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Q-FUZZY IDEAL OF ORDERED Γ-SEMIRING

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Abstract – The notion of Q-fuzzy ideal in ordered Γ-semiring is introduced and studied along with some operations. Among all other results, it is shown that the set of all Q-fuzzy ideals of a Γ-semiring forms a complete lattice. They also form a zerosumfree Γ-semiring under the operations of sum and composition of Q-fuzzy ideals.

Keywords – *Γ-semiring, Q-fuzzy, intersection, lattice, normal.*

1 Introduction

The fundamental concept of fuzzy set, introduced by Zadeh [10], provides a natural frame-work for generalizing several basic notions of algebra . Jun and Lee [5] applied the concept of fuzzy sets to the theory of Γ-rings. The notion of Γ-semiring was introduced by Rao [9] as a generalization of Γ-ring as well as of semiring [3].
Majumder [8] introduced and studied the concept of Q-fuzzifcation of ideals of Γ-semigroups. Akram et al [1], Lekkoksung [6, 7] extended this concept in case of Γ-semigroup and ordered semigroups [4] and investigated some important properties.
Main object of the present paper is to define ordered Γ-semiring and study its ideals using the concept of Q-fuzzification.

2 Preliminary

Definition 2.1. A semiring is a system consisting of a non-empty set S on which operations addition and multiplication (denoted in the usual manner) have been defined such that $(S, +)$ is a semigroup, $(S, \cdot)$ is a semigroup and multiplication distributes over addition from either side.
A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring S is zerosumfree if and only if $s + s' = 0$ implies that $s = s' = 0$.

Definition 2.2. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ($(a, \alpha, b) \mapsto a\alpha b$) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,

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(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,

(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

(v) $0_S\alpha a = 0_S = a\alpha 0_S$,

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Example 2.3. Let S be the set of all $m \times n$ matrices over $\mathbf{Z}_0^-$(the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbf{Z}_0^-$, then S forms a Γ-semiring with usual addition and multiplication of matrices.

Definition 2.4. A left ideal I of Γ-semiring S is a nonempty subset of S satisfying the following conditions:

(i) If $a, b \in I$ then $a + b \in I$,

(ii) If $a \in I$, $s \in S$ and $\gamma \in \Gamma$ then $s\gamma a \in I$.

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.5. An ordered Γ-semiring is a Γ-semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.6. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e., $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Now we recall the example of ordered ideal from [2].

Example 2.7. Let $S = ([0,1], \vee, \cdot, 0)$ where [0,1] is the unit interval, $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0,1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.8. A fuzzy subset f of a non-empty set S is defined as a mapping from S to [0,1].

Definition 2.9. A function μ from $S \times Q$ to the real closed interval [0, 1] is called Q-fuzzy subset of S, where Q is a non-empty set.

Definition 2.10. Let μ be a Q-fuzzy subset of a set S and $t \in$[0,1]. The set

$$\mu_t = \{(x,q) \in S \times Q | \mu(x,q) \geq t\}$$

is called the level subset of μ. Clearly, $\mu_t \subseteq \mu_s$, whenever $t \geq s$.

Definition 2.11. The characteristic function $\chi_{A\times Q}$ of $A \times Q$, is the mapping of $S \times Q$ to $[0,1]$ defined by

$$\begin{aligned} \chi_{A\times Q}(x,q) &= 1 \text{ if } (x,q) \in A \times Q \\ &= 0, \text{ if } (x,q) \notin A \times Q \end{aligned}$$

Definition 2.12. The union and intersection of two Q-fuzzy subsets μ and σ of a set S, denoted by $\mu \cup \sigma$ and $\mu \cap \sigma$ respectively, are defined as

$$(\mu \cup \sigma)(x,q) = \max\{\mu(x,q), \sigma(x,q)\} \text{ for all } x \in S,\ q \in Q$$

$$(\mu \cap \sigma)(x,q) = \min\{\mu(x,q), \sigma(x,q)\} \text{ for all } x \in S,\ q \in Q.$$

3 Main Results

Throughout this paper unless otherwise mentioned S denotes the ordered Γ-semiring.

Definition 3.1. Let μ and ν be two Q-fuzzy subsets of an ordered Γ-semiring S and $x,\ y,\ z \in S$, $\gamma \in \Gamma$, $q \in Q$. We define composition and sum of μ and ν as follows:

$$\begin{aligned} \mu \circ_1 \nu(x,q) &= \sup_{x \leq y\gamma z}\{\min\{\mu(y,q), \nu(z,q))\}\} \\ &= 0,\ \text{if } x \text{ cannot be expressed as } x \leq y\gamma z \end{aligned}$$

and

$$\begin{aligned} \mu +_1 \nu(x,q) &= \sup_{x \leq y+z}\{\min\{\mu(y,q), \nu(z,q))\}\} \\ &= 0,\ \text{if } x \text{ cannot be expressed as } x \leq y+z. \end{aligned}$$

Proposition 3.2. For any Q-fuzzy subset μ of an ordered Γ-semiring S, $(\chi_{S\times Q} o_1 \mu)(x,q) \geq (\chi_{S\times Q} o_1 \mu)(y,q)$ (resp. $(\chi_{S\times Q} +_1 \mu)(x,q) \geq (\chi_{S\times Q} +_1 \mu)(y,q)$) $\forall x,\ y \in S$, $q \in Q$ with $x \leq y$.

Proof. Let μ be a Q-fuzzy subset of an ordered Γ-semiring S and $x,\ y \in S$ with $x \leq y$. If y cannot be expressed as $y \leq y_1\gamma y_2$ for $y_1,\ y_2 \in S$ and $\gamma \in \Gamma$ then the proof is trivial so we omit it.
Let y have such an expression. Then

$$(\chi_{S\times Q} o_1 \mu)(y,q) = \sup_{y \leq y_1\gamma y_2}\{\min\{\chi_{S\times Q}(y_1,q), \mu(y_2,q)\}\} = \sup_{y \leq y_1\gamma y_2}\{\mu(y_2,q)\}.$$

Since $x \leq y \leq y_1\gamma y_2$, we have

$$\begin{aligned} (\chi_{S\times Q} o_1 \mu)(x,q) &= \sup_{x \leq x_1\gamma x_2}\{\min\{\chi_{S\times Q}(x_1,q), \mu(x_2,q)\}\} \\ &\geq \sup_{x \leq y_1\gamma y_2}\{\min\{\chi_{S\times Q}(y_1,q), \mu(y_2,q)\}\} \\ &= \sup_{y \leq y_1\gamma y_2}\{\mu(y_2,q)\} = (\chi_{S\times Q} o_1 \mu)(y,q). \end{aligned}$$

Similarly for $x \leq y$, we can prove that $(\chi_{S\times Q} +_1 \mu)(x,q) \geq (\chi_{S\times Q} +_1 \mu)(y,q)$. □

Definition 3.3. Let μ be a non empty Q-fuzzy subset of an ordered Γ-semiring S (i.e., $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a Q-fuzzy left ideal [resp. Q-fuzzy right ideal] of S if

(i) $\mu(x+y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$,

(ii) $\mu(x\gamma y,q) \geq \mu(y,q)$ [resp. $\mu(x\gamma y,q) \geq \mu(x,q)$] and

(iii) $x \leq y$ implies $\mu(x,q) \geq \mu(y,q)$.

for all $x, y \in S$, $\gamma \in \Gamma$ and $q \in Q$.
By a Q-fuzzy ideal we mean, it is both a Q-fuzzy left ideal as well as a Q-fuzzy right ideal.

Theorem 3.4. A Q-fuzzy subset μ of S is a Q-fuzzy ordered ideal if and only if its level subset μ_t, $t \in [0,1]$ is an ordered ideal of $S \times Q$.

Proof. We only prove the theorem only for left ordered ideal. For right ordered ideal it follows similarly.

Let μ be a Q-fuzzy left ordered ideal of S. Suppose $a \in S$ and $b \in \mu_t$ with $a \leq b$. As μ is a Q-fuzzy left ordered ideal of S, $\mu(a,q) \geq \mu(b,q) \geq t$ so that $a \in \mu_t$ i.e., μ_t is a left ordered ideal of $S \times Q$.

Conversely, if μ_t is a left ordered ideal of $S \times Q$, then μ is a Q-fuzzy ideal of S. Now suppose $x, y \in S$ with $x \leq y$. We have to show that $\mu(x,q) \geq \mu(y,q)$. Let $\mu(x,q) < \mu(y,q)$. Then there exists $t_1 \in [0,1]$ such that $\mu(x,q) < t_1 < \mu(y,q)$. Then $(y,q) \in \mu_{t_1}$ but $(x,q) \notin \mu_{t_1}$ which is a contradiction to the fact that μ_t is a left ordered ideal of $S \times Q$.

□

Definition 3.5. Let μ be a Q-fuzzy subset of an ordered Γ-semiring S and $a \in S$. We denote I_a the subset of $S \times Q$ defined as follows:

$$I_a = \{(b,q) \in S \times Q | \mu(b,q) \geq \mu(a,q)\}.$$

Proposition 3.6. Let S be an ordered Γ-semiring and μ be a Q-fuzzy right (resp. left) ideal of S. Then I_a is a right (resp. left) ideal of $S \times Q$ for every $a \in S$.

Proof. Let μ be a Q-fuzzy right ideal of S and $a \in S$, $q \in Q$. Then $I_a \neq \phi$ because $(a, q) \in I_a$ for every $(a, q) \in S \times Q$. Let (b, q), $(c, q) \in I_a$ and $x \in S$. Since (b, q), $(c, q) \in I_a$, $\mu(b, q) \geq \mu(a, q)$ and $\mu(c, q) \geq \mu(a, q)$. Now

$$\begin{aligned} \mu(b+c, q) &\geq \min\{\mu(b, q), \mu(c, q)\}[\because \mu \text{ is a } Q\text{-fuzzy right ideal}] \\ &\geq \mu(a, q). \end{aligned}$$

which implies $(b + c, q) \in I_a$.

Also $\mu(b\gamma x, q) \geq \mu(b, q) \geq \mu(a, q)$ i.e. $(b\gamma x, q) \in I_a$.

Let $(b, q) \in I_a$ and $S \ni x \leq b$. Then $\mu(x, q) \geq \mu(b, q) \geq \mu(a, q) \Rightarrow (x, q) \in I_a$.

Thus I_a is a right ideal of $S \times Q$.

Similarly, we can prove the result for left ideal also. □

Proposition 3.7. Intersection of a non-empty collection of Q-fuzzy right (resp. left) ideals is also a Q-fuzzy right (resp. left) ideal of S.

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of Q-fuzzy right ideals of S and $x, y \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned} \bigcap_{i \in I} \mu_i(x+y, q) &= \inf_{i \in I}\{\mu_i(x+y, q)\} \geq \inf_{i \in I}\{\min\{\mu_i(x, q), \mu_i(y, q)\}\} \\ &= \min\{\inf_{i \in I} \mu_i(x, q), \inf_{i \in I} \mu_i(y, q)\} = \min\{\bigcap_{i \in I} \mu_i(x, q), \bigcap_{i \in I} \mu_i(y, q)\}. \end{aligned}$$

Again

$$\bigcap_{i \in I} \mu_i(x\gamma y, q) = \inf_{i \in I}\{\mu_i(x\gamma y, q)\} \geq \inf_{i \in I}\{\mu_i(x, q)\} = \bigcap_{i \in I} \mu_i(x, q).$$

Suppose $x \leq y$. Then $\mu_i(x, q) \geq \mu_i(y, q)$ for all $i \in I$ which implies $\bigcap_{i \in I} \mu_i(x, q) \geq \bigcap_{i \in I} \mu_i(y, q)$.

Hence $\bigcap_{i \in I} \mu_i$ is a Q-fuzzy right ideal of S.

Similarly, we can prove the result for Q-fuzzy left ideal also. □

Proposition 3.8. Let $\{\mu_i : i \in I\}$ be a family of Q-fuzzy ideals of S such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for $i, j \in I$. Then $\underset{i \in I}{\cup} \mu_i$ is a Q-fuzzy ideal of S.

Proof. The proof follows by routine verification. □

Definition 3.9. Let f be a function from a set X to a set Y; μ be a Q-fuzzy subset of X and σ be a Q-fuzzy subset of Y.

Then image of μ under f, denoted by $f(\mu)$, is a Q-fuzzy subset of Y defined by

$$f(\mu)(y, q) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x, q) & \text{if } f^{-1}(y) \neq \phi \\ 0 \quad \text{otherwise} \end{cases}$$

The pre-image of σ under f, symbolized by $f^{-1}(\sigma)$, is a Q-fuzzy subset of X defined by

$$f^{-1}(\sigma)(x, q) = \sigma(f(x), q) \; \forall x \in X.$$

Proposition 3.10. Let $f : R \to S$ be a morphism of ordered Γ-semirings i.e. Γ-semiring homomorphism satisfying additional condition $a \leq b \Rightarrow f(a) \leq f(b)$. Then if ϕ is a Q-fuzzy left ideal of S, then $f^{-1}(\phi)$ is also a Q-fuzzy left ideal of R.

Proof. Let $f : R \to S$ be a morphism of ordered Γ-semirings and ϕ is a Q-fuzzy left ideal of S and $q \in Q$, $\gamma \in \Gamma$.

Now $f^{-1}(\phi)(0_R, q) = \phi(0_S, q) \geq \phi(x', q) \neq 0$ for some $x' \in S$.

Therefore $f^{-1}(\phi)$ is non-empty.

Now, for any $r, s \in R$

$$\begin{aligned} f^{-1}(\phi)(r+s,q) &= \phi(f(r+s),q) = \phi(f(r)+f(s),q) \\ &\geq \min\{\phi(f(r),q), \phi(f(s),q)\} \\ &= \min\{(f^{-1}(\phi))(r,q), (f^{-1}(\phi))(s,q)\}. \end{aligned}$$

Again

$$(f^{-1}(\phi))(r\gamma s, q) = \phi(f(r\gamma s), q) = \phi(f(r)\gamma f(s), q) \geq \phi(f(s), q) = (f^{-1}(\phi))(s,q).$$

Also if $r \leq s$, then $f(r) \leq f(s)$. Then

$$(f^{-1}(\phi))(r,q) = \phi(f(r),q) \geq \phi(f(s),q) = (f^{-1}(\phi))(s,q).$$

Thus $f^{-1}(\phi)$ is a Q-fuzzy left ideal of R. □

Definition 3.11. Let μ and ν be Q-fuzzy subsets of X. The cartesian product of μ and ν is defined by $(\mu \times \nu)((x,y),q) = \min\{\mu(x,q), \nu(y,q)\}$ for all $x, y \in X$ and $q \in Q$.

Theorem 3.12. Let μ and ν be fuzzy left ideals of an ordered Γ-semiring S. Then $\mu \times \nu$ is a Q-fuzzy left ideal of $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$, $\gamma \in \Gamma$ and $q \in Q$. Then

$$\begin{aligned} &(\mu \times \nu)((x_1,x_2)+(y_1,y_2),q) \\ &= (\mu \times \nu)((x_1+y_1, x_2+y_2),q) \\ &= \min\{\mu(x_1+y_1,q), \nu(x_2+y_2,q)\} \\ &\geq \min\{\min\{\mu(x_1,q),\mu(y_1,q)\}, \min\{\nu(x_2,q),\nu(y_2,q)\}\} \\ &= \min\{\min\{\mu(x_1,q),\nu(x_2,q)\}, \min\{\mu(y_1,q),\nu(y_2,q)\}\} \\ &= \min\{(\mu \times \nu)((x_1,x_2),q), (\mu \times \nu)((y_1,y_2),q)\} \end{aligned}$$

and

$$\begin{aligned} (\mu \times \nu)((x_1,x_2)\gamma(y_1,y_2),q) &= (\mu \times \nu)((x_1\gamma y_1, x_2\gamma y_2),q) \\ &= \min\{\mu(x_1\gamma y_1,q), \nu(x_2\gamma y_2,q)\} \\ &\geq \min\{\mu(y_1,q), \nu(y_2,q)\} \\ &= (\mu \times \nu)((y_1,y_2),q). \end{aligned}$$

Also if $(x_1,x_2) \leq (y_1,y_2)$, then

$$(\mu \times \nu)((x_1,x_2),q) = \min\{\mu(x_1,q),\nu(x_2,q)\} \geq \min\{\mu(y_1,q),\nu(y_2,q)\}.$$

Therefore $\mu \times \nu$ is a Q-fuzzy left ideal of $S \times S$. □

Proposition 3.13. For any three Q-fuzzy subset $\mu_1,\ \mu_2,\ \mu_3$ of an ordered Γ-semiring S, $\mu_1 o_1 (\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$.

Proof. Let $\mu_1,\ \mu_2,\ \mu_3$ be any three fuzzy subset of an ordered Γ-semiring S and $x \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned} &(\mu_1 o_1 (\mu_2 +_1 \mu_3))(x,q) \\ &= \sup_{x \leq y\gamma z} \{\min\{\mu_1(y,q), (\mu_2 +_1 \mu_3)(z,q)\}\} \\ &= \sup_{x \leq y\gamma z} \{\min\{\mu_1(y,q), \sup_{z \leq a+b} \{\min\{\mu_2(a,q), \mu_3(b,q)\}\}\}\} \\ &= \sup\{\min\{\sup\{\min\{\mu_1(y,q),\mu_2(a,q)\}\}, \sup\{\min\{\mu_1(y,q),\mu_3(b,q)\}\}\}\} \\ &\leq \sup_{x \leq y\gamma a + y\gamma b}\{\min\{(\mu_1 o_1 \mu_2)(y\gamma a,q), (\mu_1 o_1 \mu_3)(y\gamma b,q)\}\} \\ &\leq ((\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3))(x,q). \end{aligned}$$

Also

$$((\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3))(x,q)$$
$$= \sup_{x \le x_1 + x_2}\{\min\{(\mu_1 o_1 \mu_2)(x_1,q), (\mu_1 o_1 \mu_3)(x_2,q)\}\}$$
$$= \sup_{x \le x_1 + x_2}\{\min\{\sup_{x_1 \le c_1\gamma_1 d_1}\{\min\{\mu_1(c_1,q),\mu_2(d_1,q)\}\}, \sup_{x_2 \le c_2\gamma_2 d_2}\{\min\{\mu_1(c_2,q),\mu_3(d_2,q)\}\}\}\}$$
$$\le \sup_{x \le x_1+x_2 \le c_1\gamma_1 d_1 + c_2\gamma_2 d_2 < (c_1+c_2)\gamma(d_1+d_2)}\{\min\{\mu_1(c_1+c_2,q), \sup\{\min\{\mu_2(d_1,q),\mu_3(d_2,q)\}\}\}\}$$
$$\le \sup_{x \le c\gamma d}\{\min\{\mu_1(c,q),(\mu_2 +_1 \mu_3)(d,q)\}\}$$
$$= (\mu_1 o_1(\mu_2 +_1 \mu_3))(x,q).$$

Therefore $\mu_1 o_1(\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$. □

Theorem 3.14. If $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals of an ordered Γ-semiring S, then $\mu_1 +_1 \mu_2$ is also so.

Proof. Assume that $\mu_1,\ \mu_2$ are any two fuzzy ideals of an ordered Γ-semiring S and $x,\ y \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned}(\mu_1 +_1 \mu_2)(x+y,q) &= \sup_{x+y \le c+d}\{\min\{\mu_1(c,q),\mu_2(d,q)\}\}\\ &\ge \sup_{x+y \le (a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)}\{\min\{\mu_1(a_1+a_2,q),\mu_2(b_1+b_2,q)\}\}\\ &\ge \sup\{\min\{\mu_1(a_1,q),\mu_1(a_2,q),\mu_2(b_1,q),\mu_2(b_2,q)\}\}\\ &= \min\{\sup_{x \le a_1+b_1}\{\min\{\mu_1(a_1,q),\mu_2(b_1,q)\}\}, \sup_{y \le a_2+b_2}\{\min\{\mu_1(a_2,q),\mu_2(b_2,q)\}\}\}\\ &= \min\{(\mu_1 +_1 \mu_2)(x,q),(\mu_1 +_1 \mu_2)(y,q)\}.\end{aligned}$$

Now assume $\mu_1,\ \mu_2$ are as Q-fuzzy right ideals and we have

$$\begin{aligned}(\mu_1 +_1 \mu_2)(x\gamma y,q) &= \sup_{x\gamma y \le c+d}\{\min\{\mu_1(c,q),\mu_2(d,q)\}\}\\ &\ge \sup_{x\gamma y \le (x_1+x_2)\gamma y}\{\min\{\mu_1(x_1\gamma y,q),\mu_2(x_2\gamma y,q)\}\}\\ &\ge \sup_{x \le x_1+x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &= (\mu_1 +_1 \mu_2)(x,q).\end{aligned}$$

Similarly assuming $\mu_1,\ \mu_2$ are as fuzzy left ideal, we can show that

$$(\mu_1 +_1 \mu_2)(x\gamma y,q) \ge (\mu_1 +_1 \mu_2)(y,q).$$

Now suppose $x \le y$. Then $\mu_1(x) \ge \mu_1(y)$ and $\mu_2(x) \ge \mu_2(y)$.

$$\begin{aligned}(\mu_1 +_1 \mu_2)(x,q) &= \sup_{x \le x_1+x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &\ge \sup_{x \le y \le y_1+y_2}\{\min\{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= \sup_{y \le y_1+y_2}\{\min\{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= (\mu_1 +_1 \mu_2)(y,q).\end{aligned}$$

Hence $\mu_1 +_1 \mu_2$ is a Q-fuzzy ideal of S. □

Theorem 3.15. If $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals an ordered Γ-semiring S, then $\mu_1 o_1 \mu_2$ is also so.

Proof. Let $\mu_1,\ \mu_2$ be any two Q-fuzzy ideals of an ordered Γ-semiring S and $x,\ y \in S$, $\gamma \in \Gamma$, $q \in Q$. Then

$$\begin{aligned}(\mu_1 o_1 \mu_2)(x+y,q) &= \sup_{x+y \le c\gamma d}\{\min\{\mu_1(c,q),\mu_2(d,q)\}\}\\ &\ge \sup_{x+y \le c_1\gamma_1 d_1 + c_2\gamma_2 d_2 < (c_1+c_2)\gamma(d_1+d_2)}\{\min\{\mu_1(c_1+c_2,q), \mu_2(d_1+d_2,q)\}\}\\ &\ge \sup\{\min\{\mu_1(c_1,q),\mu_1(c_2,q),\mu_2(d_1,q),\mu_2(d_2,q)\}\}\\ &\ge \min\{\sup_{x \le c_1\gamma_1 d_1}\{\min\{\mu_1(c_1,q),\mu_2(d_1,q)\}\}, \sup_{y \le c_2\gamma_2 d_2}\{\min\{\mu_1(c_2,q),\mu_2(d_2,q)\}\}\}\\ &= \min\{(\mu_1 o_1 \mu_2)(x),(\mu_1 o_1 \mu_2)(y)\}.\end{aligned}$$

Now assume $\mu_1,\ \mu_2$ are as Q-fuzzy right ideals and we have

$$\begin{aligned}(\mu_1 o_1 \mu_2)(x\gamma y) &= \sup_{x\gamma y\leq c\gamma d}\{\min\{\mu_1(c,q),\mu_2(d,q)\}\}\\ &\geq \sup_{x\gamma y\leq (x_1\gamma_1 x_2)\gamma_2 y}\{\min\{\mu_1(x_1,q),\mu_2(x_2\gamma_2 y,q)\}\}\\ &\geq \sup_{x\leq x_1\gamma_1 x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &= (\mu_1 o_1 \mu_2)(x,q).\end{aligned}$$

Similarly assuming $\mu_1,\ \mu_2$ are as Q-fuzzy left ideal, we can show that $(\mu_1 o_1 \mu_2)(x\gamma y,q) \geq (\mu_1 o_1 \mu_2)(y,q)$.

Now suppose $x \leq y$. Then $\mu_1(x) \geq \mu_1(y)$ and $\mu_2(x) \geq \mu_2(y)$.

$$\begin{aligned}(\mu_1 o_1 \mu_2)(x,q) &= \sup_{x\leq x_1\gamma x_2}\{\min\{\mu_1(x_1,q),\mu_2(x_2,q)\}\}\\ &\geq \sup_{x\leq y\leq y_1\gamma y_2}\{\min\{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= \sup_{y\leq y_1\gamma y_2}\{\min\{\mu_1(y_1,q),\mu_2(y_2,q)\}\}\\ &= (\mu_1 o_1 \mu_2)(y,q).\end{aligned}$$

Hence $\mu_1 o_1 \mu_2$ is a Q-fuzzy ideal of S. □

Theorem 3.16. The set of all Q-fuzzy left ideals of S form a complete lattice.

Proof. Suppose the set of all Q-fuzzy left ideals of S is denoted by $FLI(S)$.
Now, for $\mu_1,\mu_2 \in FLI(S)$, define a relation $\leq$ such that $\mu_1 \leq \mu_2$ if and only if $\mu_1(x,q) \leq \mu_2(x,q)$ for all $x \in S,\ q \in Q$. Then $FLI(S)$ is a poset with respect to the relation $\leq$.
Now, for every pair of elements say μ_1,μ_2 of $FLI(S)$, we see that $\mu_1+\mu_2$ is the least upper bound and $\mu_1 \cap \mu_2$ is the greatest lower bound of μ_1 and μ_2. Thus $FLI(S)$ is a lattice.
Suppose ψ is a fuzzy subset of S such that $\psi(x,q)=1$ for all $x \in S,\ q \in Q$. Then $\psi \in FLI(S)$ and for all $\mu \in FLI(S)$, $\mu(x,q) \leq \psi(x,q)$ for all $x \in S,\ q \in Q$. So, ψ is the greatest element of $FLI(S)$.
Let $\{\mu_i : i \in I\}$ be a non-empty family of Q-fuzzy left ideals of S. Then $\bigcap_{i\in I}\mu_i \in FLI(S)$. Also it is the greatest lower bound of $\{\mu_i : i \in I\}$.
Hence $FLI(S)$ is a complete lattice. □

Definition 3.17. Let μ be a Q-fuzzy subset of X and $\alpha \in [0,1\text{-}\sup\{\mu(x,q) : x \in X,\ q \in Q\}]$, $\beta \in [0,1]$. The mappings $\mu^T_\alpha : X \to [0,1]$, $\mu^M_\beta : X \to [0,1]$ and $\mu^{MT}_{\beta,\alpha} : X \to [0,1]$ are called a Q-fuzzy translation, Q-fuzzy multiplication and Q-fuzzy magnified translation of μ respectively if $\mu^T_\alpha(x,q) = \mu(x,q)+\alpha$, $\mu^M_\beta(x,q) = \beta.\mu(x,q)$ and $\mu^{MT}_{\beta,\alpha}(x,q) = \beta.\mu(x,q)+\alpha$ for all $x \in X,\ q \in Q$.

Theorem 3.18. Let μ be a Q-fuzzy subset of S and $\alpha \in [0,1\text{-}\sup\{\mu(x) : x \in X\}]$, $\beta \in (0,1]$. Then μ is a Q-fuzzy left ideal of S if and only if $\mu^{MT}_{\beta,\alpha}$, the Q-fuzzy magnified translation of μ, is also a Q-fuzzy left ideal of S.

Proof. Suppose μ is a Q-fuzzy left ideal of S. Let $x,y \in S,\ \gamma \in \Gamma,\ q \in Q$. Then

$$\begin{aligned}\mu^{MT}_{\beta,\alpha}(x+y,q) &= \beta.\mu(x+y,q)+\alpha\\ &\geq \beta.\min\{\mu(x,q),\mu(y,q)\}+\alpha\\ &= \min\{\beta.\mu(x,q),\beta.\mu(y,q)\}+\alpha\\ &= \min\{\beta.\mu(x,q)+\alpha,\beta.\mu(y,q)+\alpha\}\\ &= \min\{\mu^{MT}_{\beta,\alpha}(x,q),\mu^{MT}_{\beta,\alpha}(y,q)\}\end{aligned}$$

and

$$\mu^{MT}_{\beta,\alpha}(x\gamma y) = \beta.\mu(x\gamma y,q)+\alpha \geq \beta.\mu(y,q)+\alpha = \mu^{MT}_{\beta,\alpha}(y,q).$$

Therefore $\mu^{MT}_{\beta,\alpha}$ is a Q-fuzzy left ideal of S.
Conversely, suppose $\mu^{MT}_{\beta,\alpha}$ is a Q-fuzzy left ideal of S. Then for $x,y \in S,\ \gamma \in \Gamma,\ q \in Q$,

$$\begin{aligned}&\mu^{MT}_{\beta,\alpha}(x+y,q) \geq \min\{\mu^{MT}_{\beta,\alpha}(x,q),\mu^{MT}_{\beta,\alpha}(y,q)\}\\ &\Rightarrow \beta.\mu(x+y,q)+\alpha \geq \min\{\beta.\mu(x,q)+\alpha,\beta.\mu(y,q)+\alpha\}\\ &\Rightarrow \beta.\mu(x+y,q)+\alpha \geq \min\{\beta.\mu(x,q),\beta.\mu(y,q)\}+\alpha\\ &\Rightarrow \beta.\mu(x+y,q) \geq \beta.\min\{\mu(x,q),\mu(y,q)\}\\ &\Rightarrow \mu(x+y,q) \geq \min\{\mu(x,q),\mu(y,q)\}\end{aligned}$$

and

$$\begin{aligned}&\mu^{MT}_{\beta,\alpha}(x\gamma y,q)\geq\mu^{MT}_{\beta,\alpha}(y,q)\\&\Rightarrow\beta.\mu(x\gamma y,q)+\alpha\geq\beta.\mu(y,q)+\alpha\\&\Rightarrow\mu(x\gamma y,q)\geq\mu(y,q).\end{aligned}$$

Hence μ is a Q-fuzzy left ideal of S. □

Corollary 3.19. Let μ be a Q-fuzzy subset of S and $\alpha\in$ [0,1- $\sup\{\mu(x):x\in X,\ q\in Q\}]$, $\beta\in(0,1]$. Then the following are equivalent

(i) μ is a Q-fuzzy left ideal of S

(ii) μ^T_α, the Q-fuzzy translation of μ, is a Q-fuzzy left ideal of S

(iii) μ^M_β, the Q-fuzzy multiplication of μ, is a Q-fuzzy left ideal of S.

Definition 3.20. A Q-fuzzy left ideal μ of an ordered Γ-semiring S, is said to be normal Q-fuzzy left ideal if there exists $x\in S,\ q\in Q$,such that $\mu(x,q)=1$.

Proposition 3.21. Given a Q-fuzzy left ideal μ of an ordered Γ-semiring S, let μ_+ be a Q-fuzzy set in S obtained by $\mu_+(x,q)=\mu(x,q)+1-\mu(0,q)$ for all $x\in S,\ q\in Q$. Then μ_+ is a normal Q-fuzzy left ideal of S, which contains μ.

Proof. For all $x,y\in S\ \gamma\in\Gamma,\ q\in Q$, we have $\mu_+(0,q)=\mu(0,q)+1-\mu(0,q)=1$. Now,

$$\begin{aligned}\mu_+(x+y,q)&=\mu(x+y,q)+1-\mu(0,q)\\&\geq\min\{\mu(x,q),\mu(y,q)\}+1-\mu(0,q)\\&=\min\{\{\mu(x,q)+1-\mu(0,q)\},\{\mu(y,q)+1-\mu(0,q)\}\}\\&=\min\{\mu_+(x),\mu_+(y)\}\end{aligned}$$

and

$$\mu_+(x\gamma y,q)=\mu(x\gamma y,q)+1-\mu(0,q)\geq\mu(y,q)+1-\mu(0,q)=\mu_+(y,q).$$

Suppose $x\leq y$. Then

$$\mu(x,q)\geq\mu(y,q)\Rightarrow\mu(x,q)+1-\mu(0,q)\geq\mu(y,q)+1-\mu(0,q)\Rightarrow\mu_+(x)\geq\mu_+(y).$$

Therefore, μ_+ is a normal Q-fuzzy left ideal of S and from definition of μ_+, $\mu\subseteq\mu_+$. □

Let $\mathcal{NQ}(S)$ denote the set of all normal Q-fuzzy left ideals of S. Then $\mathcal{NQ}(S)$ is a poset under inclusion.

Theorem 3.22. Let $\mu\in\mathcal{NQ}(S)$ be non-constant such that it is a maximal element of $(\mathcal{NQ}(S),\subseteq)$. Then μ takes only two values 0 and 1.

Proof. Since μ is normal, we have $\mu(0,q)=1$. Let $x_0(\neq 0)\in S$ with $\mu(x_0,q)\neq 1$. We claim that $\mu(x_0,q)=0$. If not, then $0<\mu(x_0)<1$. Define on S a Q-fuzzy set ν by $\nu(x,q)=\frac{1}{2}[\mu(x,q)+\mu(x_0,q)]$ for all $x\in S,q\in Q$. Then ν is well-defined and for all $x,y\in S$, $\gamma\in\Gamma$ and $q\in Q$ we have

$$\begin{aligned}\nu(x+y,q)&=\tfrac{1}{2}[\mu(x+y,q)+\mu(x_0,q)]\\&\geq\tfrac{1}{2}[\min[\mu(x,q),\mu(y,q)]+\mu(x_0,q)]\\&=\min[\tfrac{1}{2}[\mu(x,q)+\mu(x_0,q)],\tfrac{1}{2}[\mu(y,q)+\mu(x_0,q)]]\\&=\min[\nu(x,q),\nu(y,q)]\end{aligned}$$

and

$$\nu(x\gamma y,q)=\frac{1}{2}[\mu(x\gamma y,q)+\mu(x_0,q)]\geq\frac{1}{2}[\mu(y,q)+\mu(x_0,q)]=\nu(y,q).$$

Hence ν is a Q-fuzzy left ideal of S. Hence by Proposition 3.21, ν_+ is a normal Q-fuzzy left ideal of S. Now,

$$\nu_+(x,q)=\nu(x,q)+1-\nu(0,q)=\tfrac{1}{2}[\mu(x,q)+\mu(x_0,q)]+1-\tfrac{1}{2}[\mu(0,q)+\mu(x_0,q)]=\tfrac{1}{2}[\mu(x)+1]......(1)$$

In particular, $\nu_+(0,q)=\frac{1}{2}[\mu(0,q)+1]=1$ and $\nu_+(x_0,q)=\frac{1}{2}[\mu(x_0,q)+1]......(2)$.
From (1) we see that ν_+ is non-constant as μ is non-constant. From (2) we see that $\mu(x_0,q)<\nu_+(x_0,q)$. This violates the maximality of μ and so we get a contradiction. This completes the proof. □

Theorem 3.23. Let S be an ordered Γ-semiring. Then set of all Q-fuzzy ideals of S (in short $FI(S)$) is zerosumfree Γ-semiring with infinite element $\mathbf{1}$ under the operations of sum and composition of Q-fuzzy ideals of S.

Proof. Clearly $\phi \in FI(S)$. Suppose μ_1, μ_2, μ_3 to be three Q-fuzzy ideals of S. Then
(i) $\mu_1 +_1 \mu_2 \in FI(S)$,
(ii) $\mu_1 o_1 \mu_2 \in FI(S)$,
(iii) $\mu_1 +_1 \mu_2 = \mu_2 +_1 \mu_1$,
(iv) $\phi +_1 \mu_1 = \mu_1$,
(v) $\mu_1 +_1 (\mu_2 +_1 \mu_3) = (\mu_1 +_1 \mu_2) +_1 \mu_3$,
(vi) $\mu_1 o_1 (\mu_2 o_1 \mu_3) = (\mu_1 o_1 \mu_2) o_1 \mu_3$,
(vii) $\mu_1 o_1 (\mu_2 +_1 \mu_3) = (\mu_1 o_1 \mu_2) +_1 (\mu_1 o_1 \mu_3)$,
(viii) $(\mu_2 +_1 \mu_3) o_1 \mu_1 = (\mu_2 o_1 \mu_1) +_1 (\mu_3 o_1 \mu_1)$.
Also $\phi +_1 \mu_1 = \mu_1 +_1 \phi = \mu_1$.

Thus $FI(S)$ is a Γ-semiring under the operations of sum and composition of Q-fuzzy ideals of S.

Now $\mathbf{1} \subseteq \mathbf{1} +_1 \mu_1$ for $\mu_1 \in FI(S)$. Also
$(\mathbf{1} +_1 \mu)(x,q) = \sup\limits_{x \leq y+z} \{\min\{\mathbf{1}(y,q), \mu(z,q)\} : y, z \in S,\ q \in Q\} \leq 1 = \mathbf{1}(x,q)$ for all $x \in S,\ q \in Q$.

Therefore $\mathbf{1} +_1 \mu_1 \subseteq \mathbf{1}$ and hence $\mathbf{1} +_1 \mu_1 = \mathbf{1}$ for all $\mu_1 \in FI(S)$.

Thus $\mathbf{1}$ is an infinite element of $FI(S)$.

Next let $\mu_1 +_1 \mu_2 = \phi$ for $\mu_1, \mu_2 \in FI(S)$.

Then $\mu_1 \subseteq \mu_1 +_1 \mu_2 = \phi \subseteq \mu_1$ and so $\mu_1 = \phi$.

Similarly, it can be shown that $\mu_2 = \phi$.

Hence the Γ-semiring $FI(S)$ is zerosumfree. □

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