http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 09.07.2015 *Year*: **2015**, *Number*: **7**, *Pages*: **22-28**
Published: 09.10.2015 *Original Article*[**]

ON THE ZEROS OF THE DERIVATIVES OF FIBONACCI AND LUCAS POLYNOMIALS

Nihal Yılmaz Özgür[1] <nihal@balikesir.edu.tr>
Öznur Öztunç Kaymak[2,*] <oztunc@balikesir.edu.tr>

[1] *Department of Mathematics, University of Balıkesir, 10145 Balıkesir, Turkey*
[2] *University of Balıkesir, 10145 Balıkesir, Turkey*

Abstract − The purpose of this article is to derive some functions which map the zeros of Fibonacci polynomials to the zeros of Lucas polynomials. Also we find some equations which are satisfied by $F_n^{'}(x)$ and so $L_n^{''}(x)$. To obtain these equations, formulizations which are made up of hyperbolic functions for Fibonacci and Lucas polynomials are used.

Keywords − *Fibonacci polynomial, Lucas polynomial.*

1 Introduction

As it is well known, studying zeros of polynomials plays an increasingly important role in Mathematical research. Fibonacci polynomials $F_n(x)$ are defined recursively by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \tag{1}$$

by initial conditions $F_1(x) = 1$, $F_2(x) = x$. Similarly, Lucas polynomials $L_n(x)$ are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{2}$$

with the initial values $L_1(x) = x$ and $L_2(x) = x^2 + 2$ (see [7]). In [6], V. E. Hoggat and M. Bicknell found the zeros of these polynomials using hyperbolic trigonometric

** *Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
* *Corresponding Author.*

functions defined by $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$. They found the general form of Fibonacci polynomials as follows:

$$F_{2n}(x) = \frac{\sinh 2nz}{\cosh z} \tag{3}$$

and

$$F_{2n+1}(x) = \frac{\cosh(2n+1)z}{\cosh z}, \tag{4}$$

where $x = 2\sinh z$ (see [6] for more details).

Furthermore, in [4], M. X. He, P. E. Ricci and D. Simon determined the location and distribution of the zeros of the Fibonacci polynomials.

There are several papers on the derivatives of the Fibonacci and Lucas polynomials (see [1], [2], [8], [10] and [11]). For any fixed n, in [10], Jun Wang proved the following equation

$$L_n^{(t)}(x) = nF_n^{(t-1)}(x), n \geq 1. \tag{5}$$

For the first order derivatives we have $L_n^{'}(x) = nF_n(x)$. Thus we have the zeros of $L_{2n}^{'}(x)$ and $L_{2n+1}^{'}(x)$ as follows:

$$\pm 2i \sin \frac{k\pi}{2n}, k = 0, 1, 2, ..., n-1 \tag{6}$$

and

$$\pm 2i \sin\left(\frac{2k+1}{2n+1}\right)\frac{\pi}{2}, k = 1, 2, ..., n, \tag{7}$$

respectively in [6]. It will be very nice to obtain the formulization of the zeros of $L_n^{(t)}(x)$ for $t \geq 2$. Then using (5) same formulization would also be adapted to the zeros of $F_n^{(t)}(x)$ for $t > 1$. But the successively application of chain rule makes computations difficult as t increases. The computations get difficult even for second order derivative. Our aim is to obtain the zeros of $L_n^{(t)}(x)$ as iterates of some function which map the zeros of $L_n(x)$ to the zeros of $L_n^{(t)}(x)$. In this paper we get some results for $t = 1$. Also we prove some equations which are satisfied by $F_n^{'}(x)$ and so $L_n^{''}(x)$.

2 Mapping from the Modulus of the Zeros of $L_n(x)$ to the Modulus of the Zeros of $L_n^{'}(x)$

First we give some functions which map the modulus of the zeros of $L_{2n}(x)$ and $L_{2n+1}(x)$ to the modulus of the zeros of $L_{2n}^{'}(x)$ and $L_{2n+1}^{'}(x)$, respectively. It is known that the zeros of Fibonacci and Lucas polynomials are pure imaginary and come in complex conjugates. Hence we consider their magnitudes.

For the polynomials $L'_{2n}(x)$, we denote the modulus of the zeros of these polynomials by x_n $(x_1 = 0 < x_2 < ... < x_n)$ and the modulus of the zeros of $L_{2n}(x)$ by y_n $(y_1 < y_2 < ... < y_n)$.

We know that the number of positive zeros of $L'_{2n}(x)$ is n including 0 and the number of positive zeros of $L_{2n}(x)$ is n. In the following theorem, we give a function which map the modulus of the zeros of $L_{2n}(x)$ to the modulus of the zeros of $L'_{2n}(x)$ in ascending order.

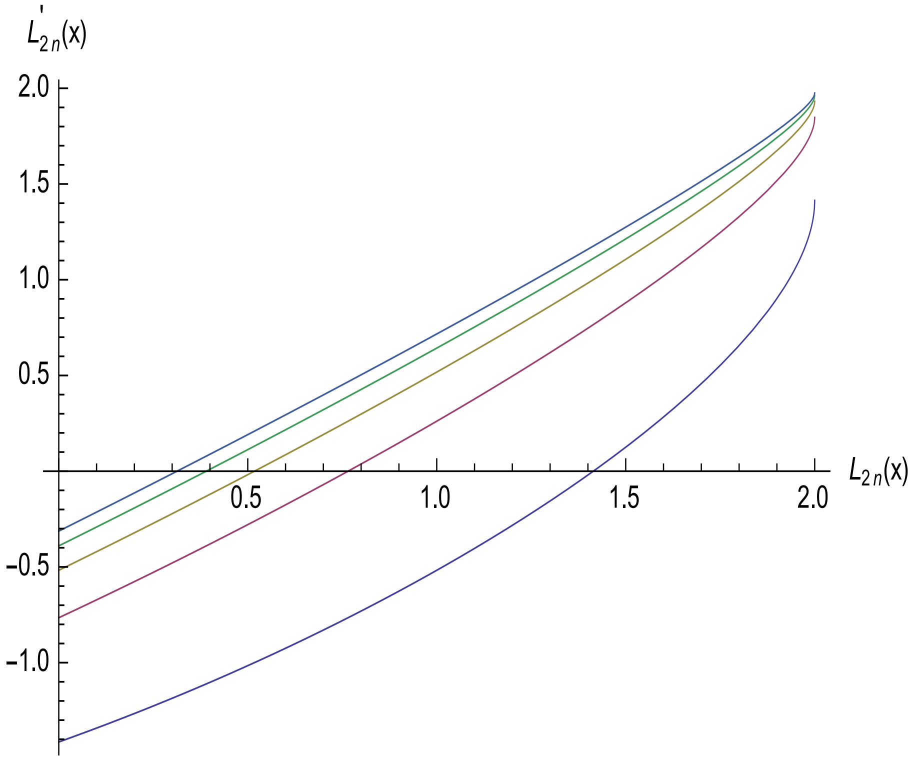

Figure 1: The graph of the function $\alpha_n(x)$ from $n = 1$ to 5.

Theorem 2.1. The modulus of the zeros of $L_{2n}(x)$ are mapped to the modulus of the zeros of $L'_{2n}(x)$ by the following function

$$\alpha_n(x) = 2\sin(\arcsin\frac{x}{2} - \frac{\pi}{4n} + k\pi),$$

where k is an integer.

Proof. From [6], we know that the magnitudes of zeros of $L'_{2n}(x)$ and $L_{2n}(x)$ are

$$x = 2\sin\frac{k\pi}{2n}, k = 0, 1, ..., n-1$$

and

$$y = 2\sin\left(\frac{2k+1}{2n}\right)\frac{\pi}{2}, k = 0, 1, ..., n-1,$$

respectively. For $\theta = \frac{k\pi}{2n}$, if we write $x = 2\sin\theta$ and $y = 2\sin\left(\theta + \frac{\pi}{4n}\right)$, the equality of the values of θ gives us

$$\arcsin(\frac{y}{2}) - \frac{\pi}{4n} + \pi k_1 = \arcsin(\frac{x}{2}) + \pi k_2,$$

where $k_1$and k_2 are integers. Then, taking $k = k_1 - k_2$, the desired function can be found easily. □

The results have been controlled numerically for $L_2(x)$ through $L_{10}(x)$ (see Figure 1).

Now we focus on the magnitudes of the zeros of odd indexed Lucas polynomials. We denote them by x_n ($x_1 = 0 < x_2 < ... < x_n$). The zeros of $L'_{2n+1}(x)$ are denoted by y_n ($y_1 < y_2 < ... < y_{n-1} < y_n$). As well as being well known that the number of the modulus of the zeros of $L_{2n+1}(x)$ is n, the number of the modulus of the zeros of $L'_{2n+1}(x)$ is n (see [3]).

Theorem 2.2. The modulus of the zeros of $L_{2n+1}(x)$ are mapped to the modulus of the zeros of $L'_{2n+1}(x)$, arranging from small to large, by the following function

$$\beta_n(x) = 2\sin(\arcsin\frac{x}{2} + \frac{\pi}{2(2n+1)} + \pi k),$$

where k is an integer.

Proof. Since we know that the magnitudes of zeros of $L_{2n+1}(x)$ and $L'_{2n+1}(x)$ are

$$x = 2\sin\frac{k\pi}{2n+1}, k = 0, 1, ..., n-1$$

and

$$y = 2\sin\left(\frac{2k+1}{2n+1}\right)\frac{\pi}{2}, k = 0, 1, ..., n-1,$$

respectively. If we take $\theta = \frac{k\pi}{2n+1}$ then we find

$$\arcsin(\frac{x}{2}) + \frac{\pi}{2(2n+1)} + \pi k_1 = \arcsin(\frac{y}{2}) + \pi k_2,$$

where $k_1$and k_2 are integers. Then, taking $k = k_1 - k_2$, the desired function can be found easily. □

Figure 2 shows the graphics of the functions $\beta_n(x)$ for $1 \leq n \leq 5$.

By (5), we know that the zeros of $F_n(x)$ are identical with the zeros of $L'_n(x)$. The functions $\beta_n(x)$ and $\alpha_n(x)$ map the zeros of $L_n(x)$ to the zeros of both $F_n(x)$ and $L'_n(x)$.

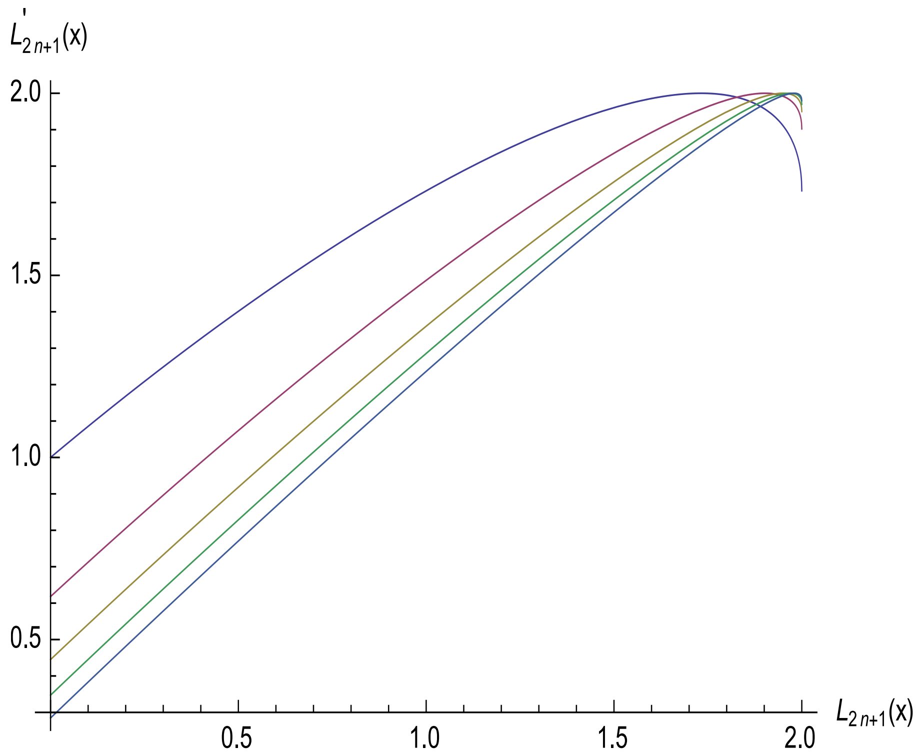

Figure 2: The graph of the function $\beta_n(x)$ from $n = 1$ to 5.

If we use the following well-known equations $\cosh z = \cos(iz)$ and $\sinh z = -i\sin(iz)$ we can rewrite the equations (3) and (4) as follows:

$$F_{2n}(x) = -i\frac{\sin(2niz)}{\cos(iz)} \tag{8}$$

and

$$F_{2n+1}(x) = \frac{\cos[i(2n+1)z]}{\cos(iz)}, \tag{9}$$

where $x = 2\sinh z$.

Then we can give the following theorems.

Theorem 2.3. The zeros of $F'_{2n}(x)$, and so the zeros of $L''_{2n}(x)$, satisfy the following equation

$$2n = -\tan(2niz)\tan(iz), \tag{10}$$

where $x = 2\sinh z$.

Proof. Using the equation (8), from the equation $F'_{2n}(x) = 0$, we can obtain

$$2n(\cos(2niz))(\cos iz) = -(\sin(2niz))(\sin iz), \tag{11}$$

where $\cos(iz) \neq 0$ and $x = 2\sinh z$. Then by rearranging the equation (11) we find the desired result (10). $\square$

Example 2.4. Let us find the zeros of the polynomial $F_6^{'}(x) = 5x^4 + 12x^2 + 3$. By (10), we obtain

$$6 = -\tan(6iz)\tan(iz).$$

Using half angle formulas we have

$$3 = -\frac{\tan(3iz)\tan(iz)}{1 - \tan^2(3iz)}. \tag{12}$$

The solutions of the equation (12), the zeros of $F_6^{'}(x)$ and $L_6^{''}(x)$, can be found easily using the equation $x = 2\sinh z$. So we have the roots

$$-i\sqrt{\frac{1}{5}(6-\sqrt{21})}, i\sqrt{\frac{1}{5}(6-\sqrt{21})}, -i\sqrt{\frac{1}{5}(6+\sqrt{21})}, -i\sqrt{\frac{1}{5}(6+\sqrt{21})}.$$

Theorem 2.5. The zeros of $F_{2n+1}^{'}(x)$, and so the zeros of $L_{2n+1}^{''}(x)$, satisfy the following equation

$$2n + 1 = \tan(iz)\cot(i(2n+1)z), \tag{13}$$

where $x = 2\sinh z$.

Proof. The proof follows from (9). $\square$

Acknowledgement

This work was supported by Balikesir University Research Grant No: 2013/02.

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