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COMMUTATIVE SOFT INTERSECTION GROUPS

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Abstract – In this paper, we first present the soft sets and soft intersection groups. We then define commutative soft sets, commutative soft intersection groups and investigate their properties.

Keywords – *Soft sets, Soft intersection groups, Commutative soft intersection groups.*

1 Introduction

In 1999, Molodtsov [23] defined the notion of soft sets to deal with uncertainties. After that the operations of soft sets have been studied Maji *et al.* [22], Ali *et al.* [3] and Çağman *et al.* [13]. By using these operations, some researchers have applied soft sets theory to many different areas, such as decison making [6, 13, 15], algebras [2, 4, 8, 14], topology [9, 25], fuzzy sets [5, 10, 11, 29] and matrix theory [7, 12].

To start the algebraic structures on soft set theory, Aktaş and Çağman [2] defined soft groups in 2007. Afterward, soft intersection groups [8, 19], soft rings [1, 21], soft fields and modules [4], soft semirings [14], soft BCK/BCI-algebras [16], soft p-ideals of soft BCI-algebras [17], soft WS-algebras [24] and soft intersection near-rings [27, 28] have been studied. In this paper, we first present the soft sets and soft intersection groups. We then define commutative soft sets, commutative soft intersection groups and investigate their properties.

2 Soft Sets

In this section, we present basic definitions of soft sets and their operations. For more detailed explanations of the soft sets, we refer to the earlier studies [13, 22, 23].

Definition 2.1. [23] Let U and E be two non empty set and $P(U)$ is the power set of U. Then, a soft set f over U is a function defined by

$$f : E \to P(U),$$

where U refer to an initial universe and E is a set of parameters.

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In other words, the soft set is a parametrized family of subsets of the set U. Every set $f(e)$, $e \in E$, from this family may be considered as the set of e-elements of the soft set f, or as the set of e-approximate elements of the soft set.

As an illustration, let us consider the following examples.

A soft set f describes the attractiveness of the houses which Mr. X is going to buy.

U - is the set of houses under consideration.

E - is the set of parameters. Each parameter is a word or a sentence.

$E = \{$expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair $\}$

In this case, to define a soft set means to point out *expensive* houses, *beautiful* houses, and so on.

It is worth noting that the sets $f(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

A soft set over U can be represented by the set of ordered pairs

$$f = \{(x, f(x)) : x \in E\}$$

Note that the set of all soft sets over U will be denoted by $S_E(U)$. From here on, "soft set" will be used without over U.

Definition 2.2. [13] Let $f \in S_E(U)$. Then,

- f is called an empty soft set, denoted by Φ_E, if $f(x) = \phi$, for all $x \in E$.
- f is called a universal soft set, denoted by $f_{\widetilde{E}}$, if $f(x) = U$, for all $x \in E$.
- The set $\mathrm{Im}(f) = \{f(x) : x \in E\}$ is called image of f.

Definition 2.3. [13] Let $f, g \in S_E(U)$. Then,

- f is a soft subset of g, denoted by $f \widetilde{\subseteq} g$, if $f(x) \subseteq g(x)$ for all $x \in E$.
- f and g are soft equal, denoted by $f = g$, if and only if $f(x) = g(x)$ for all $x \in E$.

Definition 2.4. [13] Let $f, g \in S_E(U)$. Then,

- the set $(f \widetilde{\cup} g)(x) = f(x) \cup g(x)$ for all $x \in E$ is called union of f and g.
- the set $(f \widetilde{\cap} g)(x) = f(x) \cap g(x)$ for all $x \in E$ is called intersection of f and g.
- the set $f^c(x) = U \setminus f(x)$ for all $x \in E$ is called complement of f.

3 Soft Intersection Groups

In this section, we introduce the concepts of soft intersection groups (soft int-groups) and soft product with their basic properties. For more detailed explanations of the soft int-groups, we refer to the earlier studies [8, 19].

Definition 3.1. [8] Let G be a group and $f \in S_G(U)$. Then, f is called a soft intersection groupoid over U if $f(xy) \supseteq f(x) \cap f(y)$ for all $x, y \in G$ and is called a soft intersection group over U if it satisfies $f(x^{-1}) = f(x)$ for all $x \in G$ as well.

Throughout this paper, G denotes an arbitrary group with identity element e and the set of all soft int-groups with parameter set G over U will be denoted by $S_G^g(U)$, unless otherwise stated. For short, instead of "f is a soft int-group with the parameter set G over U" we say "f is a soft int-group".

Theorem 3.2. [8] Let $f \in S_G^g(U)$. Then, $f(e) \supseteq f(x)$ for all $x \in G$.

Definition 3.3. [8] Let $A, B \subseteq E$, φ be a function from A into B and $f, g \in S_E(U)$. Then, soft image $\varphi(f)$ of f under φ is defined by

$$\varphi(f)(y) = \begin{cases} \cup\{f(x) : x \in A, \varphi(x) = y\}, & \text{for } y \in \varphi(A) \\ \emptyset, & \text{otherwise} \end{cases}$$

and soft pre-image (or soft inverse image) of g under φ is $\varphi^{-1}(g) = f$ such that $f(x) = g(\varphi(x))$ for all $x \in A$.

Theorem 3.4. [19] Let $f \in S_G^*(U)$ and $x, y \in G$. If $f(xy^{-1}) = f(e)$, then $f(x) = f(y)$.

Definition 3.5. [19] Let G be a group and $f, g \in S_G(U)$. Then, soft product $(f * g)$ of f and g is defined by

$$(f * g)(x) = \bigcup\{f(u) \cap g(v) : uv = x,\ u, v \in G\}$$

and inverse f^{-1} of f is defined by

$$f^{-1}(x) = f(x^{-1})$$

for all $x \in G$.

Definition 3.6. [20] Let G be a group. If $f \in S_G^g(U)$, then the set $N(f)$ defined by

$$N(f) = \{x \in G : f(xy) = f(yx) \text{ for all } y \in G\}$$

is called normalizer of f in G.

4 Commutative Soft Intersection Groups

In this section, we first define the notion of commutative soft sets and then define commutative soft intersection groups. We also investigate their related properties.

Definition 4.1. Let H be a semigroup and $f \in S_H(U)$. Then the set

$$Z(f) = \{x \in H : y, z \in H, f(xy) = f(yx), f(xyz) = f(yxz)\ \}$$

is called centralizer of f in H.

Here, if the semigroup H has right identity then the equality $f(xyz) = f(yxz)$ is reduced to $f(xy) = f(yx)$ for $z = e$, so the condition $f(xy) = f(yx)$ is redundant.

Definition 4.2. Let H be a semigroup and $f \in S_H(U)$. Then f is called commutative in H if $Z(f) = H$.

Definition 4.3. Let H be a group and f be a soft intersection group. Then f is called commutative soft intersection group in H if $Z(f) = H$.

Theorem 4.4. Let G be a group and $f \in S_G(U)$. Then, $Z(G) \subseteq Z(f) \subseteq N(f)$.

Proof. The proof is straightforward. □

Now, we can give an exempla for $Z(G) \neq Z(f) \neq N(f)$ as follows.

Example 4.5. Let $D_3 = \{e, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$ be the symmetric group and a soft set $f \in S_{D_3}(U)$ is defined as,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

for $\alpha_1, \alpha_0 \in P(U)$.

Now we show that $Z(f) \neq N(f)$.

$$\left.\begin{array}{l} f(\sigma\tau) = f(\tau\sigma^2) = \alpha_1 \\ f(\tau\sigma) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma\tau) = f(\tau\sigma)$$

$$\left.\begin{array}{l} f\left(\tau\sigma^2\right)=\alpha_1 \\ f\left(\sigma^2\tau\right)=f\left(\tau\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\left(\tau\sigma\right)\right)=f\left(\left(\tau\sigma\right)\tau\right)$$

$$\left.\begin{array}{l} f\left(\tau\left(\tau\sigma\right)\right)=f\left(\sigma\right)=\alpha_1 \\ f\left(\left(\tau\sigma\right)\tau\right)=f\left(\sigma^2\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\sigma^2\right)=f\left(\sigma^2\tau\right)$$

$$\left.\begin{array}{l} f\left(\tau\left(\tau\sigma^2\right)\right)=f\left(\sigma^2\right)=\alpha_1 \\ f\left(\left(\tau\sigma^2\right)\tau\right)=f\left(\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\left(\tau\sigma^2\right)\right)=f\left(\left(\tau\sigma^2\right)\tau\right)$$

so $\tau \in N(f)$. But,

$$\left.\begin{array}{l} f\left(\tau\sigma\sigma^2\right)=f\left(\tau\right)=\alpha_o \\ f\left(\sigma\tau\sigma^2\right)=f\left(\tau\sigma^2\sigma^2\right)=f\left(\tau\sigma\right)=\alpha_1 \end{array}\right\} \Rightarrow f\left(\tau\sigma\sigma^2\right)\neq f\left(\sigma\tau\sigma^2\right)$$

so $\tau \notin Z(f)$. Thus $Z(f) \neq N(f)$.

Theorem 4.6. Let H be a semigroup and $f \in S_H(U)$. Then,

$$x \in Z(f) \Leftrightarrow f(xy_1y_2...y_n) = f(y_1xy_2...y_n) = ... = f(y_1y_2...y_nx)$$

for all $y_1, y_2, ..., y_n \in H$.

Proof. Proof is by induction on n. Suppose $x \in Z(f)$. Then, for all $y_{1,}y \in H$

$$f(xy_1y_2) = f(y_1xy_2)$$

by the definition $Z(f)$. Assume,

$$f(xy_1y_2 \dots y_n) = f(y_1xy_2...y_n) = ... = f(y_1y_2...y_nx)$$

for all $y_{1,}y_{2,}..., y_n \in H$. Then,

$$f(xy_1y_2...(y_ny_{n+1})) = f(y_1xy_2...(y_ny_{n+1})) = ... = f(y_1y_2...(y_ny_{n+1})x) \qquad (1)$$

for all $y_1, y_2, ..., y_n, y_{n+1} \in H$. This can be done for any successive two y's in (1). So the proof is completed by hypothesis. □

Theorem 4.7. Let H be a semigroup and $f \in S_H(U)$. Then, f is commutative in H if and only if $x_{1,}x_{2,}..., x_n \in H$ and $f(x_1x_2 \cdots x_n) = f(x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)})$ for all $n \in N$ and for any permutation σ of $\{1, 2, ..., n\}$.

Proof. The proof is easy consequence of Theorem 4.6 □

Theorem 4.8. Let H be a semigroup and $f \in S_H(U)$. Then,

1. if $Z(f)$ is nonempty, then $Z(f)$ is a subsemigroup of H.
2. if H is a group, then $Z(f)$ is a normal subgroup of H.

Proof. *1.* Let $x_1, x_2 \in Z(f)$. Then for all $y, z \in H$, we have

$$\begin{aligned} f((x_1x_2)yz) &= f(x_1(x_2y)z) \\ &= f((x_2y)x_1z) \\ &= f(x_2(yx_1)z) \\ &= f((yx_1)x_2z) \\ &= f(y(x_1x_2)z) \end{aligned}$$

by Lemma 4.6 and clearly $f((x_1x_2)y) = f(y(x_1x_2))$. Hence $x_1x_2 \in Z(f)$. Thus $Z(f)$ is a subsemigroup of H If $Z(f)$ is nonempty.

2. Suppose H is a group. Then $Z(f)$ is nonempty since $e \in Z(f)$. If $x \in Z(f)$, then

$$\begin{aligned} f(x^{-1}yz) &= f(x^{-1}y(xx^{-1})z) \\ &= f((x^{-1}y)x(x^{-1}z)) \\ &= f(x(x^{-1}y)(x^{-1}z)) \\ &= f(yx^{-1}z) \end{aligned}$$

for all $y, z \in H$ and so $x^{-1} \in Z(f)$. Hence $Z(f) \leq H$.
Next, let $x \in Z(f)$ and $x \in H$. Then for all $y, z \in H$,

$$\begin{aligned} f((g^{-1}xg)yz) &= f(g^{-1}x(gyz)) \\ &= f(xg^{-1}(gyz)) \\ &= f(xyz) \\ &= f(xy(g^{-1}g)z) \\ &= f(y(g^{-1}xg)z) \end{aligned}$$

by Lemma 4.6 and so $g^{-1}xg \in Z(f)$. Thus $Z(f) \lhd H$, if H is a group. □

Theorem 4.9. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_G(U)$. Then,

$$\varphi(Z(f)) \subseteq Z(\varphi(f)).$$

Proof. Let $x \in \varphi(Z(f))$. Then, there exists $u \in Z(f)$ such that $\varphi(u) = x$. So for all $y \in H$,

$$\begin{aligned} \varphi(f)(xy) &= \cup\{f(a) : \varphi(a) = xy, a \in G\} \\ &= \cup\{f(uv) : a = uv, \varphi(v) = y \text{ and } a, v \in G\} \\ &= \cup\{f(vu) : b = vu, \varphi(v) = y \text{ and } v, b \in G\} \\ &= \cup\{f(b) : \varphi(b) = yx \text{ and } b \in G\} \\ &= \varphi(f)(yx) \end{aligned}$$

Similarly, for all $y, z \in H$, we obtain

$$\begin{aligned} \varphi(f)(xyz) &= \cup\{f(a) : \varphi(a) = xyz, a \in G\} \\ &= \cup\{f(uvw) : a = uvw, \varphi(v) = y, \varphi(w) = z \text{ and } v, w \in G\} \\ &= \cup\{f(vuw) : b = vuw, \varphi(v) = y, \varphi(w) = z \text{ and } v, w \in G\} \\ &= \cup\{f(b) : \varphi(b) = yxz\} \\ &= \varphi(f)(yxz) \end{aligned}$$

Thus $x \in Z(\varphi(f))$ and the result follows. □

Theorem 4.10. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_H(U)$. Then,

$$\varphi^{-1}(Z(f)) = Z(\varphi^{-1}(f)).$$

Proof. Let $x \in \varphi^{-1}(Z(f))$. Then for all $y, z \in G$,

$$\begin{aligned} \left(\varphi^{-1}(f)\right)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(\varphi(yxz)) \\ &= \left(\varphi^{-1}(f)\right)(yxz) \end{aligned}$$

and we have,

$$\left(\varphi^{-1}(f)\right)(xyz) = \left(\varphi^{-1}(f)\right)(yxz). \tag{2}$$

Similarly, $\left(\varphi^{-1}(f)\right)(xy) = \left(\varphi^{-1}(f)\right)(yx)$ and so $x \in Z\left(\varphi^{-1}(f)\right)$. Hence $\varphi^{-1}(Z(f)) \subseteq Z\left(\varphi^{-1}(f)\right)$.

On the other hand, let $x \in Z\left(\varphi^{-1}(f)\right)$ and $\varphi(x) = u$. Then for all $v, w \in H$,

$$\begin{aligned} f(uvw) &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(xyz)) \\ &= \left(\varphi^{-1}(f)\right)(xyz) \\ &= \left(\varphi^{-1}(f)\right)(yxz) \text{ (by 2)} \\ &= f(\varphi(yxz)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(vuw) \end{aligned}$$

where $y, z \in G$ are such that $\varphi(y) = v$ and $\varphi(z) = w$. Similarly, $f(uv) = f(vu)$. Thus $u \in Z(f)$, so $x \in \varphi^{-1}(Z(f))$. Hence $Z(\varphi^{-1}(f)) \subseteq \varphi^{-1}(Z(f))$ and the result follows. □

Theorem 4.11. Let G and H be two groups, $\varphi : G \to H$ be an epimorphism and $f \in S_G(U)$. Then, if f is commutative in G, then $\varphi(f)$ is commutative in H.

Proof. Let $x \in H$. Then there exists $u \in G$ such that $\varphi(u) = x$.

$$\begin{aligned} \varphi(f)(xyz) &= \cup\{f(a) : \varphi(a) = xyz\} \\ &= \cup\{f(uvw) : v, w \in G,\ \varphi(v) = y,\ \varphi(w) = z\} \\ &= \cup\{f(vuw) : v, w \in G,\ \varphi(v) = y,\ \varphi(w) = z\} \\ &= \cup\{f(b) : \varphi(b) = yxz\} \\ &= \varphi(f)(yxz) \end{aligned}$$

for all $y, z \in H$.
Similarly $\varphi(f)(xy) = \varphi(f)(yx)$.

So $x \in Z(\varphi(f))$ and $H \subseteq Z(\varphi(f))$. Thus $H = Z(\varphi(f))$ and $\varphi(f)$ is commutative in H by Definition 4.2. □

The next example shows that converse of Theorem 4.11 do not hold.

Example 4.12. Let $D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$ be the dihedral group, $N = \{e, \sigma^2\}$ and $\varphi : D_4 \longrightarrow D_4/N$ be the naturel homomorfizm. Let $f \in S_{D_4}(U)$ as,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

for $\alpha_0, \alpha_1 \in P(U)$. D_4/N is a commutative group. Then,

$$D_4/N = Z(\varphi(f))$$

so $\varphi(f)$ commutative in D_4/N. But for $\sigma, \sigma^3 \in D_4$,

$$\left.\begin{array}{l} f(\sigma(\tau\sigma)) = f(\tau) = \alpha_0 \\ f((\tau\sigma)\sigma) = f(\tau\sigma^2) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma(\tau\sigma)) \neq f((\tau\sigma)\sigma)$$

$$\left.\begin{array}{l} f(\sigma^3(\tau\sigma^3)) = f(\tau) = \alpha_0 \\ f((\tau\sigma^3)\sigma^3) = f(\tau\sigma^2) = \alpha_1 \end{array}\right\} \Rightarrow f(\sigma^3(\tau\sigma^3)) \neq f((\tau\sigma^3)\sigma^3)$$

so $\sigma, \sigma^3 \notin Z(f)$. Thus

$$D_4 \neq Z(f).$$

That is, f is not commutative in G.

Theorem 4.13. Let G and H be two semigroups, $\varphi : G \to H$ be an epimorphism and $f \in S_H(U)$. Then, if f is commutative in H, then $\varphi^{-1}(f)$ is commutative in G.

Proof. Let $x \in \varphi^{-1}(H) = G$. Then for all $y, z \in G$,

$$\begin{aligned} \varphi^{-1}(f)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &= f(\varphi(y)\varphi(x)\varphi(z)) \\ &= f(\varphi(yxz)) \\ &= (\varphi^{-1}(f))(yxz) \end{aligned}$$

Similarly, $\varphi^{-1}(f)(xy) = (\varphi^{-1}(f))(yx)$. So $x \in Z(\varphi^{-1}(f))$ and $G \subseteq Z(\varphi^{-1}(f))$. Thus $G = Z(\varphi^{-1}(f))$ and so $\varphi^{-1}(f)$ is commutative in G by Definition 4.2. □

Theorem 4.14. Let $f \in S_G^*(U)$. Then the set defined by

$$T = \{x \in G\ : f(xyx^{-1}y^{-1}) = f(e) \text{ for all } y \in G\}$$

is equal to $Z(f)$.

Proof. Let $x \in T$. Then, for any $y \in G$, we have

$$\begin{aligned} f(e) &= f\left(xyx^{-1}y^{-1}\right) \\ &= f\left((xy)(yx)^{-1}\right) \end{aligned}$$

and $f(xy) = f(yx)$ by Theorem 3.4. Now, for all $y, z \in G$, we have

$$\begin{aligned} f\left((xyz)(yxz)^{-1}\right) &= f\left(xyzz^{-1}x^{-1}y^{-1}\right) \\ &= f\left(xyx^{-1}y^{-1}\right) \\ &= f(e) \end{aligned}$$

and by Theorem 3.4, we obtain $f(xyz) = f(yxz)$ and so $x \in Z(f)$. Therefore, $T \subseteq Z(f)$.

Conversely, if $x \in Z(f)$, then for all $y \in G$

$$\begin{aligned} f\left(xyx^{-1}y^{-1}\right) &= f\left(yxx^{-1}y^{-1}\right) \\ &= f(e) \end{aligned}$$

by Lemma 4.6. Thus $x \in T$ and so $Z(f) \subseteq T$. Hence $Z(f) = T$. □

Theorem 4.15. Let H be a semigroup and $f, g \in S_H(U)$. Then,

$$Z(f) \cap Z(g) \subseteq Z\left(f \widetilde{\cap} g\right)$$

Proof. Let $x \in Z(f) \cap Z(g)$. Then $x \in Z(f)$ and $x \in Z(g)$. For all $y \in H$,

$$\begin{aligned} \left(f \widetilde{\cap} g\right)(xy) &= f(xy) \cap g(xy) \\ &= f(yx) \cap g(yx) \\ &= \left(f \widetilde{\cap} g\right)(yx) \end{aligned}$$

and for all $y, z \in H$,

$$\begin{aligned} \left(f \widetilde{\cap} g\right)(xyz) &= f(xyz) \cap g(xyz) \\ &= f(yxz) \cap g(yxz) \\ &= \left(f \widetilde{\cap} g\right)(yxz) \end{aligned}$$

Thus $x \in Z\left(f \widetilde{\cap} g\right)$. □

Theorem 4.16. Let H be a semigroup and $f, g \in S_H(U)$. If f and g are commutative, then $f \widetilde{\cap} g$ is commutative.

Proof. The proof is straightforward. □

Theorem 4.17. Let $f, g \in S_G^*(U)$ such that $f(e) = g(e)$. Then,

$$Z(f) \cap Z(g) = Z\left(f \widetilde{\cap} g\right).$$

Proof. By Lemma 4.14, for all $y \in G$,

$$\begin{aligned} x &\in Z\left(f \widetilde{\cap} g\right) \\ &\Longleftrightarrow \left(f \widetilde{\cap} g\right)(e) = \left(f \widetilde{\cap} g\right)\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow f(e) = g(e) = \left(f \widetilde{\cap} g\right)(e) = f\left(xyx^{-1}y^{-1}\right) \cap g\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow f(e) = f\left(xyx^{-1}y^{-1}\right) \text{ and } g(e) = g\left(xyx^{-1}y^{-1}\right) \\ &\Longleftrightarrow x \in Z(f) \text{ and } x \in Z(g) \\ &\Longleftrightarrow x \in Z(f) \cap Z(g) \end{aligned}$$

Thus, $Z(f) \cap Z(g) = Z\left(f \widetilde{\cap} g\right)$. □

Theorem 4.18. Let $f, g \in S_G^*(U)$ such that $f(e) = g(e)$. If f and g are commutative in G if and only if $f \widetilde{\cap} g$ is commutative in G.

Proof. The proof is straightforward. □

Theorem 4.19. If $f, g \in S_G(U)$, then $Z(f)\, Z(g) \subseteq Z(f * g)$.

Proof. Let $x_1 \in Z(f)$ and $x_2 \in Z(g)$. Then for all $y, z \in G$

$$\begin{aligned}
(f * g)((x_1 x_2) yz) &= \cup \{f(a) \cap g(b) : ab = x_1 x_2 yz,\ a, b \in G\} \\
&= \cup \left\{f\left(x_1 x_2 yz b^{-1}\right) \cap g(b) : b \in G\right\} \\
&= \cup \left\{f\left(x_2 y x_1 z b^{-1}\right) \cap g(b) : b \in G\right\} \\
&= \cup \{f(c) \cap g(b) : cb = x_2 y x_1 z,\ c, b \in G\} \\
&= \cup \left\{f(c) \cap g\left(c^{-1} x_2 y x_1 z\right) : c \in G\right\} \\
&= \cup \left\{f(c) \cap g\left(c^{-1} y x_1 x_2 z\right) : c \in G\right\} \\
&= \cup \{f(c) \cap g(d) : cd = y x_1 x_2 z,\ c, d \in G\} \\
&= (f * g)(y(x_1 x_2) z)
\end{aligned}$$

by Theorem 4.6. Similarly, $(f * g)((x_1 x_2) y) = (f * g)(y(x_1 x_2))$. Hence $x_1 x_2 \in Z(f * g)$ and $Z(f)\, Z(g) \subseteq Z(f * g)$.

□

Theorem 4.20. Let $f, g \in S_G(U)$. If either f or g is commutative in G, then $f * g$ is commutative in G.

Proof. Let f is commutative in G. Then we have $Z(f) = G$. Now,

$$\begin{aligned}
G &= G(Z(g)) \\
&= (Z(f))(Z(g)) \\
&\subseteq Z(f * g) \\
&\subseteq G
\end{aligned}$$

by Proposition 4.19, so $Z(f * g) = G$. Thus $f * g$ is commutative in G. □

The Theorem 4.19 , and the converse of Theorem 4.20, in general, do not hold, even if f and g are soft int-groups, as the next example demonstrate.

Example 4.21. Let $S_3 = \left\{e, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\right\}$ symmetric group and $f, g \in S_{S_3}(U)$. defined, respectively as, for $\alpha_1 \subset \alpha_0 \subset U$,

$$f(x) = \begin{cases} \alpha_0, & \text{for } x \in \left\{e, \sigma, \sigma^2\right\} \\ \alpha_1, & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} \alpha_0, & \text{for } x \in \{e, \tau\sigma\} \\ \alpha_1, & \text{otherwise .} \end{cases}$$

Theorem 4.22. If $f, g \in S_G^*(U)$ such that $f \subseteq g$ and $f(e) = g(e)$, then

$$Z(f) \subseteq Z(g).$$

Proof. Let $x \in Z(f)$, $f \subseteq g$ and $f(e) = g(e)$. Then for all $y \in G$,

$$\begin{aligned}
f(e) &= f\left(xyx^{-1}y^{-1}\right) \\
&\subseteq g\left(xyx^{-1}y^{-1}\right) \\
&\subseteq f(e) = f(e)
\end{aligned}$$

Hence $g\left(xyx^{-1}y^{-1}\right) = g(e)$ so $x \in Z(g)$ by Theorem 4.14. Thus $Z(f) \subseteq Z(g)$. □

Theorem 4.23. Let $f, g \in S_G^*(U)$ such that $f \subseteq g$ and $f(e) = g(e)$. If g is commutative in G, then f is commutative in G.

Proof. The proof is easy. □

Theorem 4.24. Let $f \in S_G(U)$. Then, $f\left(xyx^{-1}y^{-1}\right) = f(e)$ for all $x, y \in G$ if and only if f is commutative in G.

Proof. The proof is easy by Theorem 4.14. □

5 Conculusions

In this paper, we defined commutative soft int-groups and study some of its properties. As a future works, by using this study one can develop the nilpotent and the solvable groups.

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