 5					Columns 29	through 30
0.0257	-0.0483	-0.0673	0.0844	0.1018	...	0.0468	0.0248
0.0511	-0.0945	-0.1282	0.1548	0.1773	...	-0.0917	-0.0494
0.2281	-0.4102	-0.5310	0.5979	0.6209	...	0.3989	0.2205
0.1001	-0.1732	-0.2091	0.2105	0.1831	...	-0.1689	-0.0968
0.1232	-0.2023	-0.2213	0.1866	0.1120	...	0.1979	0.1192
0.1449	-0.2229	-0.2127	0.1316	0.0119	...	-0.2190	-0.1404
0.1651	-0.2341	-0.1839	0.0545	-0.0913	...	0.2313	0.1601
0.1835	-0.2353	-0.1377	-0.0316	-0.1709	...	-0.2342	-0.1782
0.2000	-0.2267	-0.0785	-0.1125	-0.2063	...	0.2278	0.1946
0.2144	-0.2084	-0.0119	-0.1745	-0.1884	...	-0.2121	-0.2089
0.2265	-0.1814	0.0558	-0.2075	-0.1218	...	0.1879	0.2212
0.2362	-0.1466	0.1183	-0.2058	-0.0237	...	-0.1561	-0.2312
0.2434	-0.1057	0.1696	-0.1697	0.0805	...	0.1180	0.2389
0.2480	-0.0603	0.2048	-0.1053	0.1639	...	-0.0752	-0.2442
0.2500	-0.0124	0.2207	-0.0234	0.2049	...	0.0294	0.2470
0.2493	0.0361	0.2157	0.0625	0.1930	...	0.0294	0.2470
0.2460	0.0830	0.1903	0.1379	0.1312	...	-0.0640	0.2451
0.2401	0.1264	0.1469	0.1904	0.0355	...	0.1077	-0.2404
0.2316	0.1645	0.0896	0.2110	-0.0694	...	-0.1472	0.2333
0.2207	0.1956	0.0238	0.1964	-0.1564	...	0.1806	-0.2239
0.2075	0.2185	-0.0442	0.1490	-0.2029	...	-0.2068	0.2122
0.1920	0.2321	-0.1081	0.0768	-0.1970	...	0.2247	-0.1984
0.1746	0.2360	-0.1617	-0.0083	-0.1402	...	-0.2335	0.1825
0.1552	0.2298	-0.2000	-0.0920	-0.0471	...	0.2329	-0.1648
0.1342	0.2140	-0.2194	-0.1603	0.0581	...	-0.2229	0.1454
0.1118	0.1890	-0.2181	-0.2019	0.1483	...	0.2040	-0.1246
0.0883	0.1562	-0.1961	-0.2097	0.2002	...	-0.1768	0.1025
0.0637	0.1167	-0.1556	-0.1826	0.2003	...	0.1425	-0.0794
0.0385	0.0722	-0.1003	-0.1249	0.1487	...	-0.1024	0.0555
0.0129	0.0248	-0.0356	-0.0464	0.0586	...	0.0583	-0.0310
-0.0128	-0.0238	0.0325	0.0398	-0.0466	...	-0.0117	0.0062

The eigenvalues are

n	1	2	3	.....	29	30
$\lambda_n$	1.032	4.1148	9.2101	....	385.5555	388.4954

The first two eigenfunctions are shown in Figure 3.

## 4 Conclusion

We have described a numerical method for approximating the eigenvalues and eigenfunctions of a discontinuous Sturm-Liouville problem. Since the present method gives real and positive eigenvalues at all nodes, if higher accuracy is required we can simply increase the number of nodes  $N$ .

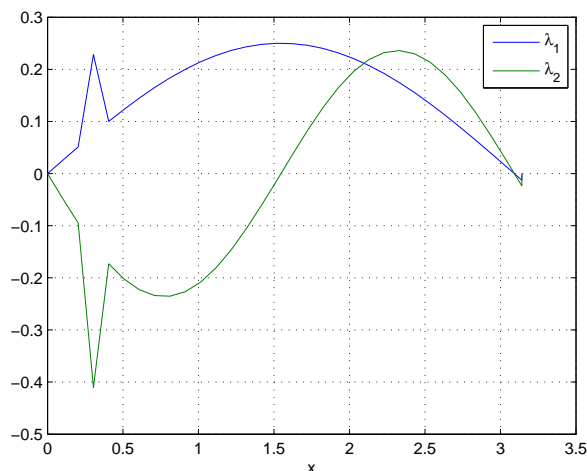


Figure 3: The first two eigenfunctions with a Neumann boundary condition

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