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ON ABELIAN FUZZY MULTI GROUPS AND ORDERS OF FUZZY MULTI GROUPS

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Abstract – As a continuation of the study of various algebraic structures of Fuzzy Multisets, in this paper the concept of Abelian fuzzy multigroups, left and right cosets of fuzzy multi groups and fuzzy multi order of an element of a group are introduced and its various properties are discussed. In the last section some of the homomorphic properties between two Fuzzy multigroups are discussed.

Keywords – *fuzzy multi group, abelian fuzzy multi group, left and right cosets of fuzzy multi groups, fuzzy multi orders, homomorphism.*

1 Introduction

Modern set theory formulated by George Cantor is fundamental for the whole Mathematics. But to represent imprecise, vague data classical set theory is insufficient. So many non classical sets were put forward to overcome this problem. Some of them are fuzzy sets, soft sets, rough sets, multisets etc. To make these non classical sets even more powerful combinations of them were also introduced in time. One of them is Fuzzy Multisets.Fuzzy Multisetsis a powerful tool for modelling quantitative and qualitative properties of objects simultaneously.

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and allow repeated occurrences of any element, then we can get a mathematical structure that is known as Multisets or Bags. For example, the prime

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factorization of an integer n>0 is a Multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the Multiset {2, 2, 2, 3, 5}. A complete account of the development of multiset theory can be seen in [1,2, 9, 10,11,12,13]. As a generalization of multiset, Yager [6] introduced the concept of Fuzzy Multiset (FMS). An element of a Fuzzy Multiset can occur more than once with possibly the sameor different membership values.

2 Preliminaries

Definition 2.1.[11] Let X be a set. A multiset (mset) M drawn from X is represented by a function Count M or C_M defined as $C_M : X \to \{0,1, 2, 3,\ldots\}$.For each $x \in X$, $C_M(x)$ is the characteristic value of x in M. Here $C_M(x)$ denotes the number of occurrences of x in M.

Definition 2.2.[10] Let X be a group. A multi set G over X is a multi group over X if the count of G satisfies the following two conditions

1. $C_G(xy) \geq C_G(x) \wedge C_G(y) \quad \forall x,y \in X$;
2. $C_G(x^{-1}) \geq C_G(x) \ \forall x \in X$

Definition 2.3.[12] If $X$is a collection of objects, then a fuzzy set A in X is a set of ordered pairs: $A = \{(x,\mu_A(x)) : x \in X, \mu_A : X \to [0,1]\}$ where $\mu_A$is called the membership function of A, and is defined from X into [0, 1].

Definition 2.4.[2] Let G be a group and $\mu \in FP(G)$ (fuzzy power set of G), then μ is called fuzzy subgroup of G if

1. $\mu(xy) \geq \mu(x) \wedge \mu(y) \ \forall x, y \in G$ and
2. $\mu(x^{-1}) \geq \mu(x) \forall x \in G$

Definition 2.5.[9] Let X be a nonempty set. A Fuzzy Multiset (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1].

Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by$\{ \mu_A^1(x), \mu_A^2(x), \mu_A^3(x), \ldots, \mu_A^p(x) \}$; $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x) \geq \ldots \geq \mu_A^p(x)$.

When every $x \in X$ is mapped to a finite multiset of Q under the count membership function CM_A, then A is called a finite fuzzy multiset of X.The collection of all finite multisets of X is denoted by $FM(X)$. Throughout this paper fuzzy multisets are taken from $FM(X)$.

Definition 2.6.[7] Let $A \in FM(X)$ and $x \in A$. Then $L(x;\ A) = Max\{ j : \mu_A^j(x) \neq 0 \}$
When we define an operation between two fuzzy multisets, the length of their membership sequences should be set to equal. So if $A$and B are FMS at consideration, take $L(x;\ A,B) = Max\ \{L(x;A), L(x;B)\}$. When no ambiguity arises we denote the length of membership by $L(x)$.

Basic relations and operations, assuming that A and B are two fuzzy multisets of X is taken from [7] and is given below.

a) Inclusion
$A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x),\ j = 1,2,\ldots\ldots, L(x)\ \forall x \in X$.

b) Equality
$A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x),\ j = 1,2,\ldots\ldots, L(x)\ \forall x \in X$.

c) Union
$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1,2,\ldots\ldots, L(x)$ where $\vee$ is the maximum operation.

d) Intersection
$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1,2,\ldots\ldots, L(x)$where $\wedge$ is the minimum operation.

By $CM_A(x) \geq CM_A(y)$ it is taken that $\mu_A^i(x) \geq \mu_A^i(y)$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$. And $CM_A(x) \wedge CM_A(y)$ means that $\{\mu_A^i(x) \wedge \mu_A^i(y)\}$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$. And by $CM_A(x) \vee CM_A(y)$ we mean $\{\mu_A^i(x) \vee \mu_A^i(y)\}$ $\forall i = 1, \ldots, \text{Max}\{L(x), L(y)\}$.

Definition 2.7.[8] Let $A \in FM(X)$. Then A^{-1} is defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$.

Definition 2.8.[8] Let $A, B \in FM(X)$. Then define $A \ o\ B$ as

$$CM_{AoB}(x) = \vee\{CM_A(y) \wedge CM_B(z)\ ;\ y,z \in X \text{ and } yz = x\}.$$ Also
$$CM_{AoB}(x) = \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\}\ \forall x \in X$$
$$= \vee_{y\in X}\{CM_A(xy^{-1}) \wedge CM_B(y)\}\ \forall x \in X.$$

Definition 2.9.[8] Let Xbe a group. A fuzzy multiset G over X is a fuzzy multi group (FMG) over X if the count (count membership) of G satisfies the following two conditions.

1. $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)$ $\forall x, y \in X$.
2. $CM_G(x^{-1}) \geq CM_G(x) \forall x \in X$.

Definition 2.10.[8] Let $A \in FM(X)$. Then
$A[\alpha, n] = \{x \in X: \mu_A^j(x) \geq \alpha\ ; L(x) \geq j \geq n \text{ and} j, n \in \mathbb{N}\}$. This is called n-α level set of A.

Definition 2.11.[8] Let $A \in FM(X)$. Then define $A^* = \{x \in X: CM_A(x) = CM_A(e)\}$.

Proposition 2.12.[8] Let $A \in FMG(X)$. Then

a) $CM_A(e) \geq CM_A(x)$ $\forall x \in X$.
b) $CM_A(x^n) \geq CM_A(x)$ $\forall x \in X$.
c) $A^{-1} \supseteq A$.

Proposition 2.13.[8] Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$ $\forall x, y \in X$.

Proposition 2.14.[8] If $A \in FMG(X)$, and H is a subgroup of X, then $A|_H$ (i.e. A restricted to H)$\in FMG(H)$ and is a fuzzy multi subgroup of A .

Proposition 2.15.[8] *Let* $A \in FMG(X)$. *Then* $A[\alpha, n]$ *are subgroups of* X.

Proposition 2.16.[8] *Let* $A \in FMG(X)$. *Then* A^* *is a subgroup of* X.

Some of the basic properties of groups are given below.

Definition 2.17.[14]Let (G,*),(G',o) be two groups. A mapping $\emptyset : G \to G'$ is called a homomorphism if $\emptyset\,(a * b) = \emptyset(a) o \emptyset(b), a, b \in G$.

Definition 2.18.[14] Let $\emptyset : G \to G'$ be a homomorphism. Then the kennel of $\emptyset$ is the set of all those elements of G which are mapped to the identity element of G'. That is
$\mathrm{Ker}\ \emptyset = \mathrm{K}_{\emptyset} = \{\mathrm{x} \in \mathrm{G} : \emptyset\,(\mathrm{x}) = \mathrm{e}'\}$ where $\mathrm{e}'$is the identity element of G'.

Proposition 2.19.[14] Let $\emptyset : G \to G'$ be a homomorphism. Then

$$\emptyset(e) = e', \emptyset(x^{-1}) = [\emptyset(x)]^{-1}$$

Proposition 2.20. [14] Let $\emptyset: G \to G$' with kernel K. Then K is a normal subgroup of G.

Definition 2.21.[14] A one-one homomorphism from G onto G' is called an isomorphism.

Definition 2.22.[14] Two groups G, and G* are said tobe isomorphic if there is an isomorphism of G onto G*.

Note :- If G and G* are isomorphic then both groups will have the same properties.

Definition 2.23.[14] An isomorphism of a group G to itself is called an Automorphism.

3. Abelian Fuzzy Multi Group

Proposition 3.1. Let $A \in FMG(x)$. Then the following assertions are equivalent.

a) $CM_A(xy) = CM_A(yx), \quad x, y \in X$
b) $CM_A(xyx^{-1}) = CM_A(y), \ x, y \in X$
c) $CM_A(xyx^{-1}) \geq CM_A(y), \ x, y \in X$
d) $CM_A(xyx^{-1}) \leq CM_A(y), \ x, y \in X$

Proof. (a) $\Longrightarrow$ (b) Let $x, y \in X$. Then $CM_A(xyx^{-1}) = CM_A(x^{-1}xy) = CM_A(y)$

(b) $\Longrightarrow$ (c)Straight forward

(c) $\Longrightarrow$ (d)$CM_A(xyx^{-1}) \leq CM_A[x^{-1}(xyx^{-1})(x^{-1})^{-1}] = CM_A(y)$

(d) $\Longrightarrow$ (a)Let $x, y \in X$
Then $CM_A(xy) = CM_A[x(yx)x^{-1}] \leq CM_A(yx) = CM_A[y(xy)y^{-1}] \leq CM_A(xy)$

Hence $CM_A(xy) = CM_A(xy)$.Thus the above assertions are equivalent.

Definition 3.2. G ∈*FMG*(X)is called an *Abelian fuzzy multi groupover X*, if $CM_G(xy) = CM_G(yx) \forall x, y \in X$. Let *AFMG*(X) denote the set of all abelian fuzzy multi groups over X.

Example 3.3. Let X be an abelian group and G be a FMG of X. Then G is an abelian FMG over X.

Proposition 3.4. Let A∈*AFMG*(X). Then A^*, $A[\alpha,n]$; n∈N, α∈ [0,1] are normal subgroups of X.

Proof. By Propositions 2.15 and 2.16 A^* and $A[\alpha,n]$ are subgroups of X.

1. Let x∈X and y∈A^*. So $CM_A(y) = CM_A(e)$. Since A ∈*AFMG*(X)
$CM_A(xy) = CM_A(yx) \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y) = CM_A(e)$ by (3.1.)
So $x^{-1} \in A^*$. Hence the proof by the definition of normal subgroup.

2. Let x∈X and y∈$A[\alpha, n]$. Since A∈*AFMG*(X), $CM_A(xy) = CM_A(yx) \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y)$ by(3.1.) So $xyx^{-1} \in A[\alpha, n]$.Hence the proof by the definition of normal subgroup.

Proposition 3.5. Let A∈*AFMG*(X). Then A^j; j∈N, isnormal subgroup of X iff $\mu_A^{j+1}(xy^{-1}) = 0\ \forall x, y \in A^j$.

Proof. In [8] it is proved thatA^j is a subgroup of Xiff $\mu_A^{j+1}(xy^{-1}) = 0\ \forall x, y \in A^j$. Let x∈X and y∈A^j. So $\mu_A^j(y) > 0$ and$\mu_A^{j+1}(y) = 0$. Since A∈*AFMG*(X) , $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$. So $CM_A(xyx^{-1}) = CM_A(y)$ by (3.1.). Then $\mu_A^j(xyx^{-1}) > 0$ *and* $\mu_A^{j+1}(xyx^{-1}) = 0$. So $xyx^{-1} \in A^j$. Hence the proof by the definition of normal subgroup.

Corollary 3.6. Let A∈*AFMG*(X). Then A^j ; j∈N, isnormal subgroup of X iff $A^j$is a subgroup of X.

Definition 3.7. Let X be a group and$A$∈*FM*(X).Then$[CM_A(e)]_x$ ∈*FM*(X) with only one element x and $CM_{[CM_A(e)]_x}(x) = CM_A(e)$.

Definition 3.8. Let Xbe a group H∈*FMG*(X) and x∈X. Also let e be the identity element of X. Then

a) the FMS $[CM_H(e)]_x\ o\ H$ is called a *left fuzzy multi coset* (LFMC) of H in X and is denoted by xH, where

$$\begin{aligned} CM_{xH}(z) &= \vee \{ CM_{[CM_H(e)]_x}(u) \wedge CM_H(v)\ ; uv = z \} \\ &= CM_{[CM_H(e)]_x}(x) \wedge CM_H(x^{-1}z) \\ &= CM_H(e) \wedge CM_H(x^{-1}z) \quad \text{by (3.9.)} \\ &= CM_H(x^{-1}z) \quad \text{by (2.10)} \end{aligned}$$

b) the FMS $H\ o[CM_H(e)]_x$is called a *right fuzzy multi coset* (RFMC) of H in X and is denoted by Hx, where

$$\begin{aligned} CM_{Hx}(z) &= \vee\{\, CM_H(u) \wedge CM_{[CM_H(e)]_x}(v)\ ; uv = z\,\} \\ &= CM_H(zx^{-1}) \wedge CM_{[CM_H(e)]_x}(x) \\ &= CM_H(zx^{-1}) \wedge CM_H(e) \\ &= CM_H(zx^{-1}) \end{aligned}$$

Remark 3.9. If $H \in AFMG(X)$, then $xH = Hx$, $\forall x \in X$.

Proof. Let $H \in AFMG(X)$

$$\begin{aligned} CM_{Hx}(z) &= CM_H(zx^{-1}) \\ &= CM_H(x^{-1}z) \\ &= CM_{xH}(z) \end{aligned}$$

Proposition 3.10. Let $H \in FMG(X)$. Then $\forall x, y \in X$,

a) $xH = yH \Leftrightarrow xH^* = yH^*$
b) $Hx = Hy \Leftrightarrow H^*x = H^*y$

Proof. a) Let $xH = yH$. Then $CM_{xH}(z) = CM_{yH}(z)$ and hence
$CM_H(x^{-1}z) = CM_H(y^{-1}z)$ $\forall z \in X$.
Now since z is arbitrary, put $z = y$, we get $CM_H(x^{-1}y) = CM_H(y^{-1}y) = CM_H(e)$
Thus $x^{-1}y \in H^*$ and hence $xH^* = yH^*$
Conversely, let $xH^* = yH^*$. Thus $x^{-1}y$, $y^{-1}x \in H^*$. …….(1)
Now $CM_H(x^{-1}z) = CM_H([x^{-1}y][y^{-1}z])$ by associativity of group
$\geq CM_H(x^{-1}y) \wedge CM_H(y^{-1}z)$ by (2.9)
$= CM_H(e) \wedge CM_H(y^{-1}z)$ by (1)
$= CM_H(y^{-1}z)$ $\forall z \in X$
Similarly $CM_H(y^{-1}z) \geq CM_H(x^{-1}z)$ $\forall z \in X$.
So $CM_{xH}(z) = CM_{yH}(z)$ $\forall z \in X$. Hence the proof.

b) Proof is similar to part (a).

Proposition 3.11. Let $H \in AFMG(X)$. If $xH = yH$, then $CM_H(x) = CM_H(y)$ $\forall x, y \in X$.

Proof. Let $xH = yH$. Then $CM_{xH}(z) = CM_{yH}(z)$ and hence $CM_H(x^{-1}z) = CM_H(y^{-1}z)$ $\forall z \in X$.
Now since z is arbitrary, put $z = y$, we get $CM_H(x^{-1}y) = CM_H(y^{-1}y) = CM_H(e)$.
Thus $x^{-1}y \in H^*$. Similarly $y^{-1}x \in H^*$. ………….(1)
Since $H \in AFMG(X)$, it follows that $CM_H(x) = CM_H(y^{-1}xy)$ by (3.1.)
$\geq CM_H(y^{-1}x) \wedge CM_H(y)$ by (2.9.)
$= CM_H(e) \wedge CM_H(y)$ by (1) and (2.10)
$= CM_H(y)$
Similarly $CM_H(y) \geq CM_H(x)$ and hence the proof.

Definition 3.12. Let $A \in FMG(X)$. Then
$[A] = \{x : x \in X \text{ and } CM_A(xy) = CM_A(yx)\ \forall\ y \in X\}$ is called the *normalizer of A in X*.

Proposition 3.13. Let $A \in FMG(X)$. Then $[A]$ is a subgroup of X and $A|_{[A]} \in AFMG([A])$.

Proof. Clearly $e \in [A]$. Let $x, y \in [A]$. Then $\forall\, z \in X$,

$$
\begin{aligned}
CM_A([xy^{-1}]z) &= CM_A(x[y^{-1}z]) \\
&= CM_A([y^{-1}z]x) \quad \text{by } x \in [A] \text{ and } y^{-1}z \in X. \\
&\geq CM_A([x^{-1}z^{-1}]y) \quad \text{by (2.9)} \\
&= CM_A(y[x^{-1}z^{-1}]) \quad \text{by } y \in [A] \text{ and } x^{-1}z^{-1} \in X. \\
&\geq CM_A(z\,[xy^{-1}]) \text{by (2.9)} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1)
\end{aligned}
$$

$$
\begin{aligned}
CM_A(z\,[xy^{-1}]) &\geq CM_A(y[x^{-1}z^{-1}]) \\
&= CM_A([x^{-1}z^{-1}]y) \\
&\geq CM_A([y^{-1}z]x) \\
&\geq CM_A([xy^{-1}]z) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)
\end{aligned}
$$

From (1) & (2) $xy^{-1} \in [A]$.So $[A]$ is a subgroup of X. By (2.14) it is proved$A|_{[A]} \in FMG([A])$. And by the definition of AFMG the proof is complete.

Proposition 3.14. Let A, $B \in FMG(X)$ and $A \subseteq B$. Then the following assertions are equivalent.

a) $CM_A(xyx^{-1}) \geq CM_A(y) \wedge CM_B(x) \quad \forall \; x, y \in X$
b) $CM_A(yx) \geq CM_A(xy) \wedge CM_B(y) \quad \forall \; x, y \in X$
c) $[CM_A(e)]_x \, o \, A \supseteq (Ao[CM_A(e)]_x) \cap B$

Proof. (a) ⇒ (b)

Since $A \subseteq B CM_A(yx) = CM_A(yxyy^{-1}) \geq CM_A(xy) \wedge CM_B(y)$ by(a)

1. (b) ⇒ (c)

$$
\begin{aligned}
\forall\; z \in X, CM_{([CM_A(e)]_x \, o \, A)}(z) &= \vee_{x \in X}[CM_{[CM_A(e)]_x}(x) \wedge CM_A(x^{-1}z)] \\
&\geq CM_A(e) \wedge CM_A(x^{-1}z) \\
&= CM_A(x^{-1}z) \\
&\geq CM_A(z^{-1}x) \text{ by(2.9.)} \\
&\geq CM_A(xz^{-1}) \wedge CM_B(z^{-1}) \text{ by (b)} \\
&\geq CM_A(zx^{-1}) \wedge CM_B(z^{-1}) \text{ by (2.9.)} \\
&= CM_{Ax}(z) \wedge CM_B(z) \text{ by(3.8 and 2.9)} \\
&= \big(CM_{Ax}(z) \cap CM_B(z)\big)(z)
\end{aligned}
$$

2. (b) ⇒ (a) $\forall \; x, y \in X$

$$
\begin{aligned}
CM_A(x[yx^{-1}]) &\geq CM_A([yx^{-1}]x) \wedge CM_B(x) \\
&= CM_A(y) \wedge CM_B(x)
\end{aligned}
$$

3. (c) ⇒(b) $\forall \; x, y \in X$

$$
\begin{aligned}
CM_A(yx) &= CM_A(x^{-1}y^{-1}) \text{ by(2.9)} \\
&= CM_{xA}(y^{-1}) \\
&\geq CM_{(Ax) \cap B}(y^{-1}) \text{ by(c)} \\
&= CM_A(y^{-1}x^{-1}) \wedge CM_B(y^{-1}) \\
&\geq CM_A(xy) \wedge CM_B(y) \quad \text{Hence the proof by (2.9.) .}
\end{aligned}
$$

4 Fuzzy Multi Order

4.1 Fuzzy Multi Order of an Element of a Group

Throughout the rest of the paper we consider X as a group with finite order. And $A \in FMG(X)$. Also $x, y \in X$.

Definition 4.1.1. Let A be a FMG of X and $x \in X$. The least positive integer n such that $CM_A(x^n) = CM_A(e)$ is known as fuzzy multi order of x w.r.t. A and is denoted by ($O(x)$;A). If no such n exists, x is said to be of infinite order w.r.t. A.

Example 4.1.2. (Z_4, $+_4$) is a group. Let A = {(.6, .4, .3, .1)/2, (.9, .8, .7, .5, .1, .1)/0} is a fuzzy multi group. $CM_A(2^2) = CM_A(e)$. So (O(2);A) = 2.

Also $O(x) = O(y)$ does not imply ($O(x)$;A) = ($O(y)$;A). It is illustrated below. Consider the Klein four cycle X = {e, a, b, c}. Then A = {(.6, .4, .3, .1)/a, (.9, .8, .7, .5, .1, .1)/b, (.9, .8, .7, .5, .1, .1)/c, (.9, .8, .7, .5, .1, .1)/e}. Here O(a) = O(b) = O(c). But (O(a);A) ≠ (O(b); A) = (O(c);A).

Proposition 4.1.3. Let $A \in FMG(X)$. If $CM_A(x^m) = CM_A(e)$, for some positive integer m, then ($O(x)$;A)|m.

Proof. Let ($O(x)$; A) = n. Given $CM_A(x^m) = CM_A(e)$. Hence n≤m.
By division algorithm ∃integers s , t such that $m = ns+t$; $0 \le t < n$. The

$$\begin{aligned} CM_A(x^t) &= CM_A(x^{m-ns}) \\ &= CM_A(x^m(x^n)^{-s}) \\ &\ge CM_A(x^m) \wedge CM_A(x^n)^{-s} \text{ by (2.9)} \\ &= CM_A(e) \wedge CM_A(x^{ns})^{-1} \\ &= CM_A(x^{ns})^{-1} \text{ by 2.12(a)} \\ &\ge CM_A(x^{ns}) \text{ by 2.9} \\ &= CM_A(x^n)^s) \\ &\ge CM_A(x^n) \text{ by 2.12(b)} \\ &= CM_A(e) \end{aligned}$$

So $CM_A(x^t) = CM_A(e)$. Hence t=0 by the minimality of n. i.e. m= ns. Hence the proof.

Proposition 4.1.4. Let $A \in FMG(X)$. Then $\forall\, x \in X$, ($O(x)$;A)|$O(x)$.

Proof. $O(x) = m$, ($O(x)$;A) =n
$CM_A(x^m) = CM_A(e)$
So ($O(x)$;A) = $n \le O(x) = m$ (Since n is the least)
Let $n \nmid m$ and let $m = np + q$; $0 < q < n$. Then

$$\begin{aligned} x^m &= x^{np+q} \\ e &= x^{np}x^q \\ x^{-q} &= x^{np} \\ CM_A(x^{-q}) &= CM_A(x^{np}) \end{aligned}$$

Thus $CM_A(x^{np}) \geq CM_A(x^q)$ by 2.9
$CM_A(x^q) = CM_A((x^n)^p) = CM_A(e)$ by (2.9.)
i.e. $\exists 0 < q < n\,; CM_A(x^q) = CM_A(e)$. This is a contradiction to (*O*(*x*); *A*) =*n*.
Hence the proof.

Proposition 4.1.5. Let *A*∈ $FMG(X)$. Let *x*, *y*∈ X such that ((*O*(*x*) ;*A*), (*O*(*y*); *A*)) = 1 and *xy* = *yx*. Then if $CM_A(xy) = CM_A$(e), then $CM_A(x) = CM_A(y) = CM_A(e)$.

Proof. Let (*O*(*x*); *A*) = *n*, (*O*(*y*); *A*) = *m*. …….. (1)

$$\begin{aligned} CM_A(e) &= CM_A(xy) \quad \text{(given)} \\ &\leq CM_A((xy)^m) \text{by 2.12(b)} \\ &= CM_A(x^m y^m) \quad \ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

Hence $CM_A(e) = CM_A(x^m y^m)$ by (2.10.) Now

$$\begin{aligned} CM_A(x^m) &= CM_A(x^m y^m y^{-m}) \\ &\geq CM_A(x^m y^m) \wedge CM_A((y^m)^{-1}) \quad \text{by (2.9.)} \\ &\geq CM_A(e) \wedge CM_A(e) \quad \text{by (1) and (2) and 2.9} \\ &= CM_A(e). \end{aligned}$$

Thus $CM_A(x^m) = CM_A(e)$. Then n|m. (by 4.1.4). But (*n*, *m*) =1 (given). Thus *n* = 1.
i.e. $CM_A(x) = CM_A(x^n) = CM_A(e)$. Similarly $CM_A(y) = CM_A(e)$.

Corollary 4.1.6. Let *A*∈ $FMG(X)$. Let *x*, *y*∈ X such that (*O*(*x*), *O*(*y*)) = 1and *xy* = *yx*. Then if $CM_A(xy) = CM_A$(e), then $CM_A(x) = CM_A(y) = CM_A(e)$.

Proof. (*O*(*x*), *O*(*y*)) = 1
(*O*(*x*) ;*A*)\ *O*(*x*). by (4.1.5.). Then
((*O*(*x*) ;*A*), (*O*(*y*); *A*)) = 1 Then the proof by (4.1.5)

Theorem 4.1.7. Let A ∈ $FMG(X)$. Let (*O*(*x*); *A*) = *n* ; *x*∈ X. If *m* is an integer with (*m*,*n*) = *d*, then (*O*(x^m); *A*) = *n*/*d*.

Proof. Let (*O*(x^m); *A*) = *t*. Now

$$\begin{aligned} CM_A\left((x^m)^{n/d}\right) &= CM_A(x^{nk})\,; \quad m/d = k \in \mathbb{Z}^+ \\ &\geq CM_A(x^n) \text{ by (2.12)} \\ &= CM_A(e) \end{aligned}$$

i.e. $CM_A\left((x^m)^{n/d}\right) = CM_A(e)$. Thus *t*|(*n*/*d*) by(4.1.3) …………(1)
Now, since (*m*, *n*) = *d*, $\exists i, j \in \mathbb{Z}$such that $ni + mj = d$.So

$$\begin{aligned} CM_A(x^{td}) &= CM_A\left(x^{t(ni+mj)}\right) \\ &\geq CM_A\left((x^n)^{ti}\right) \wedge CM_A(((x^m)^t)^j) \\ &\geq CM_A(x^n) \wedge CM_A((x^m)^t) \quad \text{by (2. 12)} \\ &= CM_A(e) \end{aligned}$$

$$CM_A(x^{td}) = CM_A(e)$$

So $n|(td)$ by(4.1.3) i.e. $(n/d)|(td/d)$.$\Longrightarrow (n/d)|t$. ……..(2)
$t = (n/d)$ by (1) and (2). Hence the proof

Proposition 4.1.8. Let $A\in FMG(X)$. Let $(O(x); A) = n$; $x\in X$. If m is an integer with $(m, n) = 1$, then $CM_A(x^m) = CM_A(x)$.

*Proof. S*ince $(m, n) = 1$, $\exists i, j \in \mathbb{Z}$ such that $ni + mj = 1$. We then have

$$\begin{aligned} CM_A(x) &= CM_A\big(x^{ni+mj}\big) \\ &\geq CM_A\big((x^n)^i\big) \wedge CM_A((x^m)^j) \quad \text{by (2.9)} \\ &\geq CM_A(x^n) \wedge CM_A(x^m) \\ &\geq CM_A(e) \wedge CM_A(x^m) \end{aligned}$$

$CM_A(x) \geq CM_A(x^m)$ So
$CM_A(x^m) = CM_A(x)$. by(2.12(b))

Theorem 4.1.9. Let $A \in FMG(X)$. Let $(O(x); A) = n$; $x \in X$. If $i \equiv j(mod n)$;$i, j \in \mathbb{Z}$. Then $(O(x^i); A) = (O(x^j); A)$.

Proof. $(O(x^i); A) = t$, $(O(x^j); A) = s$. Also $i = j + nk$; $k\in \mathbb{Z}$. So

$$\begin{aligned} CM_A\big((x^i)^s\big) &= CM_A\big((x^{j+nk})^s\big) \\ &\geq CM_A\big((x^j)^s\big) \wedge CM_A((x^n)^{ks}) \\ &\geq CM_A(e) \wedge CM_A(x^n) \\ &= CM_A(e). \end{aligned}$$

Then $CM_A\big((x^i)^s\big) = CM_A(e)$. by (2.12)
So $t|s$. Similarly by $CM_A\big((x^j)^t\big) = CM_A(e)$ we get $s|t$. Thus $t = s$.

Proposition 4.1.10. Let $A\in FMG(X)$. Let $x, y\in X$ such that $((O(x);A), (O(y); A)) = 1$ and $xy = yx$. Then $(O(xy);A) = [(O(x); A)][(O(y);A)]$.

Proof. Let $(O(xy);A) = n$, $(O(x); A) = s$, $(O(y);A) = t$. Then $(t,s) = 1$ (given)

$$\begin{aligned} CM_A((xy)^{st}) &\geq CM_A(x^{st}) \wedge CM_A(y^{st}) \\ &\geq CM_A(x^s) \wedge CM_A(y^t) \\ &= CM_A(e) \wedge CM_A(e) \\ &= CM_A(e). \end{aligned}$$

So $n|st$ by (4.1.3) ………………(1)
Now $CM_A(e) = CM_A((xy)^n) = CM_A(x^n y^n)$.
Since $n|st$ and $(t,s) = 1$, $n|s$ or $n|t$. Assume $n|t$, then $(n,s) = 1$.
So $(O(x^n); A) = s/(n,s) = s$. by (4.1.7)
Also by the same $(O(y^n); A) = t/(n,t)$. ……………..(a)
Since $(s,t) = 1$, we have $(s, (t/(n,t))) = 1$.
Thus $\big((O(x^n); A), (O(y^n); A)\big) = 1$. by (a)
Also $CM_A((xy)^n) = CM_A(x^n y^n) = CM_A(e). Since$ $(O(xy);A) = $ n.

Also $CM_A(x^n) = CM_A(y^n) = CM_A(e)$
So $s|n$ and $t|n$ by(4.1.3).
Now since $(s, t) = 1$, $(st)|n$. ……………..(2) Then from (1) and (2) $n = st$.

Proposition4.1.11. Let $A \in FMG(X)$. Let $z \in X$. $(O(z); A) = mn$ with $(m, n) = 1$, then $\exists x, y \in X$ such that $xy = yx$, $(O(x); A) = m$, and $(O(y); A) = n$.

Proof. $(m,n) = 1 \Longrightarrow \exists s, t \in \mathbb{Z}$ such that $ms + nt = 1$. ……………..(1)
So $(m,t) = (n,s) = 1$. Let $x = z^{nt}$, $y = z^{ms}$. Then $xy = z^{nt} z^{ms} = z^{ms} z^{nt} = yx = z^{nt+ms} = z$ by (1)
Given $(O(z); A) = mn$. So by (4.1.7)
$(O(x); A) = (O(z^{nt}); A) = mn/(mn, nt) = m/(m, t) = m$. (since $(m, t) = 1$)
Similarly $(O(y); A) = n$. This proves the existence of x and y.

4.2 Fuzzy Multi Order in Cyclic Groups

In this section we consider X as a cyclic group with finite order. And $A \in FMG(X)$.

Lemma4.2.1. Let $A \in FMG(X)$. And let a, b be two generators of X. Then

$$(O(a); A) = (O(b); A).$$

Proof. Let $|X| = n$. $O(a) = O(b) = n$. Now $b = a^p$; $p \in \mathbb{N}$. So $(p,n) = 1$. Let $(O(a);\ A) = m$. Then $m|n$ by (4.1.4). Then
$(O(b);\ A) = (O(a^p);\ A) = m/(p, m) = m = (O(a);\ A)$. by (4.1.7)
and since $(p,n) = 1$. So $(O(a);A) = (O(b);A)$.

Theorem 4.2.2. Let $A \in FMG(X)$, with $|X| = n$. Then the following assertions hold $\forall\ x, y \in X$.

a) If $O(x)|\ O(y)$, then $(O(x);A)|\ (O(y);A)$.
b) If $O(x) = O(y)$, then $(O(x);A) = (O(y);A)$.
c) If $O(x) > O(y)$, then $(O(x);A) \geq (O(y);A)$.

Proof. Let $X = (a)$, $x = a^s$, $y = a^t$ and $(O(a);A) = m$. Hence $O(a) = n$. Now by (4.2.1.) m is independent of a particular choice of a generator a of X. Thus $O(x) = n/(s,n)$, By the property of a cyclic group $O(y) = n/(t,n)$ ………….(1)

$(O(x);A) = (O(a^s);A) = m/(s,m)$ by(4.1.7) Similarly $(O(y);A) = (O(a^t);A) = m/(t,m)$
By (4.1.4) $m|n$. …………..(2)

a) If $O(x)|\ O(y)$, then by (1) $\{n/(s,n)\} \mid \{n/(t,n)\} = (t,n)|(s,n)$.
Now by (2) $m|n \Rightarrow n = km$; $k \in Z$. So $(t, mk)|(s, mk)$. i.e. $(t,m)|(s,m)$.
Hence $m/(s,m)|\ m/(t,m)$. Hence the proof.

b) Result follows from (a)

c) $O(x) > O(y)$, then $n/(s,n) > n/(t,n)$ So $(s,n) < (t,n)$. So $(s,m) \leq (t,m)$ by $m|n$.

5 Homomorphism between Fuzzy Multigroups

Proposition 5.1 Let x, y be two groups and f: x→y be a homomorphism. If A∈ FMG$(x) then\ f(A)$ ∈ FMG(y)

Proof. Let U,V,∈Y

Case I:
Let u,$v \in f(x)$. Then
$CM_{f(A)}(u) \wedge CM_{f(A)}(v) = 0 \wedge 0 \leq CM_{f(A)}(uv)$
$CM_{f(A)}(u^{-1}) \geq 0 = CM_{f(A)}(u)$

Case II:
$u \notin f(x), (\&v \in f(x).$ similarly vice versa).
$CM_{f(A)}(u) \wedge CM_{f(A)}(v) = 0 \wedge CM_{f(A)}(v) = 0 \leq CM_{f(A)}(uv)$

Case III:
Let $u, v \in f(x)$ Then there exist $x, y\ \in$ X such that $f(x) = u, f(y) = v$
Now $CM_{f(A)}(u) = \vee \left\{CM_{(A)}(w): w \in X; f(w) = uv\right\}$ ⟶(1)
$\geq \{CM_{(A)}(xy): x, y\ \in X, f(x) = u, f(y) = v\}$
(Since $xy \in$ (1) by the definition of homomorphism. i.e,
$f(xy) = f(x)f(y) = uv) \geq \left\{CM_{(A)^{(U)}} \wedge CM_{(A)^{(y)}}: x, y\ \in X, f(x) = u, f(y) = v\right\}$
$since\ A \in FMG(X) =$ $[\vee \{CM_A(x): x \in X, f(x) = u\}] \wedge [\vee \{CM_A(y): y \in X: f(y) = v\}]$
$= CM_{f(A)}(u) \wedge CM_{f(A)}(v)$ ⟶(2)
Also
$CM_{f(A)}(u^{-1}) = \vee \{CM_A(z^{-1}) :\ z^{-1} \in X, f(z^{-1}) = u\}$
$\geq \vee \{CM_A(z): z \in X, f(z^{-1}) = u^{-1}\}\ (A \in FMG(X))$
$= \{CM_A(z): z \in X; f(z) = u\}$

(f(z^{-1}) = u^{-1}⟹(f(z^{-1}))$^{-1}$ =(u)-1 ⟹ (f(z^{-1}))$^{-1}$ =u⟹ f(z)=u, property of h-ism)
= $CM_{f(A)}(u) \to (3)$

From (2) and (3) $f(A)\ \in\ FMG(y)$

Proposition 5.2 Let x,y, be two groups and f: x →y be a homomorphism. If

$$B \in FMG(y) than\ f^{-1} \in FMG(x).$$

Proof. Let x,y ∈ X

Case I, Case II, is similar to proposition 5.1.

Case III
Let x,y ∈ $f^{-1}\ (B)$. Then there exist u,v,∈ $y\ such\ that\ f^{-1}\ (u)$ = x and $f^{-1}\ (v) = y$.
Now $CM_{f^{-1}(B)}(xy)$ $= CM_B f(xy)$ (By definition of inverse)
$= CM_B(f)f(y)$ (Definition of homomorphism)
$\geq\ CM_B f(x) \wedge\ CM_B(f(y)$ (Since $B \in FMG(y)$

$= CM_{f^{-1}(B)}(x) \wedge CM_{f^{-1}(B)}(y)$

Now $CM_{f^{-1}(B)}(x^{-1}) = CM_B(f(x^{-1}))$ (By definition of inverse)

$= CM_B(f(x))^{-1}$ (By definition of homomorphism)

$\geq CM_B(f(x))$ (Since $B \in FMG(y)$)

$= CM_{f^{-1}(B)}(x)$

Proposition 5.3 Let $H \in AFMG(X)$ and Y be a group. Suppose that $f: X \to Y$ be an onto homomorphism. Then $f(H) \in AFMG(Y)$.

*Proof.*By proposition 5.1, $f(H) \in FMG(Y)$. Now let $y, z \in Y$. Since f is onto, there exist $u \in X$. Such that $f(u) = z$. Thus

$$CM_{f(H)}(zyz^{-1}) = \vee \{CM_H(w): w \in X, f(w) = zyz^{-1}\}$$
$$= \vee \{CM_H(u^{-1}wu): w \in X, f(u^{-1}wu) = y\}$$

$CM_H(w) = CM_H(u^{-1}wu)$ *since* $H \in AFMG(X)$byProposition 3.1

Now $f(w) = zyz^{-1} = f(u)y(f(u))^{-1}$
$\Rightarrow \left(f(u)\right)^{-1} f(w)f(u) = y \Rightarrow f(u^{-1})f(w)f(u) = y$
$\Rightarrow \vee \{CM_A(v): \vee \in X, f(\vee) = y\} = CM_{f(H)}(y)$

Hence by proposition 3.1, $f(H) \in AFMG(y)$

Proposition 5.4 Let $H \in AFMG(y)$ and X be a group. Suppose that $f: x \to y$ be an into homomorphism. Then $f^{-1}(H) \in AFMG(X)$.

Proof. By proposition 3.1, $f^{-1}(H) \in AFMG(X)$. Let $x, z \in X$. Then

$$CM_{f^{-1}(H)}(xz) = CM_H[f(xz)] = CM_H[f(x)f(z)] = CM_H[f(z)f(x)]$$

Since f is a homomorphism and $H \in AFMG(X) = CM_H[f(zx)] = CM_{f^{-1}(H)}(zx)$
Hence $f^{-1}(H) \in AFMG(X)$.

6 **Conclusions**

In this paperwe introduced the concept of Abelian fuzzy multi groups and find out some of the normal subgroups of X. Also left and right cosets of fuzzy multi groups and fuzzy multi order of an element of groups are introduced and its various properties are discussed. And it became evident that Fuzzy multi order of an element of a group has some properties similar to that of order of an element in a group. And finally we discussed some of the homomorphic properties of Fuzzy multigroups.

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