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THE NEW SOLUTION OF TIME FRACTIONAL WAVE EQUATION WITH CONFORMABLE FRACTIONAL DERIVATIVE DEFINITION

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Abstract – In this paper, we used new fractional derivative definition, the conformable fractional derivative, for solving two and three dimensional time fractional wave equation. This definition is simple and very effective in the solution procedures of the fractional differential equations that have complicated solutions with classical fractional derivative definitions like Caputo, Riemann-Liouville and etc. The results show that conformable fractional derivative definition is usable and convenient for the solution of higher dimensional fractional differential equations.

***Keywords* –** *Time fractional wave equation, conformable fractional derivative.*

1 Introduction

Fractional differential equations sometimes called as extraordinary differential equations because of their nature and easily find in various fields of applied sciences [1-8]. So the scientific and engineering problems which involve fractional calculus are very large and still very effective. In recent years, scientists have proposed many efficient and powerful methods to obtain exact or numerical solutions of fractional differential equations [9-10].

In addition to this, many researchers have been trying to form a new definition of fractional derivative. Most of these definitions include integral form for fractional derivatives. Two of these definitions which are most popular:

i. *Riemann-Liouville definition:*

If n is a positive integer and $\alpha \in [n-1, n)$, α derivative of f is given by

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$$D_a^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx.$$

ii. *Caputo definition:*

If n is a positive integer and $\alpha\in[n-1,n)$, α derivative of f is given by

$$D_a^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx.$$

In [11] R.Khalil and et al. give a new definition of fractional derivative called “conformable fractional derivative”.

Definition 1.1. Let $f:[0,\infty)\to$ □ be a function. α^{th} order *“conformable fractional derivative”* of f is defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0,\ \alpha\in(0,1)$. If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$.

This new definition satisfies the properties which are given in the following theorem [11].

Theorem 1.1. Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at point $t>0$. Then

a) $T_\alpha(cf+dg)=cT_\alpha(f)+dT_\alpha(g)$, for all $a,b\in$ □ ,
b) $T_\alpha(t^p)=pt^{p-\alpha}$ for all $p\in$ □ ,
c) $T_\alpha(\lambda)=0$ for all constant functions $f(t)=\lambda$,
d) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$,
e) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(g)-fT_\alpha(f)}{g^2}$,
f) If, in addition to f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}$.

2 Time Fractional Wave Equation in Rectangular Domain

In this section we investigate the solution of two dimensional wave equation with conformable fractional derivative,

$$\frac{\delta^\alpha}{\delta t^\alpha}\frac{\partial^\alpha u}{\partial t^\alpha}=\kappa^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\ 0<x<a,\ 0<y<b, t>0 \tag{1}$$

in the rectangular domain $D = \{(x, y) : 0 < x < a,\ 0 < y < b, x, y \in \square\}$, which describes vibrating membrane or plate with the conditions

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \tag{2}$$

$$\begin{aligned} u(x, y, 0) &= f(x, y);\ 0 \le x \le a,\ 0 \le y \le b \\ u_t(x, y, 0) &= g(x, y); 0 \le x \le a,\ 0 \le y \le b \end{aligned} \tag{3}$$

where $0 < \alpha \le 1$.

Let $u(x, y, t) = P(x)Q(y)R(t)$. Then taking necessary derivatives in equation (1) and dividing all sides by $\kappa^2 P(x)Q(y)R(t)$ we have,

$$\frac{1}{\kappa^2 R(t)} \frac{\delta^\alpha}{\delta t^\alpha} \frac{\partial^\alpha R}{\partial t^\alpha} = \frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} + \frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} = -\mu^2. \tag{4}$$

If we handle the right side of the equation we get,

$$\frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} = -\frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} - \mu^2. \tag{5}$$

For $0 < x < a$ the right side and for $0 < y < b$ the left side of the equation (5) does not change. So we obtain,

$$\frac{1}{P(x)} \frac{\partial^2 P}{\partial x^2} = -\frac{1}{Q(y)} \frac{\partial^2 Q}{\partial y^2} - \mu^2 = -\vartheta^2 \tag{6}$$

where ϑ is separation constant. From equations (4) and (6) we have the following equalities,

$$\begin{aligned} &\frac{\delta^\alpha}{\delta t^\alpha} \frac{\partial^\alpha R}{\partial t^\alpha} + \kappa^2 \mu^2 R(t) = 0, \\ &\frac{\partial^2 P}{\partial x^2} + \vartheta^2 P = 0, \\ &\frac{\partial^2 Q}{\partial y^2} + \iota^2 Q = 0. \end{aligned}$$

By using (f) in Theorem 1.1. [11] and Eq.(29) in [12] which states the sequential conformable fractional derivative the first equation becomes,

$$(1-\alpha)t^{1-2\alpha}R'(t)+t^{2-2\alpha}R''(t)+\kappa^2\mu^2R(t)=0.$$

The solution of above equation can be easily obtained as,

$$R(t)=A\cos\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right)+B\sin\left(\frac{\kappa\mu t^{\alpha}}{\alpha}\right)$$

and the solutions of other equalities can be obtained as,

$$P(x)=C\cos(\vartheta x)+D\sin(\vartheta x)$$
$$Q(y)=E\cos(\iota y)+F\sin(\iota y)$$

and the conditions (2) force these equalities to be

$$P(x)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi x}{a}\right), \tag{7}$$
$$Q(y)=\sum_{m=1}^{\infty}F_m\sin\left(\frac{m\pi y}{b}\right). \tag{8}$$

Hence from our assumption $u(x,y,t)$ can be obtained as

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\phi_{mn}\cos\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)+\gamma_{mn}\sin\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{9}$$

where $\eta_{mn}=\dfrac{m^2\pi^2}{b^2}+\dfrac{n^2\pi^2}{a^2}$.

Now with the help of conditions (3), the coefficients in equation (9) can be found as,

$$\phi_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx, \tag{10}$$
$$\gamma_{mn}=\frac{4}{\eta_{mn}ab\kappa}\int_0^a\int_0^b g(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx. \tag{11}$$

So the solution of equation (1) obtained as,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left[\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx\right]\cos\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right.$$
$$\left.+\left[\frac{4}{\eta_{mn}ab\kappa}\int_0^a\int_0^b g(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx\right]\sin\left(\frac{\eta_{mn}\kappa t^{\alpha}}{\alpha}\right)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

3 Three Dimensional Time Fractional Wave Equation

In three dimensional space, when an object or gas makes free vibration motion in a prism, without an outer effect, this motion corresponds to three dimensional wave equation. In fractional form this equation can be expressed as

$$\frac{\delta^{\alpha}}{\delta t^{\alpha}}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\kappa^2\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right),\ 0<x<a,\ 0<y<b, 0<z<d, t>0 \tag{12}$$

with the conditions

$$u(0,y,z,t)=u(a,y,z,t)=u(x,0,z,t)=u(x,b,z,t)=u(x,y,0,t)=u(x,y,d,t)=0 \tag{13}$$

$$\begin{aligned} u(x,y,z,0) &= f(x,y);\ 0\le x\le a,\ 0\le y\le b, 0\le z\le d \\ u_t(x,y,z,0) &= g(x,y); 0\le x\le a,\ 0\le y\le b, 0\le z\le d \end{aligned} \tag{14}$$

where $0<\alpha\le 1$ and the derivative is conformable fractional derivative. Assume that $u(x,y,t)=P(x)Q(y)R(z)T(t)$. Then taking derivatives which are necessary in equation (12) and dividing all sides by $\kappa^2P(x)Q(y)R(z)T(t)$ we have,

$$\frac{1}{\kappa^2T(t)}\frac{\delta^{\alpha}}{\delta t^{\alpha}}\frac{\partial^{\alpha}T}{\partial t^{\alpha}}=\frac{1}{P(x)}\frac{\partial^2P}{\partial x^2}+\frac{1}{Q(y)}\frac{\partial^2Q}{\partial y^2}+\frac{1}{R(z)}\frac{\partial^2R}{\partial z^2}=-\mu^2. \tag{15}$$

If we repeat the same procedure which is given in Section 2, we obtain the following equations,

$$\frac{\delta^\alpha}{\delta t^\alpha}\frac{\partial^\alpha T}{\partial t^\alpha}+\kappa^2\mu^2 T(t)=0,$$
$$\frac{\partial^2 P}{\partial x^2}+\vartheta^2 P=0,$$
$$\frac{\partial^2 Q}{\partial y^2}+\iota^2 Q=0,$$
$$\frac{\partial^2 R}{\partial z^2}+\rho^2 R=0.$$

By using (f) in Theorem 1.1. [11] and Eq.(29) in [12] which states the sequential conformable fractional derivative the first equation becomes,

$$(1-\alpha)t^{1-2\alpha}T'(t)+t^{2-2\alpha}T''(t)+\kappa^2\mu^2 T(t)=0$$

The solution of above equation can be easily obtained as,

$$T(t)=A\cos\left(\frac{\kappa\mu t^\alpha}{\alpha}\right)+B\sin\left(\frac{\kappa\mu t^\alpha}{\alpha}\right). \tag{16}$$

Additionally solutions of other equations can be obtained as,

$$P(x)=C\cos(\vartheta x)+D\sin(\vartheta x)$$
$$Q(y)=E\cos(\iota y)+F\sin(\iota y)$$

and the conditions (13) force these equalities to be,

$$P(x)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi x}{a}\right), \tag{17}$$
$$Q(y)=\sum_{m=1}^{\infty}F_m\sin\left(\frac{m\pi y}{b}\right), \tag{18}$$
$$R(z)=\sum_{r=1}^{\infty}K_r\sin\left(\frac{r\pi z}{d}\right). \tag{19}$$

Thus, from our assumption, we have $u(x,y,z,t)$ as,

$$\begin{aligned}u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\ &\phi_{mnr}\cos\left(\frac{\eta_{mnr}\kappa t^\alpha}{\alpha}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)\\ &+\gamma_{mnr}\sin\left(\frac{\eta_{mnr}\kappa t^\alpha}{\alpha}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{r\pi z}{d}\right)\end{aligned} \tag{20}$$

By using conditions (14) we have the coefficients ϕ_{mnr} and γ_{mnr} as,

$$\phi_{mn} = \frac{8}{abd} \int_0^a \int_0^b \int_0^d f(x, y, z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{r\pi z}{d}\right) dzdydx, \tag{21}$$

$$\gamma_{mn} = \frac{8}{\eta_{mnr} abd\kappa} \int_0^a \int_0^b \int_0^d g(x, y, z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{r\pi z}{d}\right) dzdydx. \tag{22}$$

4 Conclusions

In this paper we discuss about the solution of two and three dimensional time fractional wave equation. With the help of conformable fractional derivative definition we can easily transform fractional differential equations to the known classical differential equations. By this solution procedure there is no need any other complex methods or complex definitions to get the exact solution of the problem. From the solution procedure and the results it is easily seen that this definition is convenient and applicable for the solution of higher dimensional partial fractional differential equations.

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