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ROUGH LATTICE OVER BOOLEAN ALGEBRA

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Abstract − Rough Set Theory (RST) is a mathematical formalism for representing uncertainty that can be considered as an extension of the classical set theory. It has been used in many different research areas, including those related to inductive machine learning and reduction of knowledge in knowledge-based systems. Rough partial order relation and rough lattice are two important concepts to introduce here based on RST. This paper provides some properties of rough relations, rough lattice, rough boolean lattice and established their validity. Some results are established to illustrate the paper.

Keywords − *Rough Set, Rough Approximation, Boolean Algebra, Rough Relation, Rough Partial Order, Boolean Lattice.*

1 Introduction

Rough set plays an important role for handling situations which are not crisp and deterministic but associated with impreciseness in the form of indiscernibility between the objects of a set. So, in case of dealing with some types of knowledge representation problems, rough algebraic structures are useful. The concepts of Lattices and Boolean algebra [1] are of cardinal importance in the theory and design of computers and of circuitry in general, besides having numerous other applications in mathematical logic, probability theory and other fields of engineering and mathematics. Lattice is an algebraic structure is of considerable importance, in view of its application in fields of mathematics and computer science. The notions of rough partial order relation and rough lattice are based on RST are needed in many applications, where experimental data are processes, in particular as a theoretical basis for rough relation. In [2] Jouni Järvinen has proposed several direction of lattice theory for rough set. We have also proposed lattice theory for rough set in different direction ([8],[9],[10],[11],[12]). This paper presents the main concepts related to rough partial order relations, some of its properties and related rough boolean algebra which are different but quite related with some special cases of Järvinen's work.

The remainder of this article is organized as follows. Section 2 gives account of previous work. Our new and exciting results are described in Section 3 and Section 4. Finally, Section 5 gives the conclusions.

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2 Definitions and Notations

2.1 Rough Set

Let U be a universe of discourse and E be an equivalence relation over U, called the indiscernibility relation. By U/E, we denote the family of all equivalence classes induced by E on U. These classes are referred to as categories or concepts of E and the equivalence class of an element $x \in U$, is denoted by x/E or $[x]_E$. The basic concept of rough set theory is the notion of an approximation space, which is an ordered pair $A = (U, E)$. For $x, y \in U$, if xEy then x and y are said to be indistinguishable in A. The elements of U/E are called elementary sets in A. It is assumed that the empty set is also elementary set for every approximation space. A definable set in A is any finite union of elementary sets in A.

2.2 Rough Approximations

Theory of rough set was introduced by Z. Pawlak [4], assumed that set is chosen from a universe U, but that elements of U can be specified only upto an indiscernibility equivalence relation E on U. If a subset $X \subseteq U$ contains an element indiscernible from some elements not in X, then X is rough. Also a rough set X is described by two approximations. Basically, in rough set theory, it is assumed that our knowledge is restricted by an indiscernibility relation. An indiscernibility relation is an equivalence relation E such that two elements of an universe of discourse U are E-equivalent if we cannot distinguish these two elements by their properties known by us. By the means of an indiscernibility relation E, we can partition the elements of U into three disjoint classes respect to any set $X \subseteq U$, defined as follows:

- The elements which are certainly in X. These are elements $x \in U$ whose E-class x/E is included in X.
- The elements which certainly are not in X. These are elements $x \in U$ such that their E-class x/E is included in X^{co}, which is the complement of X
- The elements which are possibly belongs to X. These are elements whose E-class intersects with both X and X^{co}. In other words, x/E is not included in X nor in X^{co}.

From this observation, we defined lower approximation set $X \downarrow$ of X to be the set of those elements $x \in U$ whose E-class is included in X, i.e, $X \downarrow = \{x \in U : x/E \subseteq X\}$ and for the upper approximation set $X \uparrow$ of X consists of elements $x \in U$ whose E-class intersect with X, i.e, $X \uparrow = \{x \in U : x/E \cap X \neq \emptyset\}$. The difference between $X \downarrow$ and $X \uparrow$ treated as the actual area of uncertainty.

3 Rough Relation

The notion of rough relation was introduced and their properties were studied by Pawlak ([6], [7]). Stepaniuk ([13], [14]) have established some more properties of rough relations and their applications.

Definition 3.1. Let $A_1 = (U_1, E_1)$ and $A_2 = (U_2, E_2)$ be two approximation spaces. The product of A_1 by A_2 is the approximation space denoted by $A = (U, S)$, where $U = U_1 \times U_2$ and the indiscernibility relation $S \subseteq (U \times U)^2$ is defined by $(x_1, y_1), (x_2, y_2) \in S \Leftrightarrow (x_1, x_2) \in E_1$ and $(y_1, y_2) \in E_2$, $(x_1, x_2) \in U_1$ and $(y_1, y_2) \in U_2$. It can be easily seen that S is an equivalence relation on $U \times U$. The elements (x_1, y_1) and (x_2, y_2) are indiscernible in S if and only if the elements x_1 and x_2 are indiscernible in E_1 and so are the elements y_1 and y_2 in E_2.

Definition 3.2. Let $(U_1 \times U_2, E)$ be an approximation space, where U_1 and U_2 are nonempty sets and $R \subseteq (U_1 \times U_2)^2$ be an equivalence relation. For any relation $S \subseteq U_1 \times U_2$, we define two relations $L(S)$ and $U(S)$ called lower and upper approximations of S respectively given by, $L(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_E \subseteq S\}, U(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_E \cap S \neq \emptyset\}$, where $[(x_1, x_2)]_E$ denotes the equivalence class of relation E containing the pair (x_1, x_2). Rough relation of S is defined as the pair $(L(S), U(S))$.

Definition 3.3. If V and W are relations in A, then $W * V$ is a relation such that $(a,b) \in V\ and\ (b,c) \in W\ for\ some\ b \in A\}$.

Proposition 3.4. If V, W, V_1, W_1 are relations in $A, V_1 \subseteq V$ and $W_1 \subseteq W$, then $W_1 * V_1 \subseteq W * V$

Proof: Let $(a,c) \in W_1 * V_1 \Rightarrow \exists b \in A$ such that $(a,b) \in V_1$ and $(b,c) \in W_1$. Then $(a,b) \in V$ and $(b,c) \in W$ so that $(a,c) \in W * V$

Proposition 3.5. Let $A = (U, R)$ be an approximation space and $B = (U^2, S)$ the approximation product space of $A \times A$. Then:

- $[(x,y)]_S = [x]_E \times [y]_E$, and
- $[(y,z)]_S * [(x,y)]_S = [(x,z)]_S$

Proof: The first result is trivially follows from the definition of the relation S.
For the second result let $(a,c) \in [(y,z)]_S * [(x,y)]_S$. Then there exist $a, b \in U$ such that $(a,b) \in [(x,y)]_S$ and $(b,c) \in [(y,z)]_S$. It follows that $(a,b)S(x,y)$ and $(b,c)S(y,z)$. Hence aEx, bEy and cEz hold. Consequently, $(a,c) \in [(x,z)]_S$
On the other hand, let $(a,c) \in [(x,z)]_S$. This gives $(a,c)S(x,z)$. We thus get aEx and cEz. This clearly implies $(a,y)S(x,y)$ and $(y,c)S(y,z)$. Hence $(a,y) \in [(x,y)]_S$ and $(y,c) \in [(y,z)]_S$, and therefore $(a,c) \in [(y,z)]_S * [(x,y)]_S$

Definition 3.6. Let $S = (U, E)$ be an approximation space and E_g be its generated relation of E, we say that $S_g = (U \times U, E_g)$ is general approximation space of S.

Definition 3.7. [6] We consider a non-null subset M of U and a relation T on M. The rough relation $E_g(T)(M \to M)$ is said to be Reflexive: if and only if $\forall\ m \in M, (m,m) \in E_g \uparrow (T)$. Symmetric: if and only if $\forall\ m_1, m_2 \in M, (m_1, m_2) \in E_g \uparrow (T) \Rightarrow (m_2, m_1) E_g \uparrow (T)$. Transitive: if and only if $\forall\ m_1, m_2, m_3 \in M, (m_1, m_2)$ and $(m_2, m_3) E_g \uparrow (T) \Rightarrow (m_1, m_3) E_g \uparrow (T)$. Antisymmetric: if and only if $\forall m_1, m_2 \in M, (m_1, m_2), (m_2, m_1) \in E_g \uparrow (T) \Rightarrow [m_1]_E = [m_2]_E$. We only consider the upper approximation as lower approximation is always subset of upper approximation.

Definition 3.8. A relation T is said to be a rough partially ordering if $E_g(T)$ is reflexive, symmetric and transitive.

3.1 Rough Membership Function

Rough sets can also be defined by the rough membership function instead of approximation [5]. We define the membership function of X with respect to E as $\mu_X^E : X \to [0,1]$, such that $\mu_X^E = \frac{|x/E \cap X|}{|x/E|}$, where $||$ represents cardinality function on a set. The rough membership function can also be interpreted as the conditional probability, and can be interpreted as a degree of certainty to which x belongs to X. The rough membership function can be used to define the lower approximation, the upper approximation and the boundary region of a set, as follows: $E \downarrow (X) = \{x \in U : \mu_X^E(X) = 1\}, E \uparrow (X) = \{x \in U : \mu_X^E(X) > 0\}$ and
$BN_E(X) = \{x \in U : 0 < \mu_X^E(X) < 1\}$
$\mu_{A \cup B}^E(X) \geq max(\mu_A^E(X), \mu_B^E(X))$ for any $x \in U$.
$\mu_{A \cap B}^E(X) \leq min(\mu_A^E(X), \mu_B^E(X))$ for any $x \in U$.

Definition 3.9. For any rough partial ordering T on a non-null subset M of U, the dominating class of an element x in M is denoted by $T_{\geq[x]}$and is defined for every y in M as $T_{\geq[x]}(y) = r_T(x, y)$, where $r_T(x, y) = \frac{|[(x,y)]_{E_g} \cap T|}{|[(x,y)]_{E_g}|}$. For any rough partial ordering T on a non-null subset M of U, the dominating class of an element x in M is denoted by $T_{\leq[x]}$and is defined for every y in M as $T_{\leq[x]}(y) = r_T(y, x)$.

Definition 3.10. For any rough partial ordering T on a non-null subset M of U, the rough upper bound of M is the rough set denoted by $U(T, M)$ and is defined by $U(T, M) = \bigcap_{x \in M} T_{\geq[x]}$ Here, the operator "intersection" associates the minimum of the membership values in the constituents for each element in M.

Definition 3.11. For any rough partial ordering T on a non-null subset M of U, the rough greatest lower bound of M is a unique element x in $L(T, M)$ such that $L(T, M)(x) > 0$ and $r_T(y, x) > 0$ for all elements in the support of $L(T, M)$. The uniqueness of x is up to it equivalence class with respect to E

Definition 3.12. A crisp subset M of U with a rough partial ordering T is said to be a rough lattice if and only if for any subset $\{x, y\}$ in M, the least upper bound (l.u.b) and the greatest lower bound (g.l.b) exist in M. We denote the l.u.b. of $\{x, y\}$ by $x \vee y$ and the g.l.b of $\{x, y\}$ by $x \wedge y$. We say that (M, T) is a rough lattice on (U, E) and denoted it by $\mathbf{L}$.

Example-1: Let $A = (U, E)$ be an approximation space, where $U = \{a, b, c, d, e, f, g\}$ and $U/E = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g\}\}$ as shown here $B = (U \times U, S) = \{(a, g), (b, g), (g, a), (g, b), (c, g), (d, g), (e, e), (f, f), (e, f), (f, e), (a, c), (a, d), (b, d), (b, c), (d, a), (d, b), (c, a), (c, b), (a, e), (b, e), (a, f), (b, f), (e, a), (e, b), (f, a), (f, b), (c, e), (d, e), (c, f), (d, f), (e, c), (e, d), (f, c), (f, d), (c, d), (d, c), (d, d), (c, c), (b, b), (a, b), (b, a), (a, a), (g, c), (g, d), (e, g), (f, g), (g, e), (g, f), (g, g)\}$.
1. Let us consider two non empty subsets $U_1 = \{a, b, c\}$ and $U_2 = \{f, g\}$ of U. We take a subset T of $U_1 \times U_2$ as $T = \{(a, g), (b, g), (c, f), (c, g)\}$. The $E_g \downarrow (T) = \{(a, g), (b, g)\}$ and $E_g \uparrow (T) = \{(a, g), (b, g), (c, e), (d, e), (c, f), (d, f), (c, g), (d, g)\}$. $r_T(a, f) = 0$ and $r_T(c, f) = \frac{1}{4}$.
2. Let us take $T = \{(a, b), (c, d), (e, f), (g, g)\}$. Then $E_g \uparrow (T) = \{(e, e), (e, f), (f, f), (f, e), (c, d), (d, c), (d, d), (c, c), (b, b), g(a, b), (b, a), (a, a), (g, g)\}$. It is easy to see that $R_g(T)$ is a rough equivalence relation.
3. Let $T = \{(a, g), (a, c), (c, e), (g, e), (g, g)\}$. Then $E_g \uparrow (T) = \{(a, g), (b, g), (a, c), (a, d), (b, d), (b, c), (c, e), (d, e), (c, f), g(d, f), (g, e), (g, f), (g, g)\}$. So, $E_g(T)$ is antisymmetric.
4. Let $T = \{(a, g), (e, f), (c, d), (a, b), (g, g)\}$. Then $E_g \uparrow (T) = \{(a, g), (b, g), (e, e), (e, f), (f, f), (f, e), (c, d), (d, c), (d, d), (c, c), g(b, b), (a, b), (b, a), (a, a), (g, g)\}$ So, $E_g(T)$ is clearly reflexive.
$E_g(T)$ is antisymmetric as $(e, f), (f, e) \in R_g \uparrow (T)$ and $[e]_E = [f]_E; (c, d), (d, c) \in R_g \uparrow (T)$ and $[c]_E = [d]_E; (a, b), (b, a) \in R_g \uparrow (T)$ and $[a]_E = [b]_E$. It is also clearly rough transitive. So, $E_g(T)$ is a rough partially ordered relation. if the universe is partitioned into at least three non singleton equivalence classes which will give ultimately " rough boolean lattice".

4 Rough Boolean Lattice

Let R be a reflexive relation on U and $X \subseteq U$. The set $R(X) = \{y \in U : xRy, \text{ for some } x \in X\}$ is the R-neighborhood of X. If $X = \{a\}$, then we write $R(a)$ instead of $R(\{a\})$. The approximations are defined as $X_R = \{x \in U : R(x) \subseteq X\}$ and $X^R = \{x \in U : R(x) \cap X \neq \emptyset\}$. A set $X \subseteq U$ is called R-closed if $R(X) = X$, and an element $x \in U$ is R-closed, if its singleton set $\{x\}$ is R-closed. The set of R-closed points is denoted by S. Let us assume that $(U; E)$ is an indiscernibility space. The set of lower approximations $B_E(U) = \{X_E : X \subseteq U\}$ and the set of upper approximations $B^E(U) = \{X^E : X \subseteq U\}$ coincide, so we denote this set simply by $B_E(U)$. The set $B_E(U)$ is a complete Boolean sublattice of $(P(U), \subseteq)$, where $P(U)$ denotes the set of all subsets of U. This means that $B_E(U)$ forms a complete field of sets. Complete fields of sets are in one-to-one correspondence with equivalence relations, meaning that for each complete field of sets F on U, we can define an equivalence E such that $B_E(U) = F$. Note that S and all its subsets belong to $B_E(U)$, meaning that $P(S)$ is a complete sublattice of $B_E(U)$, and therefore in this sense S can be viewed to consist of completely defined objects. Each object in S can be separated from other points of U by the information provided by the indiscernibility relation E, meaning that for any $x \in S$ and $X \subseteq U, x \in X_E$ if and only if $x \in X^E$. The rough set of X is the equivalence class of all $Y \subseteq U$ such that $Y_E = X_E$ and $Y^E = X^E$. Since each rough set is uniquely determined by the approximation pair, one can represent the rough set of X as (X_E, X^E) . This is known as increasing representation [3]. This representations induce the sets $IR_E(U) = \{(X_E, X^E) : X \subseteq U\}$. The set $IR_E(U)$ can be ordered point wise $(X_E, X^E) \leq (Y_E, Y^E) \Leftrightarrow X_E \subseteq Y_E$. Therefore, $IR_E(U)$ can form completely distributive lattice. As shown in [8], $IR_E(U)$ is a complete sublattice of $P(U) \times P(U)$ ordered by the point wise set-inclusion relation, meaning that $IR_E(U)$ is an algebraic completely distributive lattice such that $\bigwedge\{(X_E, X^E) : X \subseteq H\} = (\bigcap_{X \in H} X_E, \bigcap_{X \in H} X^E)$ and $\bigvee\{(X_E, X^E) : X \subseteq H\} = (\bigcup_{X \in H} X_E, \bigcup_{X \in H} X^E)$ for all $H \subseteq IR_E(U)$.

Now we consider the rough lattice and rough boolean algebra which are parallel to fuzzy lattice and fuzzy boolean algebra [15].

Definition 4.1. A complemented distributive rough lattice (M, T) is known as rough Boolean algebra. Every complemented rough lattice need to be bounded. So every rough Boolean algebra is necessarily bounded rough lattice with bounds 0 and 1. Also every element a in M has an unique complement denoted by a^{co}

Lemma 4.2. Let $\mathbf{L}$ be a rough lattice on an approximation space (U, E) then for any two elements $a, b \in M$, and $b > a$ then $r_T(a, b) > 0 \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$

Theorem 4.3. Let $\mathbf{L}$ be a rough lattice on the approximation space (U, E). Then for all $a, b, c \in M$, $a \wedge a = a$ and $a \vee a = a, a \wedge b = b \wedge a$ and $a \vee b = b \vee a, (a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c), a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Theorem 4.4. Let $\mathbf{L}$ be a rough lattice on the approximation space (U, E). Then for all $a, b, c \in M, r_T((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) > 0$. $r_T((a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0$

Definition 4.5. A rough lattice $\mathbf{L}$ on the approximation space (U, E) is said to be complete if every subset of M has a l.u.b and a g.l.b. in (U, E)

Definition 4.6. A rough lattice $\mathbf{L}$ on the approximation space (U, E) is said to be bounded if $\exists$ two elements $0, 1 \in M$ such that $r_T(0, x) > 0$ and $r_T(x, 1) > 0$ for all $x \in M$

Definition 4.7. A rough lattice $\mathbf{L}$ on the approximation space (U, E) is said to be distributive if and only if for all $a, b, c \in M$, $P_1 : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. $P_2 : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. In this connection we can show that the statement P_1 , P_2 are equivalent

Theorem 4.8. In a rough lattice $\mathbf{L}$ on the approximation space (U, E) the cancellation laws hold, that is $a \vee b = a \vee c \Rightarrow b = c$ and $a \wedge b = a \wedge c \Rightarrow b = c$

Theorem 4.9. In a rough distributive lattice $\mathbf{L}$ on the approximation space (U, E), the De Morgan's laws hold true. That is, $(a \vee b)^{co} = a^{co} \wedge b^{co}$ and $(a \wedge b)^{co} = a^{co} \vee b^{co}$ for all $a, b \in \mathbf{L}$, where x^{co} stands for the complement of x

Definition 4.10. A rough chain is a partially ordered rough set (M, T) on the approximation space (U, E) in which for two elements $a, b \in L$, either $r_T(a, b) > 0$ *or* $r_T(b, a) > 0$

Definition 4.11. A rough lattice $\mathbf{L}$ on the approximation space (U, E) is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$, whenever $r_T(a, c) > 0$ for all $a, b, c \in \mathbf{L}$

Lemma 4.12. Every rough chain is a distributive rough lattice and every distributive rough lattice is modular.

Lemma 4.13. In a complemented distributive rough lattice $\mathbf{L}$ on the approximation space $(U, E), a, b \in L, r_T(a, b) > 0 \Leftrightarrow a \wedge b^{co} = 0 \Leftrightarrow a^{co} \vee b = 1 \Leftrightarrow r_T(b^{co}, a^{co}) > 0$

5 Conclusion

In this paper, we have presented rough lattice through a rough partial ordering relation defined on a crisp set. We have introduced some important definitions, properties and lemmas of rough lattice, rough ordering relation based on rough approximation spaces, giving interesting example. The roughness of Boolean lattice is also studied, which is an interesting topic, we will extend it further in the future.

References

[1] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, 1984.

[2] J. Järvinen, *Lattice Theory for Rough Sets*, Transactions on Rough Sets, 6, 400-498, 2007.

[3] P. Pagliani, M. K. Chakraborty, *A Geometry of Approximation*, Rough Set Theory: Logic, Algebra and Topology of Conceptual Patterns, Spinger, 5(1), 2000.

[4] Z. Pawlak, *Rough Sets*, International Journal of Computer and Information Sciences 11(5), 341-356, 1982.

[5] Z. Pawlak, A. Skowron, *Rough Membership Functions, in: Yager,R.R., Fedrizzi, M., and Kacprzyk,J.(Eds.*, Advances in the Dempster Shafer Theory of Evidence, John Wiley and Sons, 251-271, 1993.

[6] Z. Pawlak, *On Rough Relations*, Bulletin of the Polish Academy of Sciences, Tech. Sc., 34(9-10), 587-590, 1986.

[7] Z. Pawlak, *Rough Sets, Rough Relations and Rough Functions*, ICS Res. reports, Warsaw Univ. of Tech., 24/94, 1994.

[8] D. Rana, S. K. Roy, *Set Approach on Lattice*, Journal of Uncertain Systems, 5(1), 72-80, 2011.

[9] D. Rana, S. K. Roy, *Lattice for Covering Rough Approximations*, Malaya Journal of Matematik, 2(3), 222-227 2014.

[10] D. Rana, S. K. Roy, *Lattice for Rough Intervals*, Journal of New Results in Science , 2, 39-46 2013.

[11] D. Rana, S. K. Roy, *Lattice for Nested Rough Approximations*, Journal of Discrete Mathematical Sciences and Cryptography, Taylor and Francis, Forthcomming, 2015.

[12] D. Rana, S. K. Roy, *Concept Lattice - A Rough Set Approach*, Malaya Journal of Matematik, 3(1), 14-22, 2015.

[13] J. Stepaniuk, *Properties and Applications of Rough Relations*, ICS Research reports, Warsaw University of Technology, 26/96, 1996.

[14] J. Stepaniuk, *Properties of Rough Relations*, ICS Research reports, Warsaw University of Technology, 48/95, 1996.

[15] B. K. Tripathy, P. K. Choudhury, *Fuzzy Lattices and Fuzzy Boolean Algebra*, The Journal of Fuzzy Mathematics, 16(1), 185-198, 2008.