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SEPARATION AXIOMS ON MULTISET TOPOLOGICAL SPACE

Sobhy El-Sheikh[1] <elsheikh33@hotmail.com>
Rajab Omar[1] <ben_omar1955@yahoo.com>
Mahmoud Raafat[1,*] <dr_mahmoudraafat2011@yahoo.com>

[1] *Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt*

Abstract – The aim of this paper is to introduce separation axioms on multiset topological spaces and study some of their properties. Characterization of these spaces and some examples have investigated. Furthermore, we show that such separation axioms are preserved under hereditary properties.

Keywords – *Multiset, Power multiset, M-point, M-singleton, Multiset function, M-topology, Continuous multiset function.*

1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 7, 10, 11, 14, 15]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 8, 11, 12, 13, 16]. For the sake of convenience a mset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in a mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by A(x). One of the most natural and simplest examples is the mset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the mset $M = \{3/x, 2/y, 1/z\}$ where $C_M(x) = 3$, $C_M(y) = 2$, $C_M(z) = 1$.

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[*] *Corresponding Author.*

Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

A wide application of msets can be found in various branches of mathematics. Algebraic structures for multiset space have been constructed by Ibrahim et al. in [9]. Application of mset theory in decision making can be seen in [17]. In 2012, Girish and Sunil [5] introduced multiset topologies induced by multiset relations. The same authors further studied the notions of open sets, closed sets, basis, sub-basis, closure, interior, continuity and related properties in M-topological spaces in [6]. In 2015, El-Sheikh et al. [4] introduced some types of generalized open msets and their properties.

In this paper, we extend the separation axioms $T_i(i = 0, 1, 2, 3, 4, 5, 2\frac{1}{2})$ on multiset topological space (M, τ) and study some of their properties. The behaviour of these separation axioms under the hereditary property is investigated.

2 Preliminaries

Definition 2.1. [10] A mset M drawn from the set X is represented by a count function C_M defined as $C_M : X \to N$ where N represents the set of non-negative integers.

Here C_M(x) is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, x_3, ..., x_n\}$ as
$M = \{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}$ where m_i is the number of occurreneces of the element x_i, $i = 1, 2, 3, ..., n$ in the mset M. However, those elements which are not included in the mset M have zero count.

Definition 2.2. [10] A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.

The mset space $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in a mset. If $X = \{x_1, x_2, ..., x_k\}$, then $[X]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : for\ i = 1, 2, ..., k;\ m_i \in \{0, 1, 2, ..., w\}\}$.

Definition 2.3. [10] Let M and N be two msets drawn from a set X. Then:

1. $M = N$ if $C_M(x) = C_N(x)$ for all x $\in$ X.

2. M $\subseteq$ N if $C_M(x) \leq C_N(x)$ for all x $\in$ X.

3. P = M $\cup$ N if $C_P(x)$ = Max $\{C_M(x)$,$C_N(x)\}$ for all x $\in$ X.

4. P = M $\cap$ N if $C_P(x)$ = Min $\{C_M(x)$,$C_N(x)\}$ for all x $\in$ X.

5. P = M $\oplus$ N if $C_P(x)$ = Min $\{C_M(x) + C_N(x)$, w$\}$ for all x $\in$ X.

6. P = M $\ominus$ N if $C_P(x)$ = Max $\{C_M(x)$ - $C_N(x)$, 0$\}$ for all x $\in$ X where $\oplus$ and $\ominus$ represent mset addition and mset subtraction respectively.

Definition 2.4. [10] Let M be a mset drawn from the set X and if C_M(x)=0 $\forall$x$\in$X. Then, M is called empty set and denoted by ϕ i.e., ϕ(x)=0 $\forall x \in X$.

Definition 2.5. [5] (Whole submset) A submset N of M is a whole submset of M with each element in N having full multiplicity as in M i.e., $C_N(x) = C_M(x)$ for every x in N.

Definition 2.6. [5] (Partial Whole submset) A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M i.e., $C_N(x) = C_M(x)$ for some x in N.

Definition 2.7. [5] (Full submset) A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M i.e., $M^* = N^*$ with $C_N(x) \leqslant C_M(x)$ for every x in N.

Remark 2.1. [5] Empty set ϕ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M.

Example 2.1. [5] Let $M = \{2/x, 3/y, 5/z\}$ be a mset. Then:

1. A submset {2/x,3/y} is whole submset and partial whole submset of M but it is not full submset of M.

2. A submset {1/x,3/y,2/z} is partial whole submset and full submset of M but it is not a whole submset of M.

3. A submset {1/x,3/y} is partial whole submset of M which is neither whole submset nor full submset of M.

Definition 2.8. [1] (Power Whole Mset) Let $M \in [X]^w$ be a mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M i.e., for constructing power whole submsets of M, every element of M with its full multiplicity behaves like an element in a classical set. The cardinality of $PW(M)$ is 2^n where n is the cardinality of the support set (root set) of M.

Definition 2.9. [5] (Power Full Mset) Let $M \in [X]^w$ be a mset. The power full mset of M denoted by $PF(M)$ is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

Remark 2.2. [5] $PW(M)$ and $PF(M)$ are ordinary sets whose elements are msets.

If M is an ordinary set with n distinct elements, then the power set $P(M)$ of M contains exactly 2^n elements. If M is a multiset with n elements (repetitions counted), then the power set $P(M)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(M)$. In classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 2.10. [5] (Power Mset) Let $M \in [X]^w$ be a mset. The power mset $P(M)$ of M is the set of all submsets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \phi$, then N$\in^1$P(M). If N$\neq \phi$, then $N \in^k P(M)$ where $k = \prod_z \begin{pmatrix} \mid [M]_z \mid \\ \mid [N]_z \mid \end{pmatrix}$, the product $\prod_z$ is taken over by distinct elements of z of the mset N and $\mid [M]_z \mid = m$ iff z $\in^m$M, $\mid [N]_z \mid = n$ iff $z \in^n N$, then $\begin{pmatrix} \mid [M]_z \mid \\ \mid [N]_z \mid \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$

The power set of a mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of a mset.

Theorem 2.1. [12] Let $P(M)$ be a power mset drawn from the mset $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ and $P^*(M)$ be the power set of a mset M. Then, $Card(P^*(M)) = \Pi_{i=1}^n$ (1+m_i).

Definition 2.11. [5] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then, τ is called a multiset topology on M if τ satisfies the following properties:

1. The mset M and the empty mset ϕ are in τ.
2. The mset union of the elements of any subcollection of τ is in τ.
3. The mset intersection of the elements of any finite subcollection of τ is in τ.

Hence, (M, τ) is called M-topological space. Each element in τ is called open mset.

Definition 2.12. [6] Let (M, τ) be a M-topological space and N is a submset of M. The collection $\tau_N = \{U^* : U^* = N \cap U, U \in \tau\}$ is a M-topology on N, called the subspace M-topology.

Definition 2.13. [6] A submset N of a M-topological space M in $[X]^w$ is said to be closed if the mset $M \ominus N$ is open.

Remark 2.3. [5] The complement of any submset N in a M-topological space (M, τ) is mset subtraction from M i.e., $N^c = M \ominus N$.

Definition 2.14. [6] Given a submset A of a M-topological space M in $[X]^w$, the interior of A is defined as the mset union of all open msets contained in A and is denoted by $Int(A)$.
i.e., $Int(A) = \cup\{G \subseteq M : G\ is\ an\ open\ mset\ and\ G \subseteq A\}$
and $C_{Int(A)}(x) = max\{C_G(x) : G \subseteq A\}$.

Definition 2.15. [6] Given a submset A of a M-topological space M in $[X]^w$, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by $Cl(A)$.
i.e., $Cl(A) = \cap\{K \subseteq M : K\ is\ a\ closed\ mset\ and\ A \subseteq K\}$
and $C_{Cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$.

Definition 2.16. [13] Two msets A and B are said to be similar msets if for all x ($x \in A \Leftrightarrow x \in B$), where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.

Definition 2.17. [1] Let M be a mset and if $x \in^m M$, $x \in^n M$. Then, $m = n$.

3 Separation Axioms on Multiset Topological Space

3.1 M-T_o-space

Definition 3.1. A mset M is called a whole M-singleton and denoted by $\{k/x\}$ if $C_M : X \to N$ such that $C_M(x) = k$ and $C_M(x') = 0 \; \forall \; x' \in X - \{x\}$.

Note that if $x \in^k M$ means $C_M(x) = k$, so $\{k/x\}$ is called whole M-singleton submset of M and $\{m/x\}$ is called M-singleton where $0 < m < k$.

Definition 3.2. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, then there exist $V, U \in \tau$ such that ($\{k_1/x_1\} \subseteq V$ and $\{k_2/x_2\} \nsubseteq V$) or ($\{k_1/x_1\} \nsubseteq U$ and $\{k_2/x_2\} \subseteq U$). Hence, (M, τ) is M-T_o-space. i.e., there exists τ-open mset which contains one of the msets $\{k_1/x_1\}$, $\{k_2/x_2\}$ but not the other.

Theorem 3.1. The property of being M-T_o-space is a hereditary property.

Proof. Let (M, τ) be a M-T_o-space and N $\subseteq$ M s.t. (N, τ_N) is M-topology on N where $\tau_N = \{N \cap G : G \in \tau\}$. Now, we want to prove that (N, τ_N) is M-T_o-space. Let $\{k_1/x_1\}, \{k_2/x_2\} \subseteq N$ s.t. $x_1 \neq x_2$, then $\{k_1/x_1\}, \{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$. Since, (M, τ) is M-T_o-space. Then, there exist $H, G \in \tau$ such that ($\{k_1/x_1\} \subseteq H$, $\{k_2/x_2\} \nsubseteq H$) or ($\{k_1/x_1\} \nsubseteq G$, $\{k_2/x_2\} \subseteq G$). Therefore, ($\{k_1/x_1\} \subseteq N \cap H$, $\{k_2/x_2\} \nsubseteq N \cap H$) or ($\{k_1/x_1\} \nsubseteq N \cap G$, $\{k_2/x_2\} \subseteq N \cap G$) and $N \cap H, N \cap G \in \tau_N$. Hence, (N, τ_N) is M-T_o-space.

Example 3.1. Let $M = \{2/a, 3/b, 1/c\}$ be a mset and $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}, \{2/a, 1/c\}\}$. It's clear that, (M, τ) is M-T_o-space. Let $N = \{1/a, 2/b\} \subseteq M$. Then, $\tau_N = \{\phi, N, \{1/a\}, \{2/b\}\}$. Hence, (N, τ_N) is M-T_o-space.

Theorem 3.2. If (M,τ_1) is M-T_o-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is also a M-T_o-space.

Proof. Immediate.

3.2 M-T_1-space

Definition 3.3. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, then there exist $G, H \in \tau$ s.t. $\{k_1/x_1\} \subseteq H$, $\{k_2/x_2\} \nsubseteq H$ and $\{k_1/x_1\} \nsubseteq G$, $\{k_2/x_2\} \subseteq G$. Hence, (M, τ) is M-T_1-space.

Theorem 3.3. Every M-T_1-space is M-T_o-space.

Proof. Straightforward.

Remark 3.1. The converse of Theorem 3.3 is not true in general as shown in the following example.

Example 3.2. Let $M = \{2/a, 3/b, 1/c\}$ be a mset and $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}, \{2/a, 1/c\}\}$, it's clear that (M,$\tau$) is M-$T_o$-space, but not M-$T_1$-space, because $\exists\{2/a\}, \{1/c\} \subseteq M$ s.t. $a \neq c$ and all open msets contain $1/c$, contain $2/a$ in the same time.

Theorem 3.4. The property of being M-T_1-space is a hereditary property.

Proof. Similar to the proof of Theorem 3.1.

Theorem 3.5. Let (M, τ) be a M-topological space. If $\{k/x\}$ is τ-closed $\forall\, x \in M^*$, $k = C_M(x)$. Then, (M, τ) is M-T_1-space. i.e., if every whole M-singleton is closed, then (M, τ) is M-T_1-space.

Proof. Let $\{l_1/x_1\}, \{l_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, by hypothesis $\{k_1/x_1\}, \{k_2/x_2\}$ are τ-closed msets on M where $k_1 = C_M(x_1)$, $k_2 = C_M(x_2)$. Then, $\{k_1/x_1\}^c, \{k_2/x_2\}^c \in \tau$ s.t. $\{l_1/x_1\} \subseteq \{k_2/x_2\}^c$, $\{l_2/x_2\} \nsubseteq \{k_2/x_2\}^c$ and $\{l_1/x_1\} \nsubseteq \{k_1/x_1\}^c$, $\{l_2/x_2\} \subseteq \{k_1/x_1\}^c$. Hence, (M, τ) is M-T_1-space.

Corollary 3.1. Let (M, τ) be a M-topological space. If every finite whole submset of M is τ-closed mset, then (M, τ) is M-T_1-space.

Proof. Clear (by using the above theorem).

Remark 3.2. Every discrete M-topology $(M, P^*(M))$ is M-T_1-space. But, if M is a finite mset and (M, τ) is M-T_1-space $\nRightarrow \tau = P^*(M)$ [Discrete M-topology]. As shown in example 3.3.

Example 3.3. Let $M = \{2/a, 3/b, 1/c\}$, $\tau = \{\phi, M, \{2/a\}, \{3/b\}, \{1/c\}, \{2/a, 3/b\}, \{2/a, 1/c\}, \{3/b, 1/c\}\} \neq P^*(M)$. But, (M, τ) is M-T_1-space.

Remark 3.3. For a finite mset M, the smallest M-T_1-space is $PW(M)$.

Theorem 3.6. If (M, τ_1) is a M-T_1-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is a M-T_1-space.

Proof. Immediate.

3.3 M-T_2-space

Definition 3.4. Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ s.t. $x_1 \neq x_2$, then there exist $G, H \in \tau$ s.t. $\{k_1/x_1\} \subseteq G$, $\{k_2/x_2\} \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-T_2-space.

Example 3.4. Every discrete M-topology $(M, P^*(M))$ is M-T_2-space.

Example 3.5. Every indiscrete M-topology (M, τ) is not M-T_2-space where M has more than or equal two different M-points.

Theorem 3.7. The property of being M-T_2-space is a hereditary property.

Proof. Let (M,τ) be a M-T_2-space, $N \subseteq M$ and let (N, τ_N) be a subspace of (M, τ). Now, we want to prove that (N, τ_N) is M-T_2-space. Let $\{k_1/n_1\}, \{k_2/n_2\} \subseteq N$ s.t. $n_1 \neq n_2$. Since, (M, τ) is M-T_2-space. Then, there exist $G, H \in \tau$ s.t. $\{k_1/n_1\} \subseteq G$, $\{k_2/n_2\} \subseteq H$ and $G \cap H = \phi$. By a definition of the subspace, we have: $N \cap G$, $N \cap H \in \tau_N$. Therefore, $\{k_1/n_1\} \subseteq N \cap G$ and $\{k_2/n_2\} \subseteq N \cap H$. Since, $G \cap H = \phi$. Then, $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is M-T_2-space.

Theorem 3.8. If (M, τ_1) is a M-T_2-space and $\tau_1 \leqslant \tau_2$, then (M, τ_2) is also a M-T_2-space.

Proof. Immediate.

Theorem 3.9. Every M-T_2-space is a M-T_1-space.

Proof. Let (M, τ) be a M-T_2-space. Also, assume that $\{k_1/m_1\}, \{k_2/m_2\} \subseteq M$ s.t. $m_1 \neq m_2$. Then, $\exists\, G, H \in \tau$ s.t. $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ and $G \cap H = \phi$. Since, $G \cap H = \phi$, $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ then $\{k_1/m_1\} \nsubseteq H$ and $\{k_2/m_2\} \nsubseteq G$. Consequently, we have $G, H \in \tau$ s.t. $\{k_1/m_1\} \subseteq G$,$\{k_2/m_2\} \nsubseteq G$ and $\{k_1/m_1\} \nsubseteq H$,$\{k_2/m_2\} \subseteq H$. Hence, (M,τ) is M-T_1-space.

3.4 M-T_3-space

Definition 3.5. Let (M, τ) be a M-topological space. If for all $F \in \tau^c$, $\forall\ \{k/x\} \nsubseteq F$, then there exist $G, H \in \tau$ s.t. $F \subseteq G$, $\{k/x\} \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-regular space.

Definition 3.6. A M-topological space (M, τ) is said to be a M-T_3-space if:

1. (M,τ) is M-regular space.
2. (M,τ) is M-T_1-space.

Example 3.6. Every discrete M-topology $(M, P^*(M))$ is M-T_3-space.

Theorem 3.10. The property of being M-regular space is a hereditary property.

Proof. Let (M, τ) be a M-regular space and $N \subseteq M$. Let (N, τ_N) be subspace of (M, τ). Now, we want to prove that (N, τ_N) is M-regular space. Let B be τ_N-closed submset of N and $\{k/n\} \subseteq N$ s.t. $\{k/n\} \nsubseteq B$, then there exists F is τ-closed submset of M s.t. $B = F \cap N$. Since, $\{k/n\} \nsubseteq B$. Then, $\{k/n\} \nsubseteq F$. As (M, τ) be M-regular space, $\exists\, G, H \in \tau$ s.t. $F \subseteq G$, $\{k/n\} \subseteq H$ and $G \cap H = \phi$. Then, $F \cap N \subseteq G \cap N$ [i.e., $B \subseteq G \cap N$], $\{k/n\} \subseteq H \cap N$ and $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is M-regular space.

Corollary 3.2. Every subspace of M-T_3-space is also a M-T_3-space.

Theorem 3.11. Every M-T_3-space is a M-regular space.

Proof. Clear by using the definition 3.6.

Remark 3.4. The converse of Theorem 3.11 is not true in general as shown by the following example.

Example 3.7. Let $M = \{2/a, 3/b, 1/c\}$ and $\tau = \{\phi, M, \{3/b\}, \{2/a, 1/c\}\}$. Then, $\tau^c = \{\phi, M, \{2/a, 1/c\}, \{3/b\}\}$. Hence, (M, τ) is a M-regular space but not a M-T_1-space.

3.5 M-T_4-space

Definition 3.7. Let (M, τ) be a M-topological space. If for all $F_1, F_2 \in \tau^c$ s.t. $F_1 \cap F_2 = \phi$, then there exist $G, H \in \tau$ s.t. $F_1 \subseteq G$, $F_2 \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is M-normal space.

Definition 3.8. A M-topological space (M, τ) is said to be a M-T_4-space if:

1. (M,τ) is M-normal space.
2. (M,τ) is M-T_1-space.

Theorem 3.12. Every closed subspace of M-normal space is also a M-normal space.

Proof. Let (M, τ) be a M-normal space. Also, assume that $N \subseteq M$ s.t. N is τ-closed submset of M. Now, we want to prove that (N, τ_N) is also M-normal space. Let B_1, B_2 be τ_N-closed submsets of N s.t. $B_1 \cap B_2 = \phi$. Then, there exist τ-closed submsets F_1, F_2 of M s.t. $B_1 = F_1 \cap N$, $B_2 = F_2 \cap N$. Since, F_1, F_2, N are τ-closed submsets of M. Then, $F_1 \cap N$, $F_2 \cap N$ are τ-closed submsets of M i.e., B_1 *and* B_2 are τ-closed submsets of M s.t. $B_1 \cap B_2 = \phi$. Since, (M, τ) is a M-normal space. Thus, $\exists\ G, H \in \tau$ s.t. $B_1 \subseteq G$, $B_2 \subseteq H$ and $G \cap H = \phi$. Since, $B_1 \subseteq N$ and $B_1 \subseteq G$. Therefore, $B_1 \subseteq G \cap N$ and similarly $B_2 \subseteq H \cap N$. But, $G, H \in \tau$. Then, $G \cap N, H \cap N \in \tau_N$. Consequently, $\exists\ G \cap N, H \cap N \in \tau_N$ s.t. $B_1 \subseteq G \cap N$, $B_2 \subseteq H \cap N$ and $(G \cap N) \cap (H \cap N) = (G \cap H) \cap N = \phi \cap N = \phi$. Hence, (N, τ_N) is a M-normal space.

Corollary 3.3. The property of being M-T_4-space is topological property.

3.6 M-T_5-space

Definition 3.9. Let (M, τ) be a M-topological space and let $A, B \subseteq M$ be two non-empty msets. Then, we say that: A, B are separated msets if $A \cap \overline{B} = \phi$, $\overline{A} \cap B = \phi$.

Definition 3.10. A M-topological space (M, τ) is said to be M-completely normal space iff for any two separated submsets A, B of M there exist $G, H \in \tau$ s.t. $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$.

Theorem 3.13. Every M-completely normal space is M-normal space.

Proof. Let (M, τ) be a M-completely normal space. Now, we want to show that (M, τ) is a M-normal space. Let A, B be any two τ-closed submsets of M s.t. $A \cap B = \phi$. Then, $\overline{A} = A$ and $\overline{B} = B$. Hence, $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$. Consequently, A, B are separated msets. Since, (M, τ) is a M-completely normal space and A, B are separated msets. Therefore, there exist $G, H \in \tau$ s.t. $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence, (M, τ) is a M-normal space.

Theorem 3.14. The property of being M-completely normal space is a hereditary property.

Proof. Immediate.

Definition 3.11. A M-topological space (M, τ) is said to be a M-T_5-space if:

1. (M,τ) is M-completely normal space.
2. (M,τ) is M-T_1-space.

Theorem 3.15. Every M-T_5-space is a M-T_4-space.

Proof. Straightforward.

Theorem 3.16. The property of being M-T_5-space is a hereditary property.

Proof. Immediate.

3.7 M-$T_{2\frac{1}{2}}$-space

Definition 3.12. Let (M,τ) be a M-topological space. If for every two M-singleton $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, then there exist $G, H \in \tau$ such that $\{k_1/x_1\} \subseteq G$, $\{k_2/x_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence, (M,τ) is M-$T_{2\frac{1}{2}}$-space.

Example 3.8. Every discrete M-topology $(M, P^*(M))$ is M-$T_{2\frac{1}{2}}$-space.

Theorem 3.17. The property of being M-$T_{2\frac{1}{2}}$-space is a hereditary property.

Proof. Let (M,τ) be a M-$T_{2\frac{1}{2}}$-space, $N \subseteq M$ and let (N,τ_N) be a subspace of (M,τ). Now, we want to prove that (N,τ_N) is M-$T_{2\frac{1}{2}}$-space. Let $\{k_1/n_1\}, \{k_2/n_2\} \subseteq N$ such that $n_1 \neq n_2$. Since, $N \subseteq M$ and (M,τ) is M-$T_{2\frac{1}{2}}$-space. Then, there exist $G, H \in \tau$ such that $\{k_1/n_1\} \subseteq G$, $\{k_2/n_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. By a definition of the subspace, we have: $(N \cap G)$, $(N \cap H) \in \tau_N$. Therefore, $\{k_1/n_1\} \subseteq N \cap G$ and $\{k_2/n_2\} \subseteq N \cap H$. Also, $(\overline{G \cap N}) \cap (\overline{H \cap N}) \subseteq (\overline{G} \cap \overline{N}) \cap (\overline{H} \cap \overline{N}) = (\overline{G} \cap \overline{H}) \cap \overline{N} = \phi \cap \overline{N} = \phi$. Thus, $(\overline{G \cap N}) \cap (\overline{H \cap N}) = \phi$. Hence, (N,τ_N) is M-$T_{2\frac{1}{2}}$-space.

Theorem 3.18. If (M,τ_1) is a M-$T_{2\frac{1}{2}}$-space and $\tau_1 \leqslant \tau_2$, then (M,τ_2) is also a M-$T_{2\frac{1}{2}}$-space.

Proof. Straightforward.

Theorem 3.19. Every M-$T_{2\frac{1}{2}}$-space is a M-T_2-space.

Proof. Let (M,τ) be a M-$T_{2\frac{1}{2}}$-space and assume that $\{k_1/m_1\}, \{k_2/m_2\} \subseteq M$ such that $m_1 \neq m_2$. Then, there exist $G, H \in \tau$ such that $\{k_1/m_1\} \subseteq G$, $\{k_2/m_2\} \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Since, $G \subseteq \overline{G}$, $H \subseteq \overline{H}$. Therefore, $G \cap H = \phi$. Hence, (M,τ) is M-T_2-space.

4 Conclusion

In this paper we introduce the separation axioms on mset topological spaces based on the singleton mset $\{m/x\}$. This approach contains all multi-points which considered as a submset. The behavior of these axioms under some types of mapping have obtained. In the future, we study another topological property such as connected, some types of submsets and mappings on these spaces.

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References

[1] Blizard and D. Wayne, Multiset theory, *Notre Dame Journal of Formal Logic* 30 (1989 a) 36-66.

[2] Blizard and D. Wayne, Real-valued multisets and fuzzy sets, *Fuzzy Sets and Systems* 33 (1989 b) 77-97.

[3] Blizard and D. Wayne, The development of multiset theory, *Modern Logic* 1 (1991) 319-352.

[4] S. A. El-Sheikh, R. A-K. Omar and M. Raafat, γ-operation in M-topological space, *Gen. Math. Notes* 27 (2015) 40-54.

[5] K. P. Girish and Sunil Jacob John, Multiset topologies induced by multiset relations, *Information Sciences* 188 (2012) 298-313.

[6] K. P. Girish and Sunil Jacob John, On Multiset Topologies, *Theory and Applications of Mathematics & Computer Science* 2 (2012) 37-52.

[7] K. P. Girish and Sunil Jacob John, Relations and functions in multiset context, *Information Sciences* 179 (2009) 758-768.

[8] K. P. Girish and Sunil Jacob John, Rough multisets and information multisystems, *Advances in Decision Sciences* (2011) 1-17.

[9] A. M. Ibrahim, D. Singh, J. N. Singh, An outline of Multiset space Algebra, *International Journal of Algebra* 5 (2011) 1515-1525.

[10] S. P. Jena, S. K. Ghosh and B. K. Tripathy, On the theory of bags and lists, *Information Sciences* 132 (2001) 241-254.

[11] D. Singh, A note on the development of multiset theory, *Modern Logic* 4 (1994) 405-406.

[12] D. Singh and J. N. Singh, Some combinatorics of multisets, *International Journal of Mathematical Education in Science and Technology* 34 (2003) 489-499.

[13] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, An overview of the applications of multisets, *Novi Sad J. Math.* 37 (2007) 73-92.

[14] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, Complementation in multiset theory, *International Mathematical Forum* 6 (2011) 1877-1884.

[15] Slapal and Josef, Relations and topologies, *Czechoslovak Mathematical Journal* 43 (1993) 141-150.

[16] Yager, R. R., On the theory of bags, Int. General Syst. 13 (1986) 23-37.

[17] Yang, Y., Tan, X., Meng, C., The multi-fuzzy soft set and its application in decision making, Applied Mathematical Modelling 37 (2013) 4915-4923.