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THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR m-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract – In this paper, we first define operators m-convex functions for positive, bounded, self-adjoint operators in Hilbert space via m-convex functions. Secondly, we establish some new theorems for them. Finally, we obtain the Hermite-Hadamard type inequalities for the product two operators m-convex functions in Hilbert space.

Keywords – *The Hermite-Hadamard inequality, m-convex functions, operator m-convex functions, selfadjoint operator, inner product space, Hilbert space.*

1 Introduction

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

both inequalities hold in the reversed direction if f is concave.

The inequality (1) is known in the literature as the Hermite-Hadamard's inequality. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. In this paper, Firstly we defined for bounded positive self-adjoint operator m-convex functions in Hilbert space, secondly established some new

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theorems for them and finally Hermite-Hadamard type inequalities for product two bounded positive selfadjoint operators m-convex set up in Hilbert space.

2 Preliminary

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B(\text{or } B \geq A)\,\text{if}\,\langle Ax, x\rangle \leq \langle Bx, x\rangle(\text{or } \langle Bx, x\rangle \geq \langle Ax, x\rangle)$$

we call it the operator order.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H, \langle ., .\rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [1].

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

i. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

ii. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

iii. $\|\Phi(f)\| = \|f\| := sup_{t \in Sp(A)}|f(t)|$;

iv. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

If f is a continuous complex-valued functions on $C(Sp(A)$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

We denoted by $B(H)^+$ the set of all positive operators in $B(H)$.

G.H. Toader [2] defines the m-convexity, on intermediate between the usual convexity and starshaped property.

Definition 2.1. [2] The function $f : [a, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for $x, y \in [a, b]$ and $t \in [0, 1]$ we have $f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$

Denote by $K_m(b)$ the set of the m-convex functions on [a,b] for which $f(0) \leq 0$. Note that, for $m = 1$, we recapture the concept of convex functions defined on $[a, b]$ and for $m = 0$ we get the concept of starshaped functions on $[a, b]$. We recall that $f : [a, b] \to \mathbb{R}$ is starshaped if $f(tx) \leq tf(x)$, for all $t \in [0, 1]$ and $x \in [a, b]$.

3 The Hermite-Hadamard Type Inequalities for Operator m-convex Functions in Hilbert Space

3.1 Operator m-convex Functions in Hilbert Space

The following definition is firstly defined by Yeter Erdaş

Definition 3.1. Let I be an interval in $\mathbb{R}$ and K be convex subset of $B(H)^+$. A continuous function $f : I \subseteq [0, \infty) \to \mathbb{R}$ is said to be operator m-convex on I for operators in K if

$$f(tA + m(1-t)A) \leq tf(A) + m(1-t)f(A)$$

in the operator order in $B(H)^+$, for all $m, t \in [0, 1]$ and for every positive operators A and B in K whose spectra are contained in I.

Lemma 3.2. If f is operator m-convex on $[0, \infty)$ for operator in K, then $f(A)$ is positive for every $A \in K$.

Proof. For $A \in K$, we have

$$\begin{aligned} f(A) &= f\left(\frac{tA + m(1-t)A + (1-t)A + mtA}{2}\right) \\ &\leq f(tA + m(1-t)A + (1-t)A + mtA) \\ &\leq tf(A) + m(1-t)f(A) + (1-t)f(A) + mtf(A) \\ &= tf(A) + mf(A) - mtf(A) + f(A) - tf(A) + mtf(A)f(A) \\ &\leq f(A)(m+1) \\ &\leq mf(A) \end{aligned}$$

This implies that $f(A) \geq 0$.

Moslehian and Najafi [3] proved the following theorem for positive operators as follows:

Theorem 3.3. [3] Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator functions f on $[0, \infty)$.

Dragomir in [4] has proved a Hermite-Hadamard type inequality for operator convex function as following

Theorem 3.4. [4] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I we have the inequality

$$\begin{aligned} &\left(f\left(\frac{A+B}{2}\right) \leq\right)\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\Big((1-t)A + tB\Big)\Big)dt \\ &\leq \frac{1}{2}\left[f(\frac{A+B}{2}\Big) + \frac{f(A)+f(B)}{2}\right] \\ &\left(\leq \frac{f(A)+f(B)}{2}\right)\Big] \end{aligned}$$

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] :\to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R}$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x) + f(y)}{2}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex $g(x, y) : [0, 1] \to \mathbb{R}$.

Lemma 3.5. Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I the function f is operator m-convex for operators in

$$[A, B] := \{(1-t)A + mtB : t \in [0, 1]\}$$

if and only if the function $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{x,A,B}(t) = \langle f((1-t)A + mtB)x, x\rangle$$

is m-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Let f be operator m-convex for operators in $[A, B]$ then for any $t_1, t_2 \in [0, 1]$ and $\lambda, \gamma \geq 0$ with $\lambda + \gamma = 1$ we have

$$\varphi_{x,A,B}(\lambda t_1 + \gamma t_2) = \langle f((1 - (\lambda t_1 + \gamma t_2)A) + m(\lambda t_1 + \gamma t_2)B)x, x\rangle$$

$$= \langle f(\lambda A + \gamma A - \lambda A t_1 - \gamma A t_2 + m\lambda t_1 B + m\gamma t_2 B)x, x\rangle$$

$$= \langle f(\lambda[(1-t_1)A + mt_1 B] + \gamma[(1-t_2)A + mt_2 B])x, x\rangle$$

$$\leq \lambda\varphi_{x,A,B}(t_1)\varphi_{x,A,B}(t_2)$$

showing that $\varphi_{x,A,B}$ is a m-convex function on $[0, 1]$. Let now $\varphi_{x,A,B}$ be m-convex on $[0, 1]$, we show that f is operator convex for operators in $[A, B]$. For every $C :=$

$(1-t_1)A+mt_1B$ and $D:=(1-t_2)A+mt_2B$ we have

$$\begin{aligned}
\langle f((1-\lambda)C+m\lambda D)x,x\rangle &= \langle f((1-\lambda)[(1-t_1)A+mt_1B] \\
&\quad +m\lambda[(1-t_2)A+mt_2B])x,x\rangle \\
&= \langle f(A-t_1A+mt_1B-\lambda A+\lambda t_1A-m\lambda t_1B \\
&\quad +m\lambda A-m\lambda t_2A+m^2\lambda t_2B)x,x\rangle \\
&= \langle f(A(1-t_1)-\lambda A(1-t_1)+m\lambda A(1-t_2) \\
&\quad +mt_1B+m^2\lambda t_2B-m\lambda t_1B)x,x\rangle \\
&= \langle f(-\lambda((1-t_1)A+mt_1B)+A(1-t_1) \\
&\quad +mt_1B+m\lambda(A(1-t_2)+mt_2B))x,x\rangle \\
&= \langle f((1-\lambda)((1-t_1)A+mt_1B) \\
&\quad +m\lambda((1-t_2)A+mt_2B))x,x\rangle \\
&\leq (1-\lambda)\langle f(C)x,x\rangle+m\lambda\langle f(D)x,x\rangle
\end{aligned}$$

Theorem 3.6. Let $f : I \to \mathbb{R}$ be an operator m-convex function on the interval $I \subseteq [0,\infty)$ for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I we have the inequality

$$\begin{aligned}
f(\frac{A+B}{2}) &\leq \int_0^1 \Big[tf(A)+m(1-t)f(A)+tf(B)+m(1-t)f(B)\Big]dt \\
&\leq (m+1)(f(A)+f(B))
\end{aligned}$$

Proof. For $x \in H$ with $\|x\|=1$ and $t \in [0,1]$, we have

$$\langle [tA+m(1-t)B]x,x\rangle = t\langle Ax,x\rangle + m(1-t)\langle Bx,x\rangle \in I \tag{2}$$

since $\langle Ax,x\rangle \in Sp(A) \subseteq I$ and $\langle Bx,x\rangle \in Sp(B) \subseteq I$, (2) imply that the operator-valued integral $\int_0^1 f(tA+(1-t)B)dt$ exists. Since f is operator m-convex, therefore for t in $[0,1]$ and $A,B \in K$ we have

$$f(tA+m(1-t)B) \leq tf(A)+m(1-t)f(B) \tag{3}$$

integrating both sides of (3) over $[0,1]$ we get the following inequality

$$\int_0^1 \Big[f(tA+m(1-t)B)\Big]dt \leq \int_0^1 \Big[tf(A)+m(1-t)f(B)\Big]dt$$

$$\begin{aligned}
&= f(A)+mf(B)-f(B)\\
&= f(A)+(m-1)Bf(\frac{A+B}{2})\\
&= f\Big(\frac{tA+m(1-t)A+(1-t)A+mtA+tB+m(1-t)B+(1-t)B+mtB}{2(m+1)}\Big)\\
&\leq f\Big(\frac{t(A+B)+m(1-t)(A+B)+(1-t)(A+B)+mt(A+B)}{2}\Big)\\
&\leq tf(A)+tf(B)+m(1-t)f(A)+m(1-t)f(B)+f(A)+f(B)\\
&\quad -tf(A)-tf(B)+mtf(A)+mtf(B)\\
&= (m+1)[f(A)+f(B)]\int_0^1 f(tA+m(1-t)B)dt\\
&= \int_0^1 f((1-t)A+mtB)dt
\end{aligned}$$

4 The Hermite-Hadamard Type Inequalites for Product Two Operators m-convex Functions

Let $f : I \to \mathbb{R}$ be operator m-convex and $g : I \to \mathbb{R}$ operator m-convex function on the interval I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $K(A)(x)$, $L(A,B)(x)$, $R(A,B)(x)$, $S(B)(x)$, $M(A,B)(x)$, $N(A,B)(x)$ on H by

$$\begin{aligned}
K(A)(x) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle\\
L(A,B)(x) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle\\
R(A,B)(x) &= \langle f(B)x,x\rangle\langle g(A)x,x\rangle\\
S(B)(x) &= \langle f(B)x,x\rangle\langle g(B)x,x\rangle\\
M(A,B)(x) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle+\langle f(B)x,x\rangle\langle g(B)x,x\rangle\\
N(A,B)(x) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle+\langle f(B)x,x\rangle\langle g(A)x,x\rangle.
\end{aligned}$$

Theorem 4.1. Let $f : I \to \mathbb{R}$ be operator m_1-convex and $g : I \to \mathbb{R}$ operator m_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality

$$\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]dt$$

$$\leq \Big(\frac{K}{3}\Big) + \Big(\frac{m_2 L}{6}\Big) + \Big(\frac{m_1 R}{6}\Big) - \Big(\frac{m_1 m_2 S}{3}\Big)$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$ we have

$$\langle [tA + m(1-t)B]x, x\rangle = t\langle Ax, x\rangle + m(1-t)\langle Bx, x\rangle \in I \tag{4}$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Bx, x\rangle \in Sp(B) \subseteq I$. Continuity of f, g and (4) imply that the operator valued integrals $\int_0^1 f(tA+m_1(1-t)B)dt$, $\int_0^1 g(tA+m_2(1-t)B)dt$ and $\int_0^1 (fg)(tA+m(1-t)B)dt$ exist. Since f, g are operator convex, therefore for $t \in [0,1]$ and $t \in [0,1]$ we have

$$\langle f(tA + m_1(1-t)B)x, x\rangle \;\leq\; t\langle f(A)x, x\rangle + m_1(1-t)\langle f(B)x, x\rangle$$

$$\langle g(tA + m_2(1-t)B)x, x\rangle \;\leq\; t\langle g(A)x, x\rangle + m_2(1-t)\langle g(B)x, x\rangle$$

$$\Big(\langle f(tA + m_1(1-t)B)x, x\rangle\Big)\Big(\langle g(tA + m_2(1-t)B)x, x\rangle\Big)$$

$$\leq t^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle \tag{5}$$

$$+tm_1(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle$$

$$+m_1 m_2(1-t)^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle$$

Integrating both sides of (5) over $[0,1]$, we get the following inequality

$$\int_0^1 \Big[\langle f(tA + m_1(1-t)B)x, x\rangle\langle g(tA + m_2(1-t)B)x, x\rangle\Big]dt \leq$$

$$\Big(\frac{K}{3}\Big) + \Big(\frac{m_2 L}{6}\Big) + \Big(\frac{m_1 R}{6}\Big) - \Big(\frac{m_1 m_2 S}{3}\Big)$$

Theorem 4.2. Let $f : I \to \mathbb{R}$ be operator m_1-convex and $g : I \to \mathbb{R}$ operator m_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality

$$\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle$$

$$\leq \;\Big[\frac{1 - m_1 m_2}{3} + \frac{m_1 + m_2}{6}\Big]M(A,B)(x)N(A,B)(x)$$

Since f is operator m_1-convex and g is operator m_2-convex, for any $t \in I$ and any $x \in H$ with $\|x\| = 1$ we observe that

$$\begin{aligned} &\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle \\ \leq\ &\langle [tf(A) + m_1(1-t)f(A) + tf(B) + m_1(1-t)f(B)]x, x\rangle \end{aligned}$$

$$\begin{aligned} &\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle \\ \leq\ &\langle [tg(A) + m_2(1-t)g(A) + tg(B) + m_2(1-t)g(B)]x, x\rangle \end{aligned}$$

$$\begin{aligned} &\Big(\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\Big)\Big(\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle\Big) \\ \leq\ & t^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(A)x, x\rangle\langle g(A)x, x\rangle \\ +\ & t^2\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\ +\ & tm_2(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\ +\ & tm_1(1-t)\langle f(A)x, x\rangle\langle g(A)x, x\rangle + m_1m_2(1-t)^2\langle f(A)x, x\rangle\langle g(A)x, x\rangle \\ +\ & tm_1(1-t)\langle f(A)x, x\rangle\langle g(B)x, x\rangle + m_1m_2(1-t)^2\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\ +\ & t^2\langle f(B)x, x\rangle\langle g(A)x, x\rangle + tm_2(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle \\ +\ & t^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle + tm_2(1-t)\langle f(B)x, x\rangle\langle g(B)x, x\rangle \\ +\ & tm_1(1-t)\langle f(B)x, x\rangle\langle g(A)x, x\rangle + m_1m_2(1-t)^2\langle f(B)x, x\rangle\langle g(A)x, x\rangle \\ +\ & tm_1(1-t)\langle f(B)x, x\rangle\langle g(B)x, x\rangle + m_1m_2(1-t)^2\langle f(B)x, x\rangle\langle g(B)x, x\rangle \end{aligned}$$

Integrating both sides of (6) over $[0, 1]$ we get the following inequality

$$\begin{aligned} &\langle f\Big(\frac{A+B}{2}\Big)x, x\rangle\langle g\Big(\frac{A+B}{2}\Big)x, x\rangle \\ \leq\ & \Big[\frac{1-m_1m_2}{3} + \frac{m_1+m_2}{6}\Big]M(A, B)(x)N(A, B)(x) \end{aligned}$$

and this finishes the proof.

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