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NEW SUPRA TOPOLOGIES FROM OLD VIA IDEALS

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Abstract – In this paper, we define a supra topology obtained as an associated structure on a supra topological space (X, τ) induced by an ideal on X. Such a supra topology is studied in certain detail as to some of it is basic properties.

Keywords – *Ideals, Local function, Supra topology, Supra topological space.*

1 Introduction

The concept of ideal in topological space was first introduced by Kuratowski [4] and Vaidyanathswamy[9]. They also have defined local function in ideal topological space. Further Hamlett and Jankovic [2] studied the properties of ideal topological spaces and they have introduced another operator called Ψ- operator. They have also obtained a new topology from original ideal topological space. Using the local function, they defined a Kuratowski Closure operator in new topological space. Further, they showed that interior operator of the new topological space can be obtained by Ψ - operator. In [7], the authors introduced two operators $()^{*s}$ and ψ_τ in supra topological space. Mashhour et al[6] introduced the notion of supra topological space. El-Sheikh [1] studied the properties of supra topological space and he introduced the notion of supra closure operator which is generated a supra topological space. In this paper, we introduced a new supra topology from old via ideal. Further we have discussed the properties of this supra topology.

2 Preliminary

Definition 2.1. *[6] Let X be a nonempty set. A class τ of subsets of X is said to be a supra topology on X if it satisfies the following axioms:-*

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1. *$X, \emptyset \in \tau$.*
2. *The arbitrary union of members of τ is in τ.*

The members of τ are then called supra-open sets(s-open, for short). The pair (X, τ) is called a supra topological space. A subset A of a topological space (X, τ) is called a supra-closed set(s-closed, for short) if its complement A^c is an s-open set. The family of all s-closed sets is denoted by $\tau^c = \{F : F^c \in \tau\}$.

Definition 2.2. *[6] Let (X, τ) be a supra topological space and $A \subseteq X$. Then*

1. *$Scl_\tau(A) = \cap\{F \in \tau^c : A \subseteq F\}$ is called the supra-closure of $A \in P(X)$.*
2. *$Sint_\tau(A) = \cup\{M \in \tau : M \subseteq A\}$ is called the supra-interior of $A \in P(X)$.*

Definition 2.3. *[6] Let (X, τ) be a supra topological space and $x \in X$ be an arbitrary point. A set $M \subseteq X$ is called a supra-neighborhood (s-nbd, for short)of x if $x \in M \in \tau$. The family of all s-neighborhood of x is denoted by $\tau(x) = \{M \subseteq X : x \in M \in \tau\}$. We write M_x stands for the s-nbd of x.*

Theorem 2.1. *[6] Let (X, τ) be a supra topological space and $A \subseteq X$. Then*

(a) *$x \in Scl_\tau(A) \Leftrightarrow M_x \cap A \neq \phi \; \forall M_x \in \tau(x)$.*

(b) *$[Sint_\tau(A^c)]^c = Scl_\tau(A)$.*

Definition 2.4. *[6] Let τ_1 and τ_2 be two supra topologies on a set X such that $\tau_1 \subseteq \tau_2$. Then we say that τ_2 is stronger (finer) than τ_1 or τ_1 is weaker (coarser) than τ_2.*

Definition 2.5. *[6] Let (X, τ) be a supra topological space and $\beta \subseteq \tau$. Then β is called a base for the supra topology τ (s-base, for short) if every s-open set $M \in \tau$ is a union of members of β.*
Equivalently, β is a supra-base for τ if for any point p belonging to a s-open set M, there exists $B \in \beta$ with $p \in B \subseteq M$.

Definition 2.6. *[6] A mapping $c : P(X) \to P(X)$ is said to be a supra closure operator if it satisfies the following axioms:*

1. *$c(\phi) = \phi$,*
2. *$A \subseteq c(A) \;\; \forall\, A \in P(X)$,*
3. *$c(A) \cup c(B) \subseteq c(A \cup B) \;\; \forall\, A, B \in P(X)$.*
4. *$c(c(A)) = c(A) \;\forall\, A \in P(X)$. "idempotent condition",*

Theorem 2.2. *[6] Let X be a nonempty set and let the mapping $c : P(X) \to P(X)$ be a supra closure operator. Then*
the collection

$$\tau = \{G \subseteq X : c(G^c) = G^c\}$$

is a supra topology on X induced by the supra closure operator c.

Definition 2.7. *[7] Let (X, τ) be a supra topological space with an ideal $\mathcal{I}$ on X. Then*

$$A^{*^s}(\mathcal{I}) = \{x \in X : M_x \cap A \notin \mathcal{I} \;\; \forall M_x \in \tau(x)\}, \forall A \in P(X)$$

is called the supra-local function(s-local function, for short) of A with respect to $\mathcal{I}$ and τ(here and henceforth also, A^{*^s} stands for $A^{*^s}(\mathcal{I})$).

Theorem 2.3. *[7] Let (X, τ) be a supra topological space with ideals $\mathcal{I}$ and $\mathcal{J}$ on X and let A and B be two subsets of X. Then*

1. *$\phi^{*^s} = \phi$.*

2. $A \subseteq B \Rightarrow A^{*^s} \subseteq B^{*^s}$,
3. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow A^{*^s}(\mathcal{J}) \subseteq A^{*^s}(\mathcal{I})$,
4. $A^{*^s} = Scl_\tau(A^{*^s}) \subseteq Scl_\tau(A)$,
5. $(A^{*^s})^{*^s} \subseteq A^{*^s}$,
6. $A^{*^s} \cup B^{*^s} \subseteq (A \cup B)^{*^s}$,
7. $(A \cap B)^{*^s} \subseteq A^{*^s} \cap B^{*^s}$
8. $M \in \tau \Rightarrow M \cap A^{*^s} = M \cap (M \cap A)^{*^s} \subseteq (M \cap A)^{*^s}$,
9. $H \in \mathcal{I} \Rightarrow (A \cup H)^{*^s} = A^{*^s} = (A \backslash H)^{*^s}$.

3 New Supra Topologies From Old via Ideals

In this section, we generate a supra topology obtained as an associated structure on a supra topological space (X, τ), induced by an ideal on X. Such a supra topology is studied in certain details as to some of its basic properties.

Lemma 3.1. *Let (X, τ) be a supra topological space, $A \subseteq X$ and $\mathcal{I}$ be an ideal on X. Then $M \in \tau$, $M \cap A \in \mathcal{I} \Rightarrow M \cap A^{*^s} = \phi$.*

Proof. *Let $x \in M \cap A^{*^s}$.Then $x \in M$, $x \in A^{*^s} \Rightarrow M_x \cap A \notin \mathcal{I}\ \forall M_x \in \tau(x)$. Since $x \in M \in \tau$, then $M \cap A \notin \mathcal{I}$.* ■

Lemma 3.2. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then $(A \cup A^{*^s})^{*^s} \subseteq A^{*^s}\ \forall A \in P(X)$.*

Proof. *Let $x \notin A^{*^s}$. Then there exists $M_x \in \tau(x)$ such that $M_x \cap A \in \mathcal{I} \Rightarrow M_x \cap A^{*^s} = \phi$ (By Lemma 3.1). Hence, $M_x \cap (A \cup A^{*^s}) = (M_x \cap A) \cup (M_x \cap A^{*^s}) = M_x \cap A \in \mathcal{I}$. Therefore, $x \notin (A \cup A^{*^s})^{*^s}$. Hence, $(A \cup A^{*^s})^{*^s} \subseteq A^{*^s}$.* ■

Theorem 3.1. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then the operator*

$$cl^{*^s}_{\mathcal{I}} : P(X) \to P(X)$$

defined by

$$cl^{*^s}_{\mathcal{I}}(A) = A \cup A^{*^s}\ \forall A \in P(X),$$

is a supra closure operator and hence it generates a supra topology

$$\tau^*(\mathcal{I}) = \{A \in P(X) : cl^{*^s}_{\mathcal{I}}(A^c) = A^c\}$$

which is finer than τ.
When there is no ambiguity we will write cl^{*^s} for $cl^{*^s}_{\mathcal{I}}$ and τ^* for $\tau^*(\mathcal{I})$.

Proof. *(i) By Theorem 2.3, $\phi^{*^s} = \phi$, we have $cl^{*^s}(\phi) = \phi$*

(ii) Clear that, $A \subseteq cl^{*^s}(A)\ \forall A \in P(X)$.

(iii) Let $A, B \in P(X)$. Then, $cl^{*^s}(A) \cup cl^{*^s}(B) = (A \cup A^{*^s}) \cup (B \cup B^{*^s}) = (A \cup B) \cup (A^{*^s} \cup B^{*^s}) \subseteq (A \cup B) \cup (A \cup B)^{*^s} = cl^{*^s}(A \cup B)$ (by using Theorem 2.3). Hence, $cl^{*^s}(A) \cup cl^{*^s}(B) \subseteq cl^{*^s}(A \cup B)$.

(iv) Let $A \in P(X)$. Since, by (ii), $A \subseteq cl^{*^s}(A)$, then $cl^{*^s}(A) \subseteq cl^{*^s}(cl^{*^s}(A))$. On the other hand, $cl^{*^s}(cl^{*^s}(A)) = cl^{*^s}(A \cup A^{*^s}) = (A \cup A^{*^s}) \cup (A \cup A^{*^s})^{*^s} \subseteq A \cup A^{*^s} \cup A^{*^s} = cl^{*^s}(A)$(by Lemma 3.2), it follows that $cl^{*^s}(cl^{*^s}(A)) \subseteq cl^{*^s}(A)$. Hence $cl^{*^s}(cl^{*^s}(A)) = cl^{*^s}(A)$. Consequently, cl^{*^s} is a supra closure operator. Also, it is easy to show that the collection $\tau^*(\mathcal{I}) = \{A \in P(X) : cl^{*^s}(A^c) = A^c\}$ is a supra topology on X which is called the supra topology induced by the supra closure operator.
Next, from Theorem 2.3(4) we have $A^{*^s} \subseteq Scl_\tau(A) \Rightarrow A \cup A^{*^s} \subseteq A \cup Scl_\tau(A) = Scl_\tau(A) \Rightarrow cl^{*^s}(A) \subseteq Scl_\tau(A)$. Hence $\tau \subseteq \tau^*(\mathcal{I})$. ■

Example 3.1. *Let (X, τ) be a supra topological space. If $\mathcal{I} = \{\phi\}$, then $\tau = \tau^*(\mathcal{I})$. In fact, if $x \in Scl(A)$, then, (by Theorem 2.1(a)), $M_x \cap A \neq \phi\ \forall M_x \in \tau(x) \Rightarrow M_x \cap A \notin \{\phi\} = \mathcal{I}\ \forall M_x \in \tau(x) \Rightarrow x \in A^{*^s} \Rightarrow x \in A \cup A^{*^s} = cl^{*^s}(A)$. Hence $Scl(A) \subseteq cl^{*^s}(A)$, but, by Theorem 3.1 , $cl^{*^s}(A) \subseteq Scl_\tau(A)$. Hence $cl^{*^s}(A) = Scl_\tau(A)\ \forall A \in P(X)$. Consequently, $\tau = \tau^*(\mathcal{I}) = \tau^*(\{\phi\})$.* ■

Theorem 3.2. *Let (X, τ) be a supra topological space and let $\mathcal{I}_1$, $\mathcal{I}_2$ be two ideals on X. Then If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\tau^*(\mathcal{I}_1) \subseteq \tau^*(\mathcal{I}_2)$.*

Proof. *Let $M \in \tau^*(\mathcal{I}_1)$. Then $cl^{*^s}_{\mathcal{I}_1}(M^c) = M^c \Rightarrow M^c = M^c \cup M^{c*^s}(\mathcal{I}_1) \Rightarrow M^{c*^s}(\mathcal{I}_1) \subseteq M^c \Rightarrow M^{c*^s}(\mathcal{I}_2) \subseteq M^c$ (by Theorem 2.3) implies $M^c = M^c \cup M^{c*^s}(\mathcal{I}_2) \Rightarrow cl^{*^s}_{\mathcal{I}_2}(M^c) = M^c \Rightarrow M \in \tau^*(\mathcal{I}_2)$.* ■

Theorem 3.3. *Let (X, τ) be a supra topological space and let $\mathcal{I}$ be an ideal on X. Then*

(1) $H \in \mathcal{I} \Rightarrow H^c \in \tau^*(\mathcal{I})$.

(2) $A^{*^s} = cl^{*^s}(A^{*^s})\ \forall A \in P(X)$, *i.e.* A^{*^s} *is a $\tau^*(\mathcal{I})$-closed $\forall A \in P(X)$.*

Proof. (1) *In Theorem 2.3(9), put $A = \phi \Rightarrow H^{*^s} = \phi\ \forall H \in \mathcal{I}$. Hence $cl^{*^s}(H) = H \cup \phi = H \Rightarrow H^c \in \tau^*(\mathcal{I})$ i.e. H is a $\tau^*(\mathcal{I})$-closed $\forall H \in \mathcal{I}$.*
(2) *From Theorem 2.3(5), we have $(A^{*^s})^{*^s} \subseteq A^{*^s} \Rightarrow A^{*^s} = A^{*^s} \cup (A^{*^s})^{*^s} = cl^{*^s}(A^{*^s})$. Hence A^{*^s} is a $\tau^*(\mathcal{I})$-closed.* ■

Lemma 3.3. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then F is a τ^*-closed if and only if $F^{*^s} \subseteq F$.*

Proof. *Straightforward.* ■

Theorem 3.4. *Let (X, τ_1) and (X, τ_2) be two supra topological spaces and $\mathcal{I}$ be an ideal on X. Then*

$$\tau_1 \subseteq \tau_2 \Rightarrow A^{*^s}(\mathcal{I}, \tau_2) \subseteq A^{*^s}(\mathcal{I}, \tau_1).$$

Proof. *Let $x \in A^{*^s}(\mathcal{I}, \tau_2)$, then $M_x \cap A \notin \mathcal{I}\ \forall M_x \in \tau_2(x) \Rightarrow M_x \cap A \notin \mathcal{I}\ \forall M_x \in \tau_1(x) \Rightarrow x \in A^{*^s}(\mathcal{I}, \tau_1)$. Hence, $A^{*^s}(\mathcal{I}, \tau_2) \subseteq A^{*^s}(\mathcal{I}, \tau_1)$.* ■

Corollary 3.1. *Let (X, τ_1) and (X, τ_2) be two supra topological spaces and $\mathcal{I}$ be an ideal on X. Then*

$$\tau_1 \subseteq \tau_2 \Rightarrow \tau_1^*(\mathcal{I}) \subseteq \tau_2^*(\mathcal{I}).$$

Proof. *It follows from Theorem 3.4.* ■

Theorem 3.5. *Let (X, τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then the collection*

$$\beta(\mathcal{I}, \tau) = \{M - H : M \in \tau, H \in \mathcal{I}\}$$

is a base for the supra topology $\tau^*(\mathcal{I})$.

Proof. *Let $M \in \tau^*$ and $x \in M$. Then M^c is a τ^*-closed so that $cl^{*^s}(M^c) = M^c$, and hence $M^{c*^s} \subseteq M^c$(by Lemma 3.3). Then $x \notin M^{c*^s}$ and so there exists $V \in \tau(x)$ such that $V \cap M^c \in \mathcal{I}$. Putting $H = V \cap M^c$, then $x \notin H$ and $H \in \mathcal{I}$. Thus $x \in V \backslash H = V \cap H^c = V \cap (V \cap M^c)^c = V \cap (V^c \cup M) = V \cap M \subseteq M$. Hence, $x \in V \backslash H \subseteq M$, where $V \backslash H \in \beta(\mathcal{I}, \tau)$. Hence M is the union of sets in $\beta(\mathcal{I}, \tau)$.*
Note that, τ^* is a supra topology, so it is not closed under finite intersection, thus, we need only to prove that $M \in \tau^*$ is a union of sets in $\beta(\mathcal{I}, \tau)$ as done above. ■

Theorem 3.6. *For any ideal on a supra topological space (X, τ), we have*

$$\tau \subseteq \beta(\mathcal{I}, \tau) \subseteq \tau^*.$$

Proof. *Let $M \in \tau$. Then $M = M\backslash\phi \in \beta(\mathcal{I},\tau)$. Hence $\tau \subseteq \beta(\mathcal{I},\tau)$. Now, let $G \in \beta(\mathcal{I},\tau)$, then there exists $M \in \tau$ and $H \in \mathcal{I}$ such that $G = M\backslash H$. Then, $cl^{*^s}(G^c) = cl^{*^s}(M\backslash H)^c = (M\backslash H)^c \cup ((M\backslash H)^c)^{*^s} = (M^c \cup H) \cup (M^c \cup H)^{*^s}$. But, $H \in \mathcal{I}$, then, by Theorem 2.3(9), $(M^c \cup H)^{*^s} = M^{c*^s}$ and so, $cl^{*^s}(M\backslash H)^c = M^c \cup H \cup M^{c*^s} \subseteq M^c \cup H$ (by Lemma 3.3). Hence $cl^{*^s}(M\backslash H)^c \subseteq M^c \cup H = (M\backslash H)^c$, but $(M\backslash H)^c \subseteq cl^{*^s}(M\backslash H)^c$. Hence $cl^{*^s}(M\backslash H)^c = (M\backslash H)^c$. Therefore, $G = M\backslash H \in \tau^*$. Hence $\beta(\mathcal{I},\tau) \subseteq \tau^*$. Consequently, $\tau \subseteq \beta(\mathcal{I},\tau) \subseteq \tau^*$.* ■

Corollary 3.2. *Let (X,τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then If $\mathcal{I} = \{\phi\}$, then $\tau = \beta(\mathcal{I},\tau) = \tau^*$.*

Proof. *It follows from Example 3.1 and Theorem 3.6 .* ■

Theorem 3.7. *Let (X,τ) be a supra topological space and $\mathcal{I}$ be an ideal on X. Then, $\tau^{**} = \tau^*$.*

Proof. *From Theorem 3.1, we have $\tau^* \subseteq \tau^{**}$. Now, let $N \in \tau^{**}$, then N can be written as $N = \cup_{\alpha\in\Lambda}(M^*_\alpha \cap H^c_\alpha)$ such that $M^*_\alpha \in \tau^*$ and $H_\alpha \in \mathcal{I}$ $\forall\alpha \in \Lambda$. But, $M^*_\alpha = \cup_{j\in J}(M_{\alpha_j} \cap H^c_{\alpha_j})$ where $M_{\alpha_j} \in \tau$ and $H_{\alpha_j} \in \mathcal{I}$, then*
$N = \cup_{\alpha\in\Lambda}(M^*_\alpha \cap H^c_\alpha)$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap H^c_{\alpha_j}) \cap H^c_\alpha]$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap (H^c_{\alpha_j} \cap H^c_\alpha))]$
$= \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap (H_{\alpha_j} \cup H_\alpha)^c)]$
putting $S_{\alpha_j} = H_{\alpha_j} \cup H_\alpha$, then

$$N = \cup_{\alpha\in\Lambda}[\cup_{j\in J}(M_{\alpha_j} \cap S^c_{\alpha_j})].$$

Since $H_{\alpha_j}, H_\alpha(= H_{\alpha_j} \cup H_\alpha) \in \mathcal{I}$, then $S_{\alpha_j} \in \mathcal{I}$, also $\cup_{j\in J} M_{\alpha_j} \in \tau$, it follows that $\cup_{j\in J} M_{\alpha_j} \cap S^c_{\alpha_j} \in \beta(\mathcal{I},\tau)$. Consequently, $N \in \tau^*$. ■

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