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## TYPES OF FUZZY PAIRWISE S-COMPACTNESS MODULO SMOOTH IDEALS

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**Abstract** – In this paper, we introduce several types of fuzzy pairwise compactness and fuzzy pairwise compactness modulo a smooth ideal in smooth bitopological spaces by using the family of  $r$ - $(\tau_i, \tau_j)$ -fuzzy semi-open sets as cover. Several characterizations and some properties of these spaces are discussed. Preservation of fuzzy pairwise compactness modulo a smooth ideal by some types of mappings is also investigated.

**Keywords** – Fuzzy pairwise compactness, fuzzy pairwise S-compactness, smooth bitopological spaces.

### 1 Introduction

Šostak [27] introduced the fundamental concept of a ‘fuzzy topological structure’ as an extension of both crisp topology and Chang’s fuzzy topology [4], indicating that not only the objects were fuzzified, but also the axiomatics. Subsequently, Badard [3] introduced the concept of ‘smooth topological space’. Chattopadhyay et al. [5] and Chattopadhyay and Samanta [6] re-introduced the same concept, calling it ‘gradation of openness’. Ramadan [19] and his colleagues introduced a similar definition, namely a smooth topological space for lattice  $L = [0, 1]$ . Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [5, 6, 9, 28]). Thus, the terms ‘fuzzy topology’, in Šostak’s sense, ‘gradation of openness’ and ‘smooth topology’ are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Further to this, Lee et al. [16] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil’s fuzzy bitopological space [10].

The concept of fuzzy semi-open sets and fuzzy semi-continuous mapping in fuzzy topological spaces was studied by Azad [2]. Kumar [13] generalized the concepts of fuzzy semi-open sets, fuzzy semi-continuous mappings into fuzzy bitopological spaces. Kim et al. [12] as well as Lee and Lee

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[15], introduced the notion of fuzzy  $r$ -semi-open sets and fuzzy  $r$ -semi-continuous maps in smooth topological space which are generalizations of fuzzy semi-open sets and fuzzy semi-continuous maps in Chang's fuzzy topology. Ramadan and Abbas [21] introduced the notion of  $r$ -fuzzy semi-open sets in smooth bitopological spaces. El-sheikh [7] characterized the notion of  $r$ -fuzzy semi-open sets [21] and generalized the notions that introduced in smooth bitopological space [13], [20], [21]. Recently [29], we introduced the concept of  $r$ - $\tau_{12}$ -fuzzy semi-open sets in smooth supra topological space  $(X, \tau_{12})$  which were induced from smooth bitopological space  $(X, \tau_1, \tau_2)$ . We have also shown that the present notion of fuzzy semi-open sets and the notion of  $r$ - $(\tau_i, \tau_j)$ -fuzzy semi-open sets that introduced in [21] are independent.

Ideals are an important notions which was introduced into general topology by Kuratowski [14], where a nonempty family  $I$  of  $P(X)$  is called an ideal if: (1)  $A \in I$  and  $B \subseteq A$  gives  $B \in I$  (heredity) and (2)  $A, B \in I$  gives  $A \cup B \in I$  (finite additivity). Sarkar [26] introduced and studied the notion of ideal in Chang's sense. Ramadan et al. [22] introduced the notion of a smooth ideal in smooth topology.

The concept of compactness modulo an ideal was first introduced by Newcomb [18] and Rančin [23] and was studied by Hamlett and Janković [8]. Abd El-Monsef et al. [1] studied the relations between ideals and some types of weak compactness. Salama [25] defined and studied some other types of fuzzy compactness with respect to fuzzy ideals in Chang's fuzzy topologies. Saber and Abdel-Sattar [24] investigated some properties of smooth ideals and used these to introduce and study the concept of  $r$ -fuzzy ideal-compact,  $r$ -fuzzy quasi H-closed, and  $r$ -fuzzy compact modulo a smooth ideal in smooth topological spaces.

In the present paper we use the concept of  $r$ - $(\tau_i, \tau_j)$ -fuzzy semi-open sets and a smooth ideal to introduce new types of compactness in smooth bitopological spaces, namely  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compactness,  $r$ - $(\tau_i, \tau_j)$ - $FST$ -Lindelöfness and  $r$ - $(\tau_i, \tau_j)$ - $FTS$ -closedness that generalize  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compactness,  $r$ - $(\tau_i, \tau_j)$ - $FS$ -Lindelöfness and  $r$ - $(\tau_i, \tau_j)$ - $FS$ -closedness respectively. We give the relation between these types of compactness and those introduced by Saber and Abdel-Sattar [24]. Also, we study some of the properties and characterizations. Moreover, the behavior of these types of compactness under some types of mappings is also investigated.

## 2 Preliminary

In this paper,  $X$  is a non-empty set,  $I = [0, 1]$  and  $I_0 = (0, 1]$ . A fuzzy set  $\mu$  of  $X$  is a mapping with  $\mu : X \rightarrow I$ , and  $I^X$  the family of all fuzzy sets of  $X$ . For any  $\mu_1, \mu_2 \in I^X$ ,  $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\}$ ,  $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}$  and  $(\mu_1 - \mu_2)(x) = \min\{\mu_1(x), \bar{1} - \mu_2(x) : x \in X\}$ . For a fuzzy set  $\lambda$  of  $X$ ,  $supp(\lambda) = \{x \in X \mid \lambda(x) > 0\}$ . For  $\lambda \in I^X$ ,  $\bar{1} - \lambda$  denotes the complement of  $\lambda$ . For  $\alpha \in I$ ,  $\bar{\alpha}(x) = \alpha \forall x \in X$ . By  $\bar{0}$  and  $\bar{1}$ , we denote constant maps on  $X$  with values 0 and 1, respectively. For  $\mu, \lambda \in I^X$ ,  $\mu$  is called quasi-coincident with  $\lambda$ , denoted by  $\mu q \lambda$ , if  $\mu(x) + \lambda(x) > 1$  for some  $x \in X$ . Otherwise we write  $\mu \bar{q} \lambda$ . For any  $\lambda_1$  and  $\lambda_2 \in I^X$ ,  $\lambda_1 \leq \lambda_2 \iff \lambda_1 \bar{q} \bar{1} - \lambda_2$ .  $FP$  stands for fuzzy pairwise. The indices are  $i, j \in \{1, 2\}$  and  $i \neq j$ . All other notations are standard notations of fuzzy set theory.

**Definition 2.1.** [3, 5, 19, 27] A smooth topology on  $X$  is a mapping  $\tau : I^X \rightarrow I$  which satisfies the following properties:

- (1)  $\tau(\bar{0}) = \tau(\bar{1}) = 1$ ,
- (2)  $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ ,  $\forall \mu_1, \mu_2 \in I^X$ ,
- (3)  $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$ , for any  $\{\mu_i : i \in J\} \subseteq I^X$ .

The pair  $(X, \tau)$  is called a smooth topological space. The value of  $\tau(\mu)$  is interpreted as the degree of openness of fuzzy set  $\mu$ . For  $r \in I_0$ ,  $\mu$  is an  $r$ -open fuzzy set of  $X$  if  $\tau(\mu) \geq r$ , and  $\mu$  is an  $r$ -closed fuzzy set of  $X$  if  $\tau(\bar{1} - \mu) \geq r$ . Note, Šostak [27] used the term 'fuzzy topology' and Chattopadhyay [5], the term 'gradation of openness' for a smooth topology  $\tau$ .

**Definition 2.2.** [16] A triple  $(X, \tau_1, \tau_2)$  consisting of the set  $X$  endowed with smooth topologies  $\tau_1$  and  $\tau_2$  on  $X$  is called a smooth bitopological space (smooth bts). For  $\lambda \in I^X$  and  $r \in I_0$ ,  $r$ - $\tau_i$ -open (resp. closed) fuzzy set denotes the  $r$ -open (resp. closed) fuzzy set in  $(X, \tau_i)$ , for  $i = 1, 2$ .

**Theorem 2.3.** [6, 11] Let  $(X, \tau_1, \tau_2)$  be a smooth bts. For  $\lambda \in I^X$  and  $r \in I_0$ , a  $\tau_i$ -fuzzy closure of  $\lambda$  is a mapping  $C_{\tau_i} : I^X \times I_0 \longrightarrow I^X$  defined as

$$C_{\tau_i}(\lambda, r) = \bigwedge \{ \mu \in I^X \mid \mu \geq \lambda, \tau_i(\bar{1} - \mu) \geq r \}$$

And, a  $\tau_i$ -fuzzy interior of  $\lambda$  is a mapping  $I_{\tau_i} : I^X \times I_0 \longrightarrow I^X$  defined as

$$I_{\tau_i}(\lambda, r) = \bigvee \{ \mu \in I^X \mid \mu \leq \lambda, \tau_i(\mu) \geq r \}.$$

Then:

- (1)  $C_{\tau_i}$  (resp.  $I_{\tau_i}$ ) is a fuzzy closure (resp. interior) operator.
- (2)  $\tau_{C_{\tau_i}} = \tau_{I_{\tau_i}} = \tau_i$ .
- (3)  $I_{\tau_i}(\bar{1} - \lambda, r) = \bar{1} - C_{\tau_i}(\lambda, r), \forall r \in I_0, \lambda \in I^X$ .

**Definition 2.4.** [24] A mapping  $\mathcal{I} : I^X \longrightarrow I$  is called a smooth ideal on  $X$  if it satisfies the following conditions:

- (S1)  $\mathcal{I}(\bar{1}) = 0, \mathcal{I}(\bar{0}) = 1,$
- (S2)  $\mathcal{I}(\lambda \vee \mu) \geq \mathcal{I}(\lambda) \wedge \mathcal{I}(\mu),$  for  $\lambda, \mu \in I^X,$
- (S3) If  $\lambda \leq \mu,$  then  $\mathcal{I}(\mu) \leq \mathcal{I}(\lambda),$  for  $\lambda, \mu \in I^X.$

If  $\mathcal{I}$  and  $\mathcal{J}$  are smooth ideals on  $X,$  we say  $\mathcal{I}$  is finer than  $\mathcal{J}$  (or  $\mathcal{J}$  is coarser than  $\mathcal{I}$ ), denoted by  $\mathcal{J} \leq \mathcal{I},$  if and only if  $\mathcal{J}(\lambda) \leq \mathcal{I}(\lambda)$  for all  $\lambda \in I^X.$

For each smooth ideal  $\mathcal{I}$  on  $X$  and  $\alpha \in I_0, \mathcal{I}_\alpha = \{ \nu \in I^X \mid \mathcal{I}(\nu) \geq \alpha \}$  is a fuzzy ideal on  $X$  in the sense of Sarkar [26]. By a fuzzy ideal we mean a non-empty collection of fuzzy sets  $\mathfrak{J}$  of a set  $X$  satisfying the following conditions:

- (i) If  $\mu \in \mathfrak{J}$  and  $\nu \leq \mu,$  then  $\nu \in \mathfrak{J}$  [heredity],
- (ii) If  $\mu \in \mathfrak{J}$  and  $\nu \in \mathfrak{J},$  then  $\mu \vee \nu \in \mathfrak{J}$  [finite additivity].

The simplest smooth ideal on  $X$  is  $\mathcal{I}^0 : I^X \longrightarrow I$  defined by  $\mathcal{I}^0(\lambda) = 1,$  if  $\lambda = \bar{0}$  and 0 otherwise.

We denote the smooth bts  $(X, \tau_1, \tau_2)$  with a smooth ideal  $\mathcal{I}$  by the quadruple  $(X, \tau_1, \tau_2, \mathcal{I})$  and call it a smooth ideal bitopological space (smooth ideal bts).

**Definition 2.5.** [21] Let  $(X, \tau_1, \tau_2)$  be a smooth bts for  $\lambda \in I^X$  and  $r \in I_0.$  Then:

- (1)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fuzzy semi-open set ( $r$ - $(\tau_i, \tau_j)$ -fso), if there exists  $\mu \in I^X$  with  $\tau_i(\mu) \geq r$  such that  $\mu \leq \lambda \leq C_{\tau_j}(\mu, r).$
- (2)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fuzzy semi-closed set ( $r$ - $(\tau_i, \tau_j)$ -fsc), if there exists  $\mu \in I^X$  with  $\tau_i(\bar{1} - \mu) \geq r$  such that  $I_{\tau_j}(\mu, r) \leq \lambda \leq \mu.$
- (3) The  $r$ - $(i, j)$ -fuzzy semi-interior of  $\lambda$  is denoted by  $SI_{ij}(\lambda, r)$  and defined as

$$SI_{ij}(\lambda, r) = \bigvee \{ \nu \in I^X \mid \nu \leq \lambda, \nu \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso} \}.$$

- (4) The  $r$ - $(i, j)$ -fuzzy semi-closure of  $\lambda$  is denoted by  $SC_{ij}(\lambda, r)$  and defined as

$$SC_{ij}(\lambda, r) = \bigwedge \{ \nu \in I^X \mid \nu \geq \lambda, \nu \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc} \}.$$

- (5)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fuzzy preopen set ( $r$ - $(\tau_i, \tau_j)$ -fpo) if  $\lambda \leq I_{\tau_i}(C_{\tau_j}(\lambda, r), r).$
- (6)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fuzzy preclosed set ( $r$ - $(\tau_i, \tau_j)$ -fpc) if  $C_{\tau_i}(I_{\tau_j}(\lambda, r), r) \leq \lambda.$

**Theorem 2.6.** [7] Let  $(X, \tau_1, \tau_2)$  be a smooth bts for  $\lambda \in I^X$  and  $r \in I_0.$  Then:

- (1)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fso iff  $\lambda = SI_{ij}(\lambda, r).$
- (2)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fsc iff  $\lambda = SC_{ij}(\lambda, r).$
- (3)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fso iff  $\bar{1} - \lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fsc.
- (4)  $\lambda$  is an  $r$ - $(\tau_i, \tau_j)$ -fso iff  $\lambda \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r).$
- (5)  $SC_{ij}(\bar{0}, r) = \bar{0}.$

$$(6) \quad \bar{1} - SC_{ij}(\lambda, r) = SI_{ij}(\bar{1} - \lambda, r).$$

**Definition 2.7.** [7] Let  $(X, \tau_1, \tau_2)$  and  $(Y, \tau_1^*, \tau_2^*)$  be smooth bts's. Let  $f : X \rightarrow Y$  be a mapping. Then  $f$  is called:

- (1) *FP-irresolute* (resp. *FP-semi-continuous* [21]) iff  $f^{-1}(\mu)$  is an  $r$ - $(\tau_i, \tau_j)$ -fso set in  $X$  for each  $r$ - $(\tau_i^*, \tau_j^*)$ -fso set  $\mu$  in  $Y$  (resp.  $\mu \in I^Y, \tau_i^*(\mu) \geq r$ ).
- (2) *FP-irresolute open* iff  $f(\mu)$  is an  $r$ - $(\tau_i^*, \tau_j^*)$ -fso set in  $Y$  for each  $r$ - $(\tau_i, \tau_j)$ -fso set  $\mu$  in  $X$ .

**Theorem 2.8.** [7] Let  $(X, \tau_1, \tau_2)$  and  $(Y, \tau_1^*, \tau_2^*)$  be smooth bts's. Let  $f : X \rightarrow Y$  be a mapping. Then the following statements are equivalent:

- (1)  $f$  is a *FP-irresolute*.
- (2) For each  $r$ - $(\tau_i^*, \tau_j^*)$ -fsc set  $\mu \in I^Y, f^{-1}(\mu)$  is an  $r$ - $(\tau_i, \tau_j)$ -fsc set in  $X$ .
- (3)  $SC_{ij}(f^{-1}(\mu), r) \leq f^{-1}(SC_{ij}(\mu, r)), \mu \in I^Y$ .

**Lemma 2.9.** [17] Let  $f : X \rightarrow Y$  be a mapping and let  $\lambda$  and  $\mu$  be fuzzy sets in  $X$  and  $Y$ , respectively. Then the following properties hold:

- (1)  $\lambda \leq f^{-1}(f(\lambda))$  and equality holds if  $f$  is injective.
- (2)  $f(f^{-1}(\mu)) \leq \mu$  and equality holds if  $f$  is surjective.
- (3) For any fuzzy point  $x_t$  in  $X, f(x_t)$  is a fuzzy point in  $Y$  and  $f(x_t) = (f(x))_t$ .
- (4) If  $f(\lambda) \leq \mu$ , then  $\lambda \leq f^{-1}(\mu)$ .

**Definition 2.10.** [24] Let  $(X, \tau, \mathcal{I})$  be a smooth ideal topological space and  $r \in I_0$ . Then  $X$  is called:

- (1) An  $r$ -*FT-compact* (resp.  $r$ -fuzzy ideal quasi H-closed ( $r$ -*FTQHC*)) iff for every family  $\{\lambda_i \in I^X \mid \tau(\lambda_i) \geq r, i \in J\}$  such that when  $\bigvee_{i \in J} \lambda_i = \bar{1}$ , there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(\bar{1} - \bigvee_{i \in J_0} \lambda_i) \geq r$  (resp.  $\mathcal{I}(\bar{1} - \bigvee_{i \in J_0} C_\tau(\lambda_i, r)) \geq r$ ).
- (2) An  $r$ -fuzzy compact modulo fuzzy ideal space ( $r$ -fuzzy  $C(\mathcal{I})$ -compact) if for every  $\beta \in I^X, \tau(\bar{1} - \beta) \geq r$  and each family  $\{\lambda_i \in I^X \mid \tau(\lambda_i) \geq r, i \in J\}$  such that  $\beta \leq \bigvee_{i \in J} \lambda_i$ , there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(\beta \wedge [\bar{1} - \bigvee_{i \in J_0} C_\tau(\lambda_i, r)]) \geq r$ .

### 3 *FPST-compact and FPST-Lindelöf Spaces*

In this section we introduce the notion of *FPST-compact* (resp. *Lindelöf*) space in smooth bts  $(X, \tau_1, \tau_2)$  by using the family of  $r$ - $(\tau_i, \tau_j)$ -fso sets as cover. Then, we generalize the same notions via smooth ideal  $\mathcal{I}$  on  $X$  to obtain *FPST-compact* (resp. *Lindelöf*) space. We also give the relations between them and study some of their basic properties.

**Definition 3.1.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . A fuzzy set  $\rho \in I^X$  is called:

- (1) An  $r$ - $(\tau_i, \tau_j)$ -*FS-compact* if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$ , such that  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a finite set  $J_0 \subset J$ , such that  $\rho \leq \bigvee_{\alpha \in J_0} \mu_\alpha$ . The space  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS-compact* if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS-compact* as a fuzzy subset.
- (2) An  $r$ - $(\tau_i, \tau_j)$ -*FSI-compact* if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$ , such that  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha]) \geq r$ . The space  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ -*FSI-compact* if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FSI-compact* as a fuzzy subset.
- (3) An  $r$ - $(\tau_i, \tau_j)$ -*FS-Lindelöf* if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$ , such that  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a countable set  $J_0 \subset J$ , such that  $\rho \leq \bigvee_{\alpha \in J_0} \mu_\alpha$ . The space  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS-Lindelöf* if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS-Lindelöf* as a fuzzy subset.

- (4) An  $r$ - $(\tau_i, \tau_j)$ - $FST$ -Lindelöf if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$ , such that  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a countable set  $J_0 \subset J$ , such that  $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha]) \geq r$ . The space  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -Lindelöf if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -Lindelöf as a fuzzy subset.

**Definition 3.2.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then  $X$  is called:

- (1)  $FPS$ -compact (resp.  $FPSL$ -compact) if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -compact) for each  $r \in I_0$ .
- (2)  $FPS$ -Lindelöf (resp.  $FPSL$ -Lindelöf) if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -Lindelöf (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -Lindelöf) for each  $r \in I_0$ .

From Definition 3.1 we have the following remark.

**Remark 3.3.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then the following statements are true:

- (1) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -compact, then  $X$  is an  $r$ - $\tau_i$ - $FT$ -compact.
- (2) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact (resp. Lindelöf), then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -compact (resp. Lindelöf).
- (3) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -compact), then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -Lindelöf (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -Lindelöf).
- (4) If  $\mathcal{I} = \mathcal{I}^0$ , then  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact (resp.  $r$ - $(\tau_i, \tau_j)$ - $FS$ -Lindelöf) and  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -compact (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSTL$ -Lindelöf) are equivalent.

It follows from the Definition 3.1, Remark 3.3 and the fact that every  $r$ - $\tau_i$ -open fuzzy set in  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -fso set that

$$\begin{array}{ccccc} r\text{-}(\tau_i, \tau_j)\text{-}FS\text{-compact} & \implies & r\text{-}(\tau_i, \tau_j)\text{-}FSTL\text{-compact} & \implies & r\text{-}\tau_i\text{-}FT\text{-compact} \\ & & \Downarrow & & \\ & & r\text{-}(\tau_i, \tau_j)\text{-}FS\text{-Lindelöf} & & r\text{-}(\tau_i, \tau_j)\text{-}FSTL\text{-Lindelöf} \end{array}$$

**Example 3.4.** Let  $X = \mathbb{N}$ , where  $\mathbb{N}$  is the set of natural numbers. Define fuzzy set  $\lambda_n \in I^X$  as follows:

$$\lambda_n = \chi_{\{n\}}, \text{ where } \chi_{\{n\}} \text{ is the characteristic function of } \{n\}, n \in \mathbb{N}.$$

Define smooth topologies  $\tau_1 : I^X \rightarrow I$  and  $\tau_2 : I^X \rightarrow I$  as follows:

$$\tau_1(\lambda) = \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_n, n \in \mathbb{N}, \\ \frac{3}{4} & \text{otherwise.} \end{cases}$$

Define smooth ideal  $\mathcal{I} : I^X \rightarrow I$  as follows:

$$\mathcal{I}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \\ 0 & \text{if } \lambda = \bar{1}, \\ \frac{3}{4} & \text{otherwise.} \end{cases}$$

Then clearly  $(X, \tau_1, \tau_2, \mathcal{I})$  is a smooth ideal bts. Note that  $X$  is not a  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ - $FS$ -compact since there exists a family

$$\{\lambda_n \in I^X \mid \lambda_n = \chi_{\{n\}} \text{ is } \frac{1}{2}\text{-}(\tau_1, \tau_2)\text{-fso set}, n \in \mathbb{N}\} \text{ with } \bigvee_{n \in \mathbb{N}} \lambda_n = \bar{1}$$

such that there in no finite set  $J_0 \subset \mathbb{N}$  with  $\bigvee_{n \in J_0} \lambda_n = \bar{1}$ .

But  $X$  is a  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ - $FSTL$ -compact, since for any finite set  $J_0 \subset \mathbb{N}$ ,  $\mathcal{I}(\bar{1} - \bigvee_{n \in J_0} \lambda_n) = \frac{3}{4} \geq \frac{1}{2}$ .

The following example shows that the finite spaces need not to be a  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact.

**Example 3.5.** Let  $X = \{a, b, c\}$  and  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$  be fuzzy sets of  $X$  defined as

$$\lambda_1 = a_s \vee b_{0.5} \vee c_s, \quad \lambda_2 = a_{0.3} \vee b_s \vee c_s, \quad s \in [0.9, 1),$$

$$\lambda_3 = a_k \vee b_{0.5} \vee c_k, \quad \lambda_4 = a_{0.3} \vee b_k \vee c_k, \quad k \in (0, 0.1].$$

Define smooth topologies  $\tau_1 : I^X \rightarrow I$  and  $\tau_2 : I^X \rightarrow I$  by

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \wedge \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \wedge \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \vee \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly  $(X, \tau_1, \tau_2)$  is a smooth bts. Note that  $X$  is finite set but it is not a  $\frac{1}{3}$ - $(\tau_1, \tau_2)$ - $FS$ -compact since there exists a family

$$\{a_s \vee b_s \vee c_s \in I^X \mid a_s \vee b_s \vee c_s \text{ is } \frac{1}{3}\text{-}(\tau_1, \tau_2)\text{-fso sets, } s \in [0.9, 1)\} \text{ with } \bigvee_{s \in [0.9, 1)} a_s \vee b_s \vee c_s = \bar{1}.$$

But there is no finite subset  $J_0 \subset [0.9, 1)$  such that  $\bigvee_{s \in J_0} a_s \vee b_s \vee c_s = \bar{1}$ .

The following example corresponds to the concept of the ideal of finite (resp. countable) subsets of  $X$  in the ordinary sense.

**Example 3.6.** Define  $\mathcal{I}_f, \mathcal{I}_c : I^X \rightarrow I$  be two smooth ideals on  $X$  as follows:

$$\mathcal{I}_f(\mu) = \begin{cases} 1 & \text{if } \text{supp}(\mu) = \text{finite subset of } X, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{I}_c(\mu) = \begin{cases} 1 & \text{if } \text{supp}(\mu) = \text{countable subset of } X, \\ 0 & \text{otherwise.} \end{cases}$$

That is mean:

- (1)  $\mathcal{I}_f$  is a smooth ideal on  $X$  such that for every  $r \in I_0$ ,  $(\mathcal{I}_f)_r = \{\mu \in I^X \mid \mathcal{I}_f(\mu) \geq r \text{ and } \text{supp}(\mu) \text{ is a finite subset of } X\}$  is a fuzzy ideal in Sarkar's sense.
- (2)  $\mathcal{I}_c$  is a smooth ideal on  $X$  such that for every  $r \in I_0$ ,  $(\mathcal{I}_c)_r = \{\mu \in I^X \mid \mathcal{I}_c(\mu) \geq r \text{ and } \text{supp}(\mu) \text{ is a countable subset of } X\}$  is a fuzzy ideal in Sarkar's sense.

**Theorem 3.7.** A smooth ideal bts  $(X, \tau_1, \tau_2, \mathcal{I}_f)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS\mathcal{I}_f$ -compact iff  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact.

*Proof.* Let  $(X, \tau_1, \tau_2, \mathcal{I}_f)$  be an  $r$ - $(\tau_i, \tau_j)$ - $FS\mathcal{I}_f$ -compact and let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set, } \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Suppose that for any finite set  $J_0 \subset J$  we have  $\bigvee_{\alpha \in J_0} \mu_\alpha \neq \bar{1}$ . This implies  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \neq \bar{0}$ . Therefore,  $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \neq 1$ . Thus, from definition of  $\mathcal{I}_f$ , we have  $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) = 0$  meaning that for any finite set  $J_0 \subset J$ ,  $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) < r$  which contradicts the hypothesis. Hence,  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compact.

Conversely, let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set, } \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . According to  $r$ - $(\tau_i, \tau_j)$ - $FS$ -compactness of  $X$ , there exists a finite set  $J_0 \subset J$  such that  $\bigvee_{\alpha \in J_0} \mu_\alpha = \bar{1}$ . Since  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha = \bar{0}$  and  $\text{supp}(\bar{0}) = \emptyset$ , there is a finite subset of  $X$ . Then,  $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Hence,  $(X, \tau_1, \tau_2, \mathcal{I}_f)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS\mathcal{I}_f$ -compact. □

**Theorem 3.8.** A smooth ideal bts  $(X, \tau_1, \tau_2, \mathcal{I}_c)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -compact iff  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ - $F$ S-Lindelöf.

*Proof.* Let  $(X, \tau_1, \tau_2, \mathcal{I}_c)$  be an  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -compact and let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . By  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -compactness of  $X$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}_c(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ , meaning that  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha$  has a countable support. Therefore, there exists a countable set  $J_c \subset J$ , such that  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \leq \bigvee_{\alpha \in J_c} \mu_\alpha$ . Thus,  $\bar{1} = \bigvee_{\alpha \in J_0} \mu_\alpha \vee \bigvee_{\alpha \in J_c} \mu_\alpha = \bigvee_{\alpha \in J_0 \cup J_c} \mu_\alpha$ . Since  $J_0 \cup J_c$  is countable subset of  $J$ , then  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ - $F$ S-Lindelöf.

Conversely, Theorem 3.7 is a similar proof. □

**Corollary 3.9.** If  $(X, \tau_1, \tau_2, \mathcal{I}_c)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -compact, then  $(X, \tau_1, \tau_2, \mathcal{I}_c)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -Lindelöf.

**Corollary 3.10.** If  $(X, \tau_1, \tau_2, \mathcal{I}_c)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST_c$ -compact, then  $(X, \tau_1, \tau_2)$  is an  $r$ - $\tau_i$ - $F$ -Lindelöf.

**Theorem 3.11.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts. If  $\lambda_1, \lambda_2 \in I^X$  are  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact fuzzy subsets, then  $\lambda_1 \vee \lambda_2$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact.

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be a family such that  $\lambda_1 \vee \lambda_2 \leq \bigvee_{\alpha \in J} \mu_\alpha$ . Then,  $\lambda_1 \leq \bigvee_{\alpha \in J} \mu_\alpha$  and  $\lambda_2 \leq \bigvee_{\alpha \in J} \mu_\alpha$ . Since  $\lambda_1$  and  $\lambda_2$  are  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact, there exists a finite set  $J_1 \subset J$  and  $J_2 \subset J$  such that  $\mathcal{I}(\lambda_1 \wedge (\bar{1} - \bigvee_{\alpha \in J_1} \mu_\alpha)) \geq r$  and  $\mathcal{I}(\lambda_2 \wedge (\bar{1} - \bigvee_{\alpha \in J_2} \mu_\alpha)) \geq r$ . Therefore  $\mathcal{I}(\lambda_1 \vee \lambda_2 \wedge (\bar{1} - \bigvee_{\alpha \in J_1 \cup J_2} \mu_\alpha)) \geq r$ . Thus,  $\lambda_1 \vee \lambda_2$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact. □

**Theorem 3.12.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . If for each family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  with  $\mathcal{I}(\bigwedge_{\alpha \in J} \mu_\alpha) \geq r$  there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$ , then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact.

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$  implies that  $\mathcal{I}(\bigwedge_{\alpha \in J} \bar{1} - \mu_\alpha) \geq r$ . The hypothesis suggests that there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \bar{1} - \mu_\alpha) \geq r$ . Thus,  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Hence,  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact. □

**Theorem 3.13.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact and  $r \in I_0$ . If  $\mathcal{J}$  is a smooth ideal on  $X$  such that  $\mathcal{I} \leq \mathcal{J}$ , then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $F$ S $\mathcal{J}$ -compact.

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Since  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact, there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Since  $\mathcal{I} \leq \mathcal{J}$ , then  $\mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Thus,  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $F$ S $\mathcal{J}$ -compact. □

**Theorem 3.14.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then the following statements are equivalent:

- (1)  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ - $FST$ -compact.
- (2) For any collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  with  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ , there exists a finite set  $J_0 \subset J$  with  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$ .

*Proof.* (1)  $\implies$  (2) Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  with  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ . This implies,  $\bigvee_{\alpha \in J} (\bar{1} - \mu_\alpha) = \bar{1}$ . Since  $\{\bar{1} - \mu_\alpha, \alpha \in J\}$  is a family of  $r\text{-}(\tau_i, \tau_j)$ -fso sets and by  $r\text{-}(\tau_i, \tau_j)$ -FST-compactness of  $X$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bar{1} - \mu_\alpha)) \geq r$  implies that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$ .

(2)  $\implies$  (1) Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be a family with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Then,  $\bigwedge_{\alpha \in J} (\bar{1} - \mu_\alpha) = \bar{0}$ . By (2), there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} (\bar{1} - \mu_\alpha)) \geq r$ . This implies that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Therefore,  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact.  $\square$

**Definition 3.15.** [24] A family  $\{\mu_\alpha \in I^X \mid \alpha \in J\}$  has the finite intersection property ( $\mathcal{I}$ -FIP) iff  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$  for any no finite subfamily  $J_0 \subset J$ .

**Theorem 3.16.** A smooth ideal bts  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact iff every collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  having the  $\mathcal{I}$ -FIP has a non-empty intersection.

*Proof.* Suppose  $X$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact and let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  having the  $\mathcal{I}$ -FIP. Suppose  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ . Then  $\bigvee_{\alpha \in J} (\bar{1} - \mu_\alpha) = \bar{1}$ . Since  $\bar{1} - \mu_\alpha$  is an  $r\text{-}(\tau_i, \tau_j)$ -fso set for each  $\alpha \in J$ . By  $r\text{-}(\tau_i, \tau_j)$ -FST-compactness of  $X$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bar{1} - \mu_\alpha)) \geq r$ , implies that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} \mu_\alpha) \geq r$  which is a contradiction. Hence,  $\bigwedge_{\alpha \in J} \mu_\alpha \neq \bar{0}$ .

Conversely, let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be a family with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Suppose for every finite set  $J_0 \subset J$  we have  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) < r$ . That is mean  $\mathcal{I}(\bigwedge_{\alpha \in J_0} (\bar{1} - \mu_\alpha)) \geq r$  for every no finite  $J_0 \subset J$ , from hypothesis of  $\mathcal{I}$ -FIP we have,  $\bigwedge_{\alpha \in J} (\bar{1} - \mu_\alpha) \neq \bar{0}$ . This yields  $\bigvee_{\alpha \in J} \mu_\alpha \neq \bar{1}$  which is a contradiction. Hence,  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact.  $\square$

**Definition 3.17.** Let  $(X, \tau_1, \tau_2)$  be a smooth bts and  $r \in I_0$ . A fuzzy set  $\lambda$  of  $X$  is called:

- (1) an  $r\text{-}(\tau_i, \tau_j)$ -fuzzy regular(semi)open ( $r\text{-}(\tau_i, \tau_j)$ -fro (resp.  $r\text{-}(\tau_i, \tau_j)$ -frso )) if  $\lambda = I_{\tau_i}(C_{\tau_j}(\lambda, r), r)$  (resp.  $\lambda = SI_{ij}(C_{\tau_j}(\lambda, r), r)$ ).
- (2) an  $r\text{-}(\tau_i, \tau_j)$ -fuzzy regular(semi)closed ( $r\text{-}(\tau_i, \tau_j)$ -frc (resp.  $r\text{-}(\tau_i, \tau_j)$ -frsc )) if  $\lambda = C_{\tau_i}(I_{\tau_j}(\lambda, r), r)$  (resp.  $\lambda = SC_{ij}(I_{\tau_j}(\lambda, r), r)$ ).

**Theorem 3.18.** If  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact, then for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_j, \tau_i)\text{-frc set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ .

*Proof.* The proof derives from the fact that every  $r\text{-}(\tau_j, \tau_i)$ -frc set is an  $r\text{-}(\tau_i, \tau_j)$ -fso set.  $\square$

**Theorem 3.19.** If  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)$ -FST-compact, then for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-frso set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ .

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is an } r\text{-}(\tau_i, \tau_j)\text{-frso set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Then, from Definition 3.17(1), we have  $\bigvee_{\alpha \in J} SI_{ij}(C_{\tau_j}(\mu_\alpha, r), r) = \bar{1}$ . As  $\{SI_{ij}(C_{\tau_j}(\mu_\alpha, r), r) \in I^X, \alpha \in J\}$  is a family of  $r\text{-}(\tau_i, \tau_j)$ -fso sets, and by  $r\text{-}(\tau_i, \tau_j)$ -FST-compactness of  $X$  there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SI_{ij}(C_{\tau_j}(\mu_\alpha, r), r)) \geq r$ , then this  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ .  $\square$



**Theorem 3.20.** If  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ -*FST*-compact, then for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fpo set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} C_{\tau_j}(\mu_\alpha, r)) \geq r$ .

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fpo set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Since  $\mu_\alpha \leq C_{\tau_j}(\mu_\alpha, r) \leq C_{\tau_j}(I_{\tau_i}(C_{\tau_j}(\mu_\alpha, r), r), r)$ , then  $\bar{1} = \bigvee_{\alpha \in J} C_{\tau_j}(\mu_\alpha, r)$  such that  $\{C_{\tau_j}(\mu_\alpha, r) \in I^X, \alpha \in J\}$  is a family of  $r$ - $(\tau_i, \tau_j)$ -fso set of  $X$ . By  $r$ - $(\tau_i, \tau_j)$ -*FST*-compactness of  $X$ , there exists a finite  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} C_{\tau_j}(\mu_\alpha, r)) \geq r$ .  $\square$

**Corollary 3.21.** If  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ -*FST*-compact, then for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fpc set}, \alpha \in J\}$  such that  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} I_{\tau_j}((\mu_\alpha, r))) \geq r$ .

## 4 *FPSC*( $\mathcal{I}$ )-compact and *FPI-S*-closed Spaces

In this section we introduce the notions of *FPSC*-compact, *FPSC*( $\mathcal{I}$ )-compact, *FP-S*-closed and *FPI-S*-closed in a smooth bts  $(X, \tau_1, \tau_2)$  and study some of their basic properties. We give the relations between them. Furthermore, we show that *FPSC*( $\mathcal{I}$ )-compactness is not a generalization of *FPSC*-compactness.

**Definition 4.1.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then,  $X$  is called:

- (1) An  $r$ - $(\tau_i, \tau_j)$ -*FSC*-compact, if for every  $r$ - $(\tau_i, \tau_j)$ -fsc set  $\rho$  of  $X$  and every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  with  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a finite set  $J_0 \subset J$  such that  $\rho \leq \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)$ .
- (2) An  $r$ - $(\tau_i, \tau_j)$ -*FSC*( $\mathcal{I}$ )-compact, if for every  $r$ - $(\tau_i, \tau_j)$ -fsc set  $\rho$  of  $X$  and every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  with  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)]) \geq r$ .
- (3) An  $r$ - $(\tau_i, \tau_j)$ -*FS*-closed, if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r) = \bar{1}$ .
- (4) An  $r$ - $(\tau_i, \tau_j)$ -*FT-S*-closed, if for every family  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$ .

**Definition 4.2.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then  $X$  is called:

- (1) *FPSC*-compact (resp. *FPSC*( $\mathcal{I}$ )-compact), if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FSC*-compact (resp.  $r$ - $(\tau_i, \tau_j)$ -*FSC*( $\mathcal{I}$ )-compact) for each  $r \in I_0$ .
- (2) *FP-S*-closed (resp. *FPI-S*-closed), if  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS*-closed (resp.  $r$ - $(\tau_i, \tau_j)$ -*FT-S*-closed) for each  $r \in I_0$ .

From Definition 4.1 we have the following remark.

**Remark 4.3.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then the following statements are true:

- (1) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FSC*( $\mathcal{I}$ )-compact, then  $X$  is an  $r$ - $\tau_i$ -*FC*( $\mathcal{I}$ )-compact.
- (2) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FT-S*-closed, then  $X$  is an  $r$ - $\tau_i$ -*FTQHC*.
- (3) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FS*-closed, then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -*FT-S*-closed.

- (4) If  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSC$ -compact (resp.  $r$ - $(\tau_i, \tau_j)$ - $FSC(\mathcal{I})$ -compact), then  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -closed (resp.  $r$ - $(\tau_i, \tau_j)$ - $FTS$ -closed).
- (5) If  $\mathcal{I} = \mathcal{I}^0$ , then  $r$ - $(\tau_i, \tau_j)$ - $FS$ -closed and  $r$ - $(\tau_i, \tau_j)$ - $FTS$ -closed are equivalent.

It follows from the Definition 4.1, Remark 4.3 and the fact that every  $r$ - $\tau_i$ -closed fuzzy set in  $X$  is an  $r$ - $(\tau_i, \tau_j)$ -fsc set that.

$$\begin{array}{ccc}
 r\text{-}(\tau_i, \tau_j)\text{-}FSC\text{-compact} & r\text{-}(\tau_i, \tau_j)\text{-}FSC(\mathcal{I})\text{-compact} & \implies r\text{-}\tau_i\text{-}FC(\mathcal{I})\text{-compact} \\
 \Downarrow & \Downarrow & \Downarrow \\
 r\text{-}(\tau_i, \tau_j)\text{-}FS\text{-closed} & \implies r\text{-}(\tau_i, \tau_j)\text{-}FTS\text{-closed} & \implies r\text{-}\tau_i\text{-}FTQHC
 \end{array}$$

**Remark 4.4.** The notion of  $r$ - $(\tau_i, \tau_j)$ - $FSC(\mathcal{I})$ -compactness of  $X$  is not a generalization of  $r$ - $(\tau_i, \tau_j)$ - $FSC$ -compactness. We show this in the next example.

**Example 4.5.** Let  $X = \{a, b, c\}$ . Define fuzzy sets  $\lambda_1, \lambda_2$  and  $\lambda_3 \in I^X$  as follows:

$$\lambda_1 = a_{0.5} \vee b_{0.1} \vee c_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.9} \vee c_{0.5}, \quad \lambda_3 = a_{0.6} \vee b_{0.2}.$$

Define smooth topologies  $\tau_1 : I^X \rightarrow I$  and  $\tau_2 : I^X \rightarrow I$  as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \lambda_2, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \lambda_2, \lambda_3, \\ \frac{3}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_3, \lambda_1 \wedge \lambda_3, \\ \frac{3}{4} & \text{if } \lambda = \lambda_2 \vee \lambda_3, \lambda_2 \wedge \lambda_3, \\ 0 & \text{otherwise.} \end{cases}$$

Define smooth ideal  $\mathcal{I} : I^X \rightarrow I$  by

$$\mathcal{I}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \\ 0 & \text{if } \lambda = \bar{1}, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Then  $(X, \tau_1, \tau_2, \mathcal{I})$  is a smooth ideal bts. For  $r = \frac{1}{2}$ ,  $\{\bar{0}, \bar{1}, \lambda_1, \lambda_2\}$  is the family of all  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ -fso sets in  $X$ , and for any  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ -fsc set in  $X$  it easy to verify that  $(X, \tau_1, \tau_2)$  is a  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ - $FSC$ -compact space. But  $(X, \tau_1, \tau_2, \mathcal{I})$  is not a  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ - $FSC(\mathcal{I})$ -compact space, as  $\rho = a_{0.5} \vee b_{0.1} \vee c_{0.5}$  is a  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ -fsc set in  $X$ . However, for any  $J$  is  $\frac{1}{2}$ - $(\tau_1, \tau_2)$ -fso set which cover  $\rho$  and for any finite subset  $J_0$  of  $J$  we have,

$$\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{12}(\lambda_\alpha, \frac{1}{2})]) = \mathcal{I}(\rho) = \frac{1}{4} < \frac{1}{2}.$$

**Theorem 4.6.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then the following statements are equivalent:

- (1)  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSC(\mathcal{I})$ -compact.
- (2) For any collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  and every  $r$ - $(\tau_i, \tau_j)$ -fsc set  $\rho$  in  $X$  with  $\rho \bar{q} \bigwedge_{\alpha \in J} \mu_\alpha$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\rho \wedge \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r$ .
- (3)  $\rho \bar{q} \bigwedge_{\alpha \in J} \mu_\alpha$  holds for every collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  and every  $r$ - $(\tau_i, \tau_j)$ -fsc set  $\rho$  in  $X$  with  $\{\rho \wedge SI_{ij}(\mu_\alpha, r), \alpha \in J\}$  has the  $\mathcal{I}$ - $FIP$ .

*Proof.* (1)  $\implies$  (2) Let  $\rho$  be an  $r$ - $(\tau_i, \tau_j)$ -fsc set in  $X$  and  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J\}$  be any family with  $\rho \bar{q} \bigwedge_{\alpha \in J} \mu_\alpha$ . Then  $\rho \leq \bigvee_{\alpha \in J} \bar{1} - \mu_\alpha$ . For each  $\alpha \in J$ ,  $\bar{1} - \mu_\alpha$  is an  $r$ - $(\tau_i, \tau_j)$ -fso set. Since  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSC(\mathcal{I})$ -compact, there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\bar{1} - \mu_\alpha, r)]) \geq r$ . Since

$$\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\bar{1} - \mu_\alpha, r)] = \rho \wedge \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r). \text{ Thus, } \mathcal{I}(\rho \wedge \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r.$$

(2)  $\implies$  (3) This is trivial.

(3)  $\implies$  (1) Let  $\rho$  be an  $r$ - $(\tau_i, \tau_j)$ -fsc set in  $X$  and  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  where  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ . Suppose there is no finite subfamily  $J_0 \subset J$ , then  $\mathcal{I}(\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)]) \geq r$ . Since

$$\rho \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)] = \rho \wedge \bigwedge_{\alpha \in J_0} (\bar{1} - SC_{ij}(\mu_\alpha, r)) = \bigwedge_{\alpha \in J_0} \{\rho \wedge SI_{ij}(\bar{1} - \mu_\alpha, r)\}$$

the family  $\{\rho \wedge SI_{ij}(\bar{1} - \mu_\alpha, r), \alpha \in J\}$  has  $\mathcal{I}$ -FIP. By (3),  $\rho \bar{q} \bigwedge_{\alpha \in J} (\bar{1} - \mu_\alpha)$  implies that  $\bigvee_{\alpha \in J} \mu_\alpha \leq \rho$ . This is a contradiction.  $\square$

**Theorem 4.7.** If  $(X, \tau_1, \tau_2, \mathcal{I}_f)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSL_f$ -compact, then  $(X, \tau_1, \tau_2)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -closed.

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Since  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSL_f$ -compact, there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}_f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . This means  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha$  has a finite support, implying there exists a finite set  $J_k \subset J$  such that  $\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \leq \bigvee_{\alpha \in J_k} \mu_\alpha$ . Therefore,  $\bar{1} = \bigvee_{\alpha \in J_0 \cup J_k} \mu_\alpha$ . Since for any  $\alpha \in J$ ,  $\mu_\alpha \leq SC_{ij}(\mu_\alpha, r)$ . Then,  $\bar{1} = \bigvee_{\alpha \in J_0 \cup J_k} SC_{ij}(\mu_\alpha, r)$ . Hence,  $X$  is an  $r$ - $(\tau_i, \tau_j)$ - $FS$ -closed.  $\square$

**Definition 4.8.** A smooth topological space  $(X, \tau)$  is called an  $r$ - $FQHC$  if for every family  $\{\mu_\alpha \in I^X \mid \tau(\mu_\alpha) \geq r, \alpha \in J\}$  with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ , there exists a finite set  $J_0 \subset J$  such that  $\bigvee_{\alpha \in J_0} C_\tau(\mu_\alpha, r) = \bar{1}$ .

**Theorem 4.9.** If  $(X, \tau_1, \tau_2, \mathcal{I}_f)$  is an  $r$ - $(\tau_i, \tau_j)$ - $FSL_f$ -compact, then  $(X, \tau_1, \tau_2)$  is an  $r$ - $\tau_i$ - $FQHC$ .

*Proof.* The proof is similar to the proof of Theorem 4.7.  $\square$

**Theorem 4.10.** Let  $(X, \tau_1, \tau_2, \mathcal{I})$  be a smooth ideal bts and  $r \in I_0$ . Then the following statements are equivalent:

- (1)  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ - $FT$ - $S$ -closed.
- (2) For any collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J_0\}$  with  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r$ .
- (3)  $\bigwedge_{\alpha \in J} \mu_\alpha \neq \bar{0}$ , holds for any collection  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J_0\}$  such that  $\{SI_{ij}(\mu_\alpha, r), \alpha \in J\}$  has  $\mathcal{I}$ -FIP.

*Proof.* (1)  $\iff$  (2) Similar to the proof of Theorem 3.14.

(1)  $\implies$  (3) Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fsc set}, \alpha \in J_0\}$  be any family such that  $\{SI_{ij}(\mu_\alpha, r), \alpha \in J\}$  has the  $\mathcal{I}$ -FIP. If  $\bigwedge_{\alpha \in J} \mu_\alpha = \bar{0}$ , then  $\bigvee_{\alpha \in J} \bar{1} - \mu_\alpha = \bar{1}$ . Since  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r$ - $(\tau_i, \tau_j)$ - $FT$ - $S$ -closed, there exists a finite set  $J_0 \subset J$  such that

$$\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\bar{1} - \mu_\alpha, r)) \geq r.$$

Since

$$\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\bar{1} - \mu_\alpha, r) = \bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r). \text{ Then, } \mathcal{I}(\bigwedge_{\alpha \in J_0} SI_{ij}(\mu_\alpha, r)) \geq r.$$

This is a contradiction.

(3)  $\implies$  (1) Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J_0\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Suppose there is no finite set  $J_0 \subset J$  satisfying  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$ . Since  $\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r) = \bigwedge_{\alpha \in J_0} SI_{ij}(\bar{1} - \mu_\alpha, r)$ , then the family  $\{SI_{ij}(\bar{1} - \mu_\alpha, r), \alpha \in J\}$  has  $\mathcal{I}\text{-FIP}$ . By (3), we have  $\bigwedge_{\alpha \in J} \bar{1} - \mu_\alpha \neq \bar{0}$ . Then  $\bigvee_{\alpha \in J} \mu_\alpha \neq \bar{1}$ . This is a contradiction. □

**Definition 4.11.** A smooth bts  $(X, \tau_1, \tau_2)$  is called an  $r\text{-}(\tau_i, \tau_j)\text{-fuzzy semiregular space}$  iff for each  $r\text{-}(\tau_i, \tau_j)\text{-fso set } \lambda \text{ in } X \text{ and } r \in I_0, \lambda = \bigvee\{\nu \in I^X \mid \nu \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, SC_{ij}(\nu, r) = \lambda\}$ .

**Theorem 4.12.** If  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)\text{-FT-S-closed}$  and  $r\text{-}(\tau_i, \tau_j)\text{-fuzzy semiregular space}$ , then  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)\text{-FST-compact}$ .

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be any family with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . By  $r\text{-}(\tau_i, \tau_j)\text{-fuzzy semiregular of } X$ , for each  $\alpha \in J, \mu_\alpha = \bigvee_{\alpha_k \in K_\alpha} \{\lambda_{\alpha_k} \mid \lambda_{\alpha_k} \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso}, SC_{ij}(\lambda_{\alpha_k}, r) = \mu_\alpha\}$ . Hence,  $\bigvee_{\alpha \in J} \mu_\alpha = \bigvee_{\alpha \in J} (\bigvee_{\alpha_k \in K_\alpha} \lambda_{\alpha_k}) = \bar{1}$ . Since  $X$  is an  $r\text{-}(\tau_i, \tau_j)\text{-FT-S-closed}$ , there exist finite sets  $J_0 \subset J$  and  $K_{J_0} \subset K_\alpha$ , such that

$$\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r))) \geq r \text{ for each } \alpha \in J_0.$$

Since,

$$\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r) \leq \mu_\alpha.$$

This implies that,

$$\bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r)) \leq \bigvee_{\alpha \in J_0} \mu_\alpha \text{ which also implies } \bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha \leq \bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r)).$$

Therefore,

$$\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq \mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} (\bigvee_{\alpha_k \in K_{J_0}} SC_{ij}(\lambda_{\alpha_k}, r))).$$

Thus,  $(X, \tau_1, \tau_2, \mathcal{I})$  is an  $r\text{-}(\tau_i, \tau_j)\text{-FST-compact}$ . □

## 5 FPS-compactness Modulo a Smooth Ideal and Mappings

In this section we show the types of FPS-compactness via a smooth ideal that is introduced in Section 3 and 4 which are preserved under some types of mappings. Throughout this section let  $(X, \tau_1, \tau_2, \mathcal{I})$  and  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be two smooth ideal bts's.

**Theorem 5.1.** Let  $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be a surjective, FP-irresolute mapping. If  $(X, \tau_1, \tau_2, \mathcal{I})$  is FPS-compact and  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$  for each  $\rho \in I^X$ , then  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is FPS-compact.

*Proof.* Let  $\{\mu_\alpha \in I^Y \mid \mu_\alpha \text{ is } r\text{-}(\tau_i^*, \tau_j^*)\text{-fso set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Then  $\bigvee_{\alpha \in J} f^{-1}(\mu_\alpha) = \bar{1}$ . Since  $f$  is  $FP$ -irresolute, then for each  $\alpha \in J$ ,  $f^{-1}(\mu_\alpha)$  is an  $r\text{-}(\tau_i, \tau_j)$ -fso set. By  $FPS\mathcal{I}$ -compactness of  $X$ , there exists a finite set  $J_0 \subset J$  with  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha)) \geq r$ . Since  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ , then for each  $\alpha \in J_0$  we have  $\mathcal{J}(f(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha))) \geq r$ . From the surjectivity of  $f$ , we have  $f(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha)) = \bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha$ . Thus  $\mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Hence,  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is an  $FPS\mathcal{J}$ -compact.  $\square$

**Theorem 5.2.** Let  $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be a surjective,  $FP$ -irresolute mapping. If  $(X, \tau_1, \tau_2, \mathcal{I})$  is  $FPSC(\mathcal{I})$ -compact and  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$  for each  $\rho \in I^X$ , then  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is  $FPSC(\mathcal{J})$ -compact.

*Proof.* Let  $\rho$  be an  $r\text{-}(\tau_i^*, \tau_j^*)$ -fsc set in  $Y$ , and let  $\{\mu_\alpha \in I^Y \mid \mu_\alpha \text{ is } r\text{-}(\tau_i^*, \tau_j^*)\text{-fso set}, \alpha \in J\}$  with  $\rho \leq \bigvee_{\alpha \in J} \mu_\alpha$ . Then,  $f^{-1}(\rho) \leq \bigvee_{\alpha \in J} f^{-1}(\mu_\alpha)$ . Since  $f$  is  $FP$ -irresolute, for each  $\alpha \in J$ ,  $f^{-1}(\mu_\alpha)$  is an  $r\text{-}(\tau_i, \tau_j)$ -fso set of  $X$  and  $f^{-1}(\rho)$  is an  $r\text{-}(\tau_i, \tau_j)$ -fso set in  $X$ . By  $FPSC(\mathcal{I})$ -compactness in  $X$ , there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(f^{-1}(\mu_\alpha), r)]) \geq r$ . Since  $f$  is  $FP$ -irresolute mapping, then from Theorem 2.8(3), we have  $SC_{ij}(f^{-1}(\rho), r) \leq f^{-1}(SC_{ij}(\rho, r))$  for every  $\rho \in I^Y$ . Hence

$$f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(f^{-1}(\mu_\alpha), r)] \geq f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))].$$

Thus,  $\mathcal{I}(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))]) \geq r$ . Since  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ . Then, for each  $\alpha \in J_0$ ,  $\mathcal{J}(f(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))])) \geq r$ . From the surjectivity of  $f$ ,

$$f(f^{-1}(\rho) \wedge [\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))]) = f(f^{-1}(\rho) \wedge (\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r))) = \rho \wedge (\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)).$$

Thus,  $\mathcal{J}(\rho \wedge (\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r))) \geq r$ . Hence  $Y$  is an  $FPSC(\mathcal{J})$ -compact.  $\square$

**Theorem 5.3.** Let  $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be a surjective,  $FP$ -irresolute mapping. If  $(X, \tau_1, \tau_2, \mathcal{I})$  is  $FPS\mathcal{I}$ - $S$ -closed and  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$  for each  $\rho \in I^X$ , then  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is  $FPS\mathcal{J}$ - $S$ -closed.

*Proof.* Similar to proof of Theorem 5.2.  $\square$

In order to complete our study of the properties of  $FPS$ -compactness via a smooth ideal under mappings, we need now to introduce the notion of  $FP$ -weakly semi-continuous mapping.

**Definition 5.4.** Let  $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$  be a mapping. Then  $f$  is called a  $FP$ -weakly semi-continuous iff  $f^{-1}(\mu) \leq SI_{ij}(f^{-1}(SC_{ij}(\mu, r)), r)$ ,  $\mu \in I^Y$ .

**Theorem 5.5.** Let  $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be a surjective,  $FP$ -weakly semi-continuous mapping. If  $(X, \tau_1, \tau_2, \mathcal{I})$  is  $FPS\mathcal{I}$ -compact and  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$  for each  $\rho \in I^X$ , then  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is  $FPS\mathcal{J}$ - $S$ -closed.

*Proof.* Let  $\{\mu_\alpha \in I^Y \mid \mu_\alpha \text{ is } r\text{-}(\tau_i^*, \tau_j^*)\text{-fso set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Then  $\bigvee_{\alpha \in J} f^{-1}(\mu_\alpha) = \bar{1}$ . Since  $f$  is a  $FP$ -weakly semi-continuous, then for each  $\alpha \in J$ ,  $f^{-1}(\mu_\alpha) \leq SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)$ . Hence,  $\bigvee_{\alpha \in J} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r) = \bar{1}$ . Since  $\{SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r) \in I^X, \alpha \in J\}$  is a family of  $r\text{-}(\tau_i, \tau_j)$ -fso sets and by  $FPS\mathcal{I}$ -compactness of  $X$ , there exists a finite set  $J_0 \subset J$  such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)) \geq r$ . Since  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$ ,  $\mathcal{J}(f(\bar{1} - \bigvee_{\alpha \in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r))) \geq r$  for each  $\alpha \in J_0$ . From the surjectivity of  $f$ ,

$$f(\bar{1} - \bigvee_{\alpha \in J_0} SI_{ij}(f^{-1}(SC_{ij}(\mu_\alpha, r)), r)) \geq f(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(SC_{ij}(\mu_\alpha, r))) = \bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r).$$

Thus,  $\mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} SC_{ij}(\mu_\alpha, r)) \geq r$ . Hence,  $Y$  is  $FP\mathcal{J}$ - $S$ -closed. □

**Theorem 5.6.** The image of  $FPST$ -compact set under surjective,  $FP$ -weakly semi-continuous mapping such that  $\mathcal{I}(\rho) \leq \mathcal{J}(f(\rho))$  for each  $\rho \in I^X$ , is  $FP\mathcal{J}$ - $S$ -closed.

*Proof.* Similar to proof of Theorem 5.5. □

The following theorem shows that the image of a smooth ideal is a smooth ideal.

**Theorem 5.7.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be a mapping from a smooth topological space  $(X, \tau)$  into a smooth topological space  $(Y, \delta)$ . If  $\mathcal{I}$  is a smooth ideal on  $X$ , then  $f(\mathcal{I})$  is a smooth ideal on  $Y$  defined as follows:

$$f(\mathcal{I})(\mu) = \begin{cases} \bigvee_{f(\lambda)=\mu} \mathcal{I}(\lambda) & \text{if } f^{-1}(\mu) \neq \bar{0} \\ 0 & \text{if } f^{-1}(\mu) = \bar{0} \end{cases}$$

*Proof.* Direct. □

**Lemma 5.8.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be a mapping from a smooth topological space  $(X, \tau)$  into a smooth topological space  $(Y, \delta)$  and  $\mathcal{I}$  be a smooth ideal on  $X$ . If  $\mathcal{I}(\nu) \geq r$ , then  $f(\mathcal{I})(f(\nu)) \geq r$ ,  $\nu \in I^X$ .

*Proof.* Obvious. □

**Theorem 5.9.** Let  $f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$  be a surjective,  $FP$ -irresolute mapping. If  $(X, \tau_1, \tau_2, \mathcal{I})$  is  $FPST$ -compact, then  $(Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$  is  $FP Sf(\mathcal{I})$ -compact.

*Proof.* Let  $\{\mu_\alpha \in I^Y \mid \mu_\alpha \text{ is } r\text{-}(\tau_i^*, \tau_j^*)\text{-fso set}, \alpha \in J\}$  be any family such that  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . Since  $f$  is  $FP$ -irresolute, then  $\{f^{-1}(\mu_\alpha) \in I^X \mid f^{-1}(\mu_\alpha) \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  with  $\bigvee_{\alpha \in J} f^{-1}(\mu_\alpha) = \bar{1}$ . By hypothesis, there exists a finite set  $J_0 \subset J$ , such that  $\mathcal{I}(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha)) \geq r$ . By Lemma 5.8 we have,  $f(\mathcal{I})(f(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha))) \geq r$ . From the surjective of  $f$ ,  $f(\bar{1} - \bigvee_{\alpha \in J_0} f^{-1}(\mu_\alpha)) = \bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha$ . Then,  $f(\mathcal{I})(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Hence,  $(Y, \tau_1^*, \tau_2^*, f(\mathcal{I}))$  is an  $FP Sf(\mathcal{I})$ -compact. □

The following theorem shows that the inverse image of a smooth ideal is a smooth ideal.

**Theorem 5.10.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be a mapping from a smooth topological space  $(X, \tau)$  into a smooth topological space  $(Y, \delta)$ . If  $\mathcal{J}$  is a smooth ideal on  $Y$ , then  $f^{-1}(\mathcal{J})$  is a smooth ideal on  $X$  defined as follows:

$$f^{-1}(\mathcal{J})(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{1} \\ \mathcal{J}(f(\lambda)) & \text{for all } \lambda \in I^X \end{cases}$$

*Proof.* Direct. □

**Lemma 5.11.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be a surjective mapping from a smooth topological space  $(X, \tau)$  into a smooth topological space  $(Y, \delta)$  and let  $\mathcal{J}$  be a smooth ideal on  $Y$ . If  $\mathcal{J}(\nu) \geq r$ , then  $f^{-1}(\mathcal{J})(f^{-1}(\nu)) \geq r$ .

*Proof.* Obvious. □

**Theorem 5.12.** Let  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*, \mathcal{J})$  be a bijective,  $FP$ -irresolute open mapping. If  $(Y, \tau_1^*, \tau_2^*, \mathcal{J})$  is  $FP S\mathcal{J}$ -compact, then  $(X, \tau_1, \tau_2, f^{-1}(\mathcal{J}))$  is  $FP Sf^{-1}(\mathcal{J})$ -compact.

*Proof.* Let  $\{\mu_\alpha \in I^X \mid \mu_\alpha \text{ is } r\text{-}(\tau_i, \tau_j)\text{-fso set}, \alpha \in J\}$  be any family with  $\bigvee_{\alpha \in J} \mu_\alpha = \bar{1}$ . From the surjective and  $FP$ -irresolute open of  $f$  we have  $\bigvee_{\alpha \in J} f(\mu_\alpha) = \bar{1}$ , such that for each  $\alpha \in J$ ,  $f(\mu_\alpha)$  is an  $r\text{-}(\tau_i^*, \tau_j^*)\text{-fso}$  set in  $Y$ . By  $FP S\mathcal{J}$ -compactness of  $Y$ , there exists the finite set  $J_0 \subset J$  such that  $\mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} f(\mu_\alpha)) \geq r$ . From Lemma 5.11,  $f^{-1}(\mathcal{J})(f^{-1}(\bar{1} - \bigvee_{\alpha \in J_0} f(\mu_\alpha))) \geq r$ . Since  $f$  is an injective, we have:

$$f^{-1}(\mathcal{J})(f^{-1}(\bar{1} - \bigvee_{\alpha \in J_0} f(\mu_\alpha))) = \mathcal{J}(f(f^{-1}(\bar{1} - \bigvee_{\alpha \in J_0} f(\mu_\alpha)))) = \mathcal{J}(\bar{1} - \bigvee_{\alpha \in J_0} f(\mu_\alpha)) = \mathcal{J}(f(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha)).$$

Then  $f^{-1}(\mathcal{J})(\bar{1} - \bigvee_{\alpha \in J_0} \mu_\alpha) \geq r$ . Hence,  $(X, \tau_1, \tau_2, f^{-1}(\mathcal{J}))$  is  $FP Sf^{-1}(\mathcal{J})$ -compact. □

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