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ON (L, M)-FUZZY TOPOGENOUS SPACES

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Abstract – In this paper, we introduce the concept of an (L, M)-fuzzy topogenous space, where L, M are strictly two sided commutative quantales lattices. Basic properties of (L, M)-fuzzy topogenous spaces are studied, (L, M)-fuzzy topological spaces, (L, M)-fuzzy uniform spaces and (L, M)-fuzzy proximity space are characterized in the framework of (L, M)-fuzzy topogenous spaces. We study some relationships between previous spaces and give their examples. The notion of their continuity property is investigated.

Keywords – *Complete residuated lattice, (L, M)-fuzzy topogenous order, (L, M)-fuzzy uniform space, L-fuzzy topologies.*

1 Introduction

The concepts of topogenous order and topogenous space were first introduced by Császèr [8] in 1963. These concepts allow to develop a unified approach to the three spaces: topologies, proximities and uniformities. This enabled him to evolve a theory including the foundations of the three classical theories of topological spaces, uniform spaces and proximity spaces.

In the case of the fuzzy structures there are at least two notions of fuzzy topogenous structures, the first notion worked out in (Katsaras 1983 [24], 1985 [26], 1988 [27]) present a unified approach to the theories of Chang in 1968 [6] fuzzy topological spaces, Hutton fuzzy uniform spaces (Hutton, 1977 [19]), Katsaras fuzzy proximity

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spaces (Katsaras 1979 [21], 1985 [26], 1990 [28]) and Artico fuzzy proximity (Artico and Moresco 1984 [2]).

The second notion worked out in Katsaras (1990 [28],1991 [29]) agree very well with Lowen fuzzy topological spaces (Lowen 1976 [35]) and Lowen-Höhle fuzzy uniform spaces (Lowen 1981 [36]). Čimoka [7] introduced L-fuzzy topogenous structures in complete lattices. El-Dardery investigated L-fuzzy topogenous order which induced L-fuzzy topology [9].

Based on the idea of (L, M)-fuzzy topological space introduced by Kubiak [33, 34] (the motivation for this concept comes from an idea of Höhle [15] which was called fuzzifying topology in [46]).

In this paper, we introduce the concept of an (L, M)-fuzzy topogenous space, where L, M are strictly two sided commutative quantales lattices. Basic properties of (L, M)-fuzzy topogenous spaces are studied, (L, M)-fuzzy topological spaces, (L, M)-fuzzy uniform space and (L, M)-fuzzy proximity space are characterized in the framework of (L, M)-fuzzy topogenous spaces. We give some important propositions that link the previous spaces to each other. We study some relationships between previous spaces and give their examples. The notion of their continuity property is investigated.

2 Preliminary

In this paper, Let X be a non-empty set and let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with the least element 0_L and the greatest element 1_L in L.

Definition 2.1. [14, 16, 41] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,

(L2) $x = x \odot 1$, for each $x \in L$ and 1 is the universal upper bound,

(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)$.

There exists a further binary operation $\rightarrow$ (called the implication operator or residuated) satisfying the following condition

$$x \rightarrow y = \bigvee \{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence; i.e, $(x \odot z) \leq y$ iff $z \leq (x \rightarrow y)$.

Remark 2.2. Every completely distributive lattice $(L, \leq, \wedge, \vee, ^*)$ with an order reversing involution * is a stsc-quantale $(L, \leq, \odot, \oplus, ^*)$ with an order reversing involution * where $\odot = \wedge$ and $\oplus = \vee$.

In this paper, we always assume that $(L, \leq, \odot, \oplus, ^*)$ (resp. $(M, \leq, \odot, \oplus, ^*)$) is a stsc-quantale with an order reversing involution * which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow 0$$

unless otherwise specified.

Lemma 2.3. [16, 17, 42] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$ and $x \to 0 = x^*$,
(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,
(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(5) $x \odot (\bigwedge_i y_i) \leq \bigwedge_i (x \odot y_i)$,
(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$, $x \oplus (\bigvee_i y_i) = \bigvee_i (x \oplus y_i)$,
(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,
(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,
(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,
(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,
(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,
(13) $x \odot (x^* \oplus y^*) \leq y^*$, $x \odot y = (x \to y^*)^*$ and $x \oplus y = x^* \to y$,
(14) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(15) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,
(16) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

Definition 2.4. [10, 11] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)) \ \ \forall\ \lambda, \mu \in L^X,$$

then S is an L-partial order on L^X. For $\lambda, \mu \in L^X$, $S(\lambda, \mu)$ can be interpreted as the degree to which λ is a subset of μ. It is called the subsethood degree or the fuzzy inclusion order.

Lemma 2.5. [10, 11] Let S be the fuzzy inclusion order, then $\forall\ \lambda, \mu, \rho, \nu \in L^X$ and $a \in L$ the following statements hold

(1) $\mu \leq \rho \Leftrightarrow S(\mu, \rho) = 1$,
(2) $S(\lambda, \rho) \odot (\rho, \mu) \leq S(\lambda, \mu)$,
(3) $\mu \leq \rho \Rightarrow S(\lambda, \mu) \leq S(\lambda, \rho)$ and $S(\mu, \lambda) \geq S(\rho, \lambda)$ $\forall\ \lambda \in L^X$,
(4) $S(\lambda, \mu) \odot S(\rho, \nu) \leq S(\lambda \odot \rho, \mu \odot \nu)$, and $S(\lambda, \mu) \wedge S(\rho, \nu) \leq S(\lambda \wedge \rho, \mu \wedge \nu)$.

Definition 2.6. [34] A map $\mathcal{T} : L^X \to M$ is called an (L, M)-fuzzy topology on X if it satisfies the following conditions.

(O1) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1_M$,
(O2) $\mathcal{T}(\lambda_1 \odot \lambda_2) \geq \mathcal{T}(\lambda_1) \odot \mathcal{T}(\lambda_2)$ $\forall\ \lambda_1, \lambda_2 \in L^X$,
(O3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i)$ $\forall\ \lambda_i \in L^X$, $i \in I$.

The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces and $f : X \to Y$ be a map. Then f is called LF-continuous if

$$\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(f^{\leftarrow}(\lambda)) \ \ \forall\ \lambda \in L^Y.$$

Definition 2.7. [7] A map $\mathcal{F} : L^X \to M$ is called an (L, M)-fuzzy cotopology on X if it satisfies the following conditions.

(F1) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1$,

(F2) $\mathcal{F}(\lambda_1 \oplus \lambda_2) \geq \mathcal{F}(\lambda_1) \odot \mathcal{F}(\lambda_2), \;\; \forall\, \lambda_1, \lambda_2 \in L^X$,

(F3) $\mathcal{F}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{F}(\lambda_i), \; \forall\, \lambda_i \in L^X, \, i \in I$.

The pair $(X, \mathcal{F})$ is called an (L, M)-fuzzy cotopological space. Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be (L, M)-fuzzy topological spaces and $f : X \to Y$ be a map. Then f is called LF-continuous if

$$\mathcal{F}_2(\lambda) \leq \mathcal{F}_1(f^{\leftarrow}(\lambda)), \quad \forall \lambda \in L^Y.$$

3 Perfect (L, M)-fuzzy topogenous structures and (L, M)-fuzzy topologies

Definition 3.1. A mapping $\xi : L^X \times L^X \to L$ is called an (L, M)-fuzzy semi-topogenous order on X if it satisfies the following axioms.

(ST1) $\xi(1_X, 1_X) = \xi(0_X, 0_X) = 1_M$,

(ST2) $\xi(\lambda, \mu) \leq S(\lambda, \mu)$,

(ST3) If $\lambda_1 \leq \lambda, \;\; \mu \leq \mu_1$, then $\xi(\lambda, \mu) \leq \xi(\lambda_1, \mu_1)$.

Remark 3.2. If ξ is an (L, M)-fuzzy semi-topogenous order on X. Then

(1) If $\xi(\lambda, \mu) = 1_M$, then $\lambda \leq \mu$,

(2) $\xi(1_X, \lambda) \leq \bigwedge_x \lambda(x)$ and $\xi(\lambda, 0_X) \leq \bigwedge_x \lambda^*(x)$,

(3) Define a mapping $\xi^s : L^X \times L^X \to M$ as $\xi^s(\lambda, \mu) = \xi(\mu^*, \lambda^*)$. Then ξ^s is an (L, M)-fuzzy semi-topogenous order on X.

Definition 3.3. An (L, M)-fuzzy semi-topogenous order ξ on X is called symmetric if

(S) $\;\; \xi = \xi^s$.

Definition 3.4. For every $\lambda_1, \lambda_2, \mu_1, \mu_2 \in L^X$, an (L, M)-fuzzy semi-topogenous order ξ is called

(1) (L, M)-fuzzy topogenous if

$$(T) \quad \xi(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) \geq \xi(\lambda_1, \mu_1) \odot \xi(\lambda_2, \mu_2),$$

(2) (L, M)-fuzzy co-topogenous if

$$(CT) \quad \xi(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2) \geq \xi(\lambda_1, \mu_1) \odot \xi(\lambda_2, \mu_2),$$

(3) (L, M)-fuzzy bitopogenous if ξ are (L, M)-fuzzy topogenous and (L, M)-fuzzy cotopogenous.

Remark 3.5. Let $(L = M, \odot = \wedge, \oplus = \vee)$ be a complete lattice, then (L, M)-fuzzy bitopogenous order is an L-fuzzy topogenous in a Čimoka sense from:
(T) $\xi(\lambda \wedge \lambda, \mu_1 \wedge \mu_2) \geq \xi(\lambda, \mu_1) \wedge \xi(\lambda_2, \mu_2)$,
(CT) $\xi(\lambda_1 \vee \lambda_2, \mu \vee \mu) \geq \xi(\lambda_1, \mu) \wedge \xi(\lambda_2, \mu)$.

Definition 3.6. An (L, M)-fuzzy topogenous (resp. cotopogenous) order ξ on X is said to be L-fuzzy topogenous (resp. cotopogenous) space if $\xi \circ \xi \geq \xi$, where
(TS) $(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in L^X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu)$.

Definition 3.7. An (L, M)-fuzzy semi-topogenous order ξ on X is called perfect if
(ST4) $\xi(\bigvee_i \lambda_i, \mu) \geq \bigwedge_i \xi(\lambda_i, \mu)$.
An (L, M)-fuzzy semi-topogenous order ξ on X is called co-perfect if
(ST5) $\xi(\lambda, \bigwedge_i \mu_i) \geq \bigwedge_i \xi(\lambda, \mu_i)$.
An (L, M)-fuzzy semi-topogenous order ξ on X is called bi-perfect if ξ are (L, M)-fuzzy perfect and (L, M)-fuzzy co-perfect.

Theorem 3.8. Let ξ_1 and ξ_2 be (L, M)-fuzzy cotopogenous (respectively, topogenous, perfect, co-perfect) order on X. Define the composition $(\xi_1 \circ \xi_2)$ of ξ_1 and ξ_2 by

$$(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu).$$

Then $(\xi_1 \circ \xi_2)$ is (L, M)-fuzzy cotopogenous (respectively, topogenous perfect, co-perfect) order on X.

Proof. (ST2) By Lemma 2.5 (2), we have

$$(\xi_1 \circ \xi_2)(\lambda, \mu) = \bigvee_{\rho \in X} \xi_1(\lambda, \rho) \odot \xi_2(\rho, \mu) \leq \bigvee_{\rho \in X} S(\lambda, \rho) \odot S(\rho, \mu) \leq S(\lambda, \mu).$$

(CT)

$$\begin{aligned}
&(\xi_1 \circ \xi_2)(\lambda_1, \mu_1) \odot (\xi_1 \circ \xi_2)(\lambda_2, \mu_2) \\
&= \bigvee_{\rho_1 \in L^X} (\xi_1(\lambda_1, \rho_1) \odot \xi_2(\rho_1, \mu_1)) \odot \bigvee_{\rho_2 \in L^X} (\xi_1(\lambda_2, \rho_2) \odot \xi_2(\rho_2, \mu_2)) \\
&\leq \bigvee_{\rho_1, \rho_2 \in L^X} ((\xi_1(\lambda_1, \rho_1) \odot \xi_1(\lambda_2, \rho_2)) \odot (\xi_2(\rho_1, \mu_1) \odot \xi_2(\rho_2, \mu_2))) \\
&\leq \bigvee_{\rho_1, \rho_2 \in L^X} (\xi_1(\lambda_1 \oplus \lambda_2, \rho_1 \oplus \rho_2) \odot \xi_2(\rho_1 \oplus \rho_2, \mu_1 \oplus \mu_2) \leq (\xi_1 \circ \xi_2)(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).
\end{aligned}$$

Other cases are similarly proved .

Theorem 3.9. Let ξ be a co-perfect (L, M)-fuzzy cotopogenous order, then
(1) The mapping $\mathcal{F}_\xi : L^X \to M$ defined by $\mathcal{F}_\xi(\lambda) = \xi(\lambda, \lambda)$ is an (L, M)-fuzzy cotopology on X,
(2) ξ^s is a perfect (L, M)-fuzzy topogenous order.

Proof. (1) (F1) $\mathcal{F}_\xi(0_X) = \xi(0_X, 0_X) = \xi(1_X, 1_X) = \mathcal{F}_\xi(1_X) = 1$,

(F2) $\mathcal{F}_\xi(\lambda_1 \oplus \lambda_2) = \xi(\lambda_1 \oplus \lambda_2, \lambda_1 \oplus \lambda_2) \geq \xi(\lambda_1, \lambda_1) \odot \xi(\lambda_2, \lambda_2) = \mathcal{F}_\xi(\lambda_1) \odot \mathcal{F}_\xi(\lambda_2)$,

(F3) $\mathcal{F}_\xi(\bigwedge_i \lambda_i) = \xi(\bigwedge_i \lambda_i, \bigwedge_i \lambda_i) \geq \bigwedge_i \xi(\bigwedge_i \lambda_i, \lambda_i) \geq \bigwedge_i \xi(\lambda_i, \lambda_i) = \bigwedge_i \mathcal{F}_\xi(\lambda_i)$.

$$\begin{aligned}(2)\ \ (T)\quad & \xi^s(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) = \xi((\mu_1 \odot \mu_2)^*, (\lambda_1 \odot \lambda_2)^*) \\ & = \xi(\mu_1^* \oplus \mu_2^*, \lambda_1^* \oplus \lambda_2^*) \geq \xi(\mu_1^*, \lambda_1^*) \odot \xi(\mu_2^*, \lambda_2^*) \geq \xi^s(\lambda_1, \mu_1) \odot \xi^s(\lambda_2, \mu_2).\end{aligned}$$

Other cases are easily proved.

Theorem 3.10. Let $\mathcal{F}$ be an (L, M)-fuzzy cotopology on X, then
(1) The mapping $\xi_\mathcal{F} : L^X \times L^X \to M$ defined by

$$\xi_\mathcal{F}(\lambda, \mu) = \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\}$$

is a co-perfect L-fuzzy cotopogenous space. Moreover, $\mathcal{F}_{\xi_\mathcal{F}} = \mathcal{F}$,

(2) If ξ is a co-perfect (L, M)-fuzzy cotopogenous order, then $\xi_{\mathcal{F}_\xi} \leq \xi$.

Proof. (1) (ST1) $\xi_\mathcal{F}(0_X, 0_X) = \bigvee\{\mathcal{F}(\gamma) \mid 0_X \leq \gamma \leq 0_X, \gamma \in L^X\} = \mathcal{F}(0_X) = 1$,

$$\xi_\mathcal{F}(1_X, 1_X) = \bigvee\{\mathcal{F}(\gamma) \mid 1_X \leq \gamma \leq 1_X, \gamma \in L^X\} = \mathcal{F}(1_X) = 1.$$

(ST2) If $\lambda \leq \gamma$, then $S(\lambda, \mu) = 1$. If $\lambda \not\leq \gamma$, then

$$\bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\} = 0.$$

It is trivial.
(ST3) If $\lambda_1 \leq \lambda$, $\mu \leq \mu_1$, then $\lambda_1 \leq \lambda \leq \gamma \leq \mu \leq \mu_1$. So, $\lambda_1 \leq \gamma \leq \mu_1$. Thus,

$$\begin{aligned}\xi_\mathcal{F}(\lambda, \mu) &= \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\} \\ &\leq \bigvee\{\mathcal{F}(\gamma) \mid \lambda_1 \leq \gamma \leq \mu_1, \gamma \in L^X\} = \xi_\mathcal{F}(\lambda_1, \mu_1).\end{aligned}$$

(CT)

$$\begin{aligned}\xi_\mathcal{F}(\lambda_1, \mu_1) \odot \xi_\mathcal{F}(\lambda_2, \mu_2) &= \big(\bigvee\{\mathcal{F}(\gamma_1) \mid \lambda_1 \leq \gamma_1 \leq \mu_1\}\big) \odot \big(\bigvee\{\mathcal{F}(\gamma_2) \mid \lambda_2 \leq \gamma_2 \leq \mu_2\}\big) \\ &\leq \bigvee\{\mathcal{F}(\gamma_1) \odot \mathcal{F}(\gamma_2) \mid \lambda_1 \oplus \lambda_2 \leq \gamma_1 \oplus \gamma_2 \leq \mu_1 \oplus \mu_2\} \\ &\leq \bigvee\{\mathcal{F}(\gamma_1 \oplus \gamma_2) \mid \lambda_1 \oplus \lambda_2 \leq \gamma_1 \oplus \gamma_2 \leq \mu_1 \oplus \mu_2\} \\ &\leq \bigvee\{\mathcal{F}(\gamma) \mid \lambda_1 \oplus \lambda_2 \leq \gamma \leq \mu_1 \oplus \mu_2\} \\ &= \xi_\mathcal{F}(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).\end{aligned}$$

(ST5)

$$\xi_{\mathcal{F}}(\lambda, \bigwedge_i \mu_i) = \bigvee\{\mathcal{F}(\gamma)|\lambda \leq \gamma \leq \bigwedge_i \mu_i\} = \bigvee\{\mathcal{F}(\gamma)|\gamma = \bigwedge_i \gamma_i,\ \lambda \leq \gamma_i \leq \mu_i\}$$
$$\geq \bigvee\{\bigwedge_i \mathcal{F}(\gamma_i)|\lambda \leq \gamma_i \leq \mu_i\} = \bigwedge_i \big(\bigvee\{\mathcal{F}(\gamma_i)|\lambda \leq \gamma_i \leq \mu_i\}\big) = \bigwedge_i \xi_{\mathcal{F}}(\lambda, \mu_i).$$

Finally, $\mathcal{F}_{\xi_{\mathcal{F}}}(\lambda) = \xi_{\mathcal{F}}(\lambda, \lambda) = \bigvee\{\mathcal{F}(\gamma) \mid \lambda \leq \gamma \leq \lambda, \gamma \in L^X\} = \mathcal{F}(\lambda).$

(2) $\xi_{\mathcal{F}_\xi}(\lambda, \mu) = \bigvee\{\mathcal{F}_\xi(\gamma) \mid \lambda \leq \gamma \leq \mu\} = \bigvee\{\xi(\gamma, \gamma) \mid \lambda \leq \gamma \leq \mu\} \leq \xi(\lambda, \mu).$

Theorem 3.11. Let ξ be a perfect (L, M) -fuzzy topogenous order, then
(1) The mapping $\mathcal{T}_\xi : L^X \to M$ defined by $\mathcal{T}_\xi(\lambda) = \xi(\lambda, \lambda)$ is an L-fuzzy topology on X,
(2) ξ^s is a coperfect (L, M)-fuzzy cotopogenous order such that $\mathcal{F}_{\xi^s}(\lambda) = \mathcal{T}_\xi(\lambda^*)$,
(3) If ξ is a symmetric bi-perfect (L, M)-fuzzy bitopogenous order, then $\mathcal{T}_\xi = \mathcal{F}_\xi$.

Proof. (1) It is similarly proved as Theorem 3.9(1).
(2) $\mathcal{T}_\xi(\lambda^*) = \xi(\lambda^*, \lambda^*) = \xi^s(\lambda, \lambda) = \mathcal{F}_{\xi^s}(\lambda)$,
(3) $\mathcal{T}_\xi(\lambda) = \xi(\lambda, \lambda) = \xi^s(\lambda, \lambda) = \mathcal{F}_{\xi^s}(\lambda)$.

Theorem 3.12. Let $\mathcal{T}$ be an (L, M)-fuzzy topology on X.
(1) The mapping $\xi_{\mathcal{T}} : L^X \times L^X \to M$ defined by

$$\xi_{\mathcal{T}}(\lambda, \mu) = \bigvee\{\mathcal{T}(\gamma) \mid \lambda \leq \gamma \leq \mu, \gamma \in L^X\}$$

is a perfect (L, M)-fuzzy topogenous space. Moreover, $\mathcal{T}_{\xi_{\mathcal{T}}} = \mathcal{T}$,
(2) If $\mathcal{F}_{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda^*)$ is an (L, M)-fuzzy topology on X, then $\xi_{\mathcal{F}_{\mathcal{T}}} = \xi^s_{\mathcal{T}}$.

Proof. (1) It is similarly proved as Theorem 3.10 (1).

(2) $\xi_{\mathcal{F}_{\mathcal{T}}}(\lambda, \mu) = \bigvee\{\mathcal{F}_{\mathcal{T}}(\gamma)|\lambda \leq \gamma \leq \mu\} = \bigvee\{\mathcal{T}(\gamma^*)|\mu^* \leq \gamma^* \leq \lambda^*\} = \xi_{\mathcal{T}}(\mu^*, \lambda^*) = \xi^s_{\mathcal{T}}(\lambda, \mu).$

Example 3.13. Let $(L = M = [0, 1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y\}$ be a set and $u, v \in L^X$ such that

$$u(x) = 0.6, u(y) = 0.5, \quad v(x) = 0.4, v(y) = 0.7.$$

Define $\mathcal{T}, \mathcal{F} : L^X \to M$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\} \\ 0.6, & \text{if } \lambda = u, \\ 0.3, & \text{if } \lambda = u \odot u, \\ 0, & \text{otherwise} \end{cases} \quad , \mathcal{F}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\} \\ 0.7, & \text{if } \lambda = v, \\ 0.4, & \text{if } \lambda = v \oplus v, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Since $0.3 = \mathcal{T}(u \odot u) \geq \mathcal{T}(u) \odot \mathcal{T}(u) = 0.2$, $\mathcal{T}$ is an (L, M)-fuzzy topology on X. By Theorem, we obtain a perfect topogenous space $\xi_{\mathcal{T}} : L^X \times L^X \to L$ as follows

$$\xi_{\mathcal{T}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X, \\ 0.6, & \text{if } u \odot u \not\geq \lambda \leq u \leq \rho, \\ 0.3, & \text{if } 0_X \neq \lambda \leq u \odot u \leq \rho, u \not\leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.12, we obtain a co-perfect cotopogenous space $\xi^s_{\mathcal{T}} : L^X \times L^X \to L$ as follows

$$\xi^s_{\mathcal{T}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.6, & \text{if } \lambda \leq u^* \leq \rho, \rho \not\geq u^* \oplus u^* \\ 0.3, & \text{if } \lambda \leq u^* \oplus u^* \leq \rho \neq 1_X, \lambda \not\leq u^*, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{F}_{\xi^s_{\mathcal{T}}}(\lambda) = \mathcal{T}(\lambda^*)$.

(2) Since $0.4 = \mathcal{F}(v \oplus v) \geq \mathcal{F}(v) \odot \mathcal{F}(v) = 0.4$, $\mathcal{F}$ is an (L, M)-fuzzy cotopology on X. By Theorem 3.10, we obtain co-perfect cotopogenous order $\xi_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\xi_{\mathcal{F}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.7, & \text{if } v \oplus v \not\geq \lambda \leq v \leq \rho, \\ 0.5, & \text{if } 0_X \neq \lambda \leq v \oplus v \leq \rho, v \not\leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.10, we obtain perfect topogenous order $\xi_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\xi^s_{\mathcal{F}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } \rho = 1_X \\ 0.7, & \text{if } v \oplus v \not\geq \lambda \leq v^* \leq \rho, \rho \not\geq v^* \odot v^* \\ 0.5, & \text{if } \lambda \leq v^* \odot v^* \leq \rho, \lambda \not\leq v^*, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{T}_{\xi^s_{\mathcal{F}}}(\lambda) = \mathcal{F}(\lambda^*)$.

Definition 3.14. Let ξ_X and ξ_Y be two (L, M)-fuzzy semi-topogenous orders on X and Y, respectively. A mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is said to be topogenous continuous if

$$\xi_Y(\lambda, \mu) \leq \xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)), \quad \forall\ \lambda, \mu \in L^Y.$$

Theorem 3.15. Let (X, ξ_X) and (Y, ξ_Y) be perfect (L, M)-fuzzy topogenous space. If a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then the mapping $f : (X, \mathcal{T}_{\xi_X}) \to (Y, \mathcal{T}_{\xi_Y})$ is LF-continuous.

Conversely, a mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is LF-continuous iff $f : (X, \xi_{\mathcal{T}_X}) \to (Y, \xi_{\mathcal{T}_Y})$ is topogenous continuous.

Proof. Since $f : (X, \xi_X) \to (Y, \xi_Y)$ is LF-topogenous continuous, then

$$\mathcal{T}_{\xi_X}(f^{\leftarrow}(\lambda)) = \xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\lambda)) \geq \xi_Y(\lambda, \lambda) = \mathcal{T}_{\xi_Y}(\lambda).$$

Conversely, since $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is LF-continuous, then

$$\begin{aligned}\xi_{\mathcal{T}_Y}(\lambda, \mu) = \bigvee\{\mathcal{T}_Y(\gamma) \mid \lambda \leq \gamma \leq \mu\} &\leq \bigvee\{\mathcal{T}_X(f^{\leftarrow}(\gamma)) \mid f^{\leftarrow}(\lambda) \leq f^{\leftarrow}(\gamma) \leq f^{\leftarrow}(\mu)\} \\ &\leq \bigvee\{\mathcal{T}_X(\rho) \mid \rho = f^{\leftarrow}(\gamma),\ f^{\leftarrow}(\lambda) \leq \rho \leq f^{\leftarrow}(\mu)\} \\ &= \xi_{\mathcal{T}_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)).\end{aligned}$$

Conversely, since $\mathcal{T}_{\xi_{\mathcal{T}_X}} = \mathcal{T}_X$ and $\mathcal{T}_{\xi_{\mathcal{T}_X}} = \mathcal{T}_X$ from Theorem 3.12(1), it is trivial.

Corollary 3.16. Let (X, ξ_X) and (Y, ξ_Y) be co-perfect (L, M)-fuzzy cotopogenous space. If a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then the mapping $f : (X, \mathcal{F}_{\xi_X}) \to (Y, \mathcal{F}_{\xi_Y})$ is LF-continuous.

Conversely, a mapping $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is LF-continuous iff $f : (X, \xi_{\mathcal{F}_X}) \to (Y, \xi_{\mathcal{F}_Y})$ is topogenous continuous.

Lemma 3.17. Let $f : X \to Y$ be a mapping, then the following inequalities hold.
(1) $(f^{\to}(\mu^*))^* \leq f^{\to}(\mu)$,
(2) $S(f^{\to}(\lambda), (f^{\to}(\mu^*))^*) \leq S(\lambda, \mu), \quad \forall\ \lambda, \mu \in L^X$,
(3) $S(\lambda, \mu) \leq S(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \quad \forall\ \lambda, \mu \in L^Y$,
(4) $f^{\to}(\lambda \odot \mu) \leq f^{\to}(\lambda) \odot f^{\to}(\mu)$,
(5) $f^{\to}(\lambda \oplus \mu) \leq f^{\to}(\lambda) \oplus f^{\to}(\mu)$,
(6) $(f^{\to}((\lambda \odot \mu)^*))^* \geq (f^{\to}(\lambda^*))^* \odot (f^{\to}(\mu^*))^*$,
(7) $(f^{\to}((\lambda \oplus \mu)^*))^* \geq (f^{\to}(\lambda^*))^* \oplus (f^{\to}(\mu^*))^*$.

Proof. (1)

$$(f^{\to}(\mu^*))^*(y) = (\bigvee_{x \in f^{-1}(y)} \mu^*(x))^* = \bigwedge_{x \in f^{-1}(\{y\})} \mu(x) \leq \bigvee_{x \in f^{-1}(\{y\})} \mu(x) = (f^{\to}(\mu))(y).$$

(2) Let $y_\circ \in Y$, then

$$\begin{aligned}S(f^{\to}(\lambda), (f^{\to}(\mu^*))^*) &= \bigwedge_{y \in Y}(f^{\to}(\lambda) \to (f^{\to}(\mu^*))^*)(y) \\ &\leq f^{\to}(\lambda)(y_\circ) \to (f^{\to}(\mu^*))^*(y_\circ) \\ &= \bigvee_{x \in f^{-1}(y_\circ)} \lambda(x) \to (\bigvee_{x \in f^{-1}(y_\circ)} \mu^*(x))^* \\ &= \bigvee_{x \in f^{-1}(y_\circ)} \lambda(x) \to \bigwedge_{x \in f^{-1}(y_\circ)} \mu(x) \leq \lambda(x) \to \mu(x).\end{aligned}$$

Hence, $S(f^{\to}(\lambda), (f^{\to}(\mu^*))^*) \leq S(\lambda, \mu)$.

(3)

$$S(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) = \bigwedge_{x\in X}(\lambda(f(x)) \to \mu(f(x))) \geq \bigwedge_{y\in Y}(\lambda(y) \to \mu(y)) = S(\lambda,\mu).$$

(4)

$$\begin{aligned} f^{\rightarrow}(\lambda \odot \mu)(y) = \bigvee_{x\in f^{-1}(\{y\})}(\lambda \odot \mu)(x) &\leq (\bigvee_{x\in f^{-1}(\{y\})}\lambda(x)) \odot (\bigvee_{x\in f^{-1}(\{y\})}\mu(x)) \\ &\leq f^{\rightarrow}(\lambda)(y) \odot f^{\rightarrow}(\mu)(y). \end{aligned}$$

(5)

$$\begin{aligned} f^{\rightarrow}(\lambda \oplus \mu)(y) = \bigvee_{x\in f^{-1}(\{y\})}(\lambda \oplus \mu)(x) &\leq (\bigvee_{x\in f^{-1}(\{y\})}\lambda(x)) \oplus (\bigvee_{x\in f^{-1}(\{y\})}\mu(x)) \\ &\leq f^{\rightarrow}(\lambda)(y) \oplus f^{\rightarrow}(\mu)(y). \end{aligned}$$

Other cases are easily proved.

Theorem 3.18. let $f : X \to Y$ be a mapping. Let ξ be an (L,M)-fuzzy topogenous (co-topogenous, perfect, co-perfect, respectively) order on Y. We define the pre-image $f^{\leftarrow}(\xi)$ of ξ under f as

$$f^{\leftarrow}(\xi)(\lambda,\mu) = \xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*), \quad \forall\ \lambda,\mu \in L^X.$$

Then,

(1) $f^{\leftarrow}(\xi)$ is an (L,M)-fuzzy topogenous (co-topogenous, perfect, co-perfect, respectively) order on X. Moreover, if $\xi \circ \xi \leq \xi$, then $f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi) \leq f^{\leftarrow}(\xi)$.

(2) A mapping $f : (X,\xi_X) \to (Y,\xi_Y)$ is topogenous continuous if and only if $f^{\leftarrow}(\xi) \leq \xi_X$.

Proof. (1) (ST2) By Lemma 3.17, we have

$$f^{\leftarrow}(\xi)(\lambda,\mu) = \xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) \leq S(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) \leq S(\lambda,\mu).$$

(T)

$$\begin{aligned} f^{\leftarrow}(\xi)(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) &= \xi(f^{\rightarrow}(\lambda_1 \odot \lambda_2), (f^{\rightarrow}((\mu_1 \odot \mu_2)^*))^*) \\ &= \xi(f^{\rightarrow}(\lambda_1) \odot f^{\rightarrow}(\lambda_2), (f^{\rightarrow}(\mu_1^*))^* \odot (f^{\rightarrow}(\mu_2^*))^*) \\ &\geq \xi(f^{\rightarrow}(\lambda_1), (f^{\rightarrow}(\mu^*))^*) \odot \xi(f^{\rightarrow}(\lambda_2), (f^{\rightarrow}(\mu^*))^*) \\ &= f^{\leftarrow}(\xi)(\lambda_1,\mu_1) \odot f^{\leftarrow}(\xi)(\lambda_2,\mu_2). \end{aligned}$$

(CT)

$$\begin{aligned} f^{\leftarrow}(\xi)(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2) &= \xi(f^{\rightarrow}(\lambda_1 \oplus \lambda_2), (f^{\rightarrow}((\mu_1 \oplus \mu_2)^*))^*) \quad \text{(by Lemma 3.17)} \\ &= \xi(f^{\rightarrow}(\lambda_1) \oplus f^{\rightarrow}(\lambda_2), (f^{\rightarrow}(\mu_1^*))^* \oplus (f^{\rightarrow}(\mu_2^*))^*) \\ &\geq \xi(f^{\rightarrow}(\lambda_1), (f^{\rightarrow}(\mu^*))^*) \odot \xi(f^{\rightarrow}(\lambda_2), (f^{\rightarrow}(\mu^*))^*) \\ &= f^{\leftarrow}(\xi)(\lambda_1,\mu_1) \odot f^{\leftarrow}(\xi)(\lambda_2,\mu_2). \end{aligned}$$

If $\xi \circ \xi \leq \xi$, then $f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi) \leq f^{\leftarrow}(\xi)$ since

$$\begin{aligned} f^{\leftarrow}(\xi) \circ f^{\leftarrow}(\xi)(\lambda, \mu) &= \bigvee_{\rho \in L^X} (f^{\leftarrow}(\xi)(\lambda, \rho) \odot f^{\rightarrow}(\xi)(\rho, \mu)) \\ &= \bigvee_{\rho \in L^X} (\xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\rho^*))^*) \odot \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\mu^*))^*) \ (by\ Lemma\ 3.17(1)) \\ &\leq \bigvee_{\rho \in L^X} (\xi(f^{\rightarrow}(\lambda), f^{\rightarrow}(\rho)) \odot \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\mu^*))^*) \\ &\leq \xi(f^{\rightarrow}(\lambda), (f^{\rightarrow}(\mu^*))^*) = f^{\leftarrow}(\xi)(\lambda, \mu). \end{aligned}$$

(2) For any $\rho, \nu \in L^X$, we have

$$f^{\leftarrow}(\xi)(\rho, \nu) = \xi(f^{\rightarrow}(\rho), (f^{\rightarrow}(\nu^*)^*)) \leq \xi_X(f^{\leftarrow}(f^{\rightarrow}(\rho)), f^{\leftarrow}(f^{\rightarrow}(\nu^*)^*)) \leq \xi_X(\rho, \nu),$$

$$\xi_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \geq f^{\leftarrow}(\xi)(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) = \xi(f^{\rightarrow} f^{\leftarrow}(\lambda), (f^{\rightarrow}(f^{\leftarrow}(\mu))^*)^*) \geq \xi(\lambda, \mu).$$

4 Perfect (L, M)-fuzzy topogenous space and (L, M)-fuzzy quasi-proximities

Kim et al [30] introduced the concept of L-fuzzy proximities in a strictly two sided, commutative quantales. We here reintroduce them in a slightly different way as follows.

Definition 4.1. A mapping $\delta : L^X \times L^X \to M$ is called an (L, M)-fuzzy quasi-proximity on X if it satisfies the following axioms.

(QP1) $\delta(0_X, 1_X) = \delta(1_X, 0_X) = 0_M$,

(QP2) $\delta(\lambda, \mu) \geq \bigvee_{x \in X} (\lambda \odot \mu)(x)$,

(QP3) If $\lambda_1 \leq \lambda_2, \rho_1 \leq \rho_2$, then $\delta(\lambda_1, \rho_1) \leq \delta(\lambda_2, \rho_2)\ \forall\ \rho \in L^X$,

(QP4) $\delta(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \leq \delta(\lambda_1, \rho_1) \oplus \delta(\lambda_2, \rho_2)$,

(QP5) $\delta(\lambda, \mu) \geq \bigwedge_{\rho} \{\delta(\lambda, \rho) \oplus \delta(\mu, \rho^*)\}$.

The pair (X, δ) is called an (L, M)-fuzzy quasi-proximity space. We call $\delta(\lambda, \mu)$ a gradation of nearness.

Let δ_1 and δ_2 be (L, M)-fuzzy quasi-proximities on X. Then δ_1 is called coarser than δ_2 if $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$ for all $\lambda, \mu \in L^X$.

An (L, M)-fuzzy quasi-proximity is called (L, M)-fuzzy proximity on X if it satisfies the following axiom

(P) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$.

An (L, M)-fuzzy quasi-proximity is called perfect if it satisfies the following axiom

(PP) $\delta(\bigvee_{i\in\Gamma}\lambda_i,\mu)=\bigvee_{i\in\Gamma}\delta(\lambda_i,\rho)$.

An (L,M)-fuzzy quasi-proximity is called co-perfect if it satisfies the following axiom

(CPP) $\delta(\lambda,\bigvee_{i\in\Gamma}\rho_i)=\bigvee_{i\in\Gamma}\delta(\lambda,\rho_i)$.

Proposition 4.2. (1) If δ is an (L,M)-fuzzy quasi-proximity space and we define $\delta^s : L^X\times L^X\to M$ by

$$\delta^s(\lambda,\mu)=\delta(\mu^*,\lambda^*),\quad \forall\ \lambda,\mu\in L^X,$$

then δ^s is an (L,M)-fuzzy quasi-proximity space.

(2) If (X,ξ) is a perfect (L,M)-fuzzy topogenous space and we define $\delta_\xi : L^X\times L^X\to M$ by

$$\delta_\xi(\lambda,\mu)=\xi^*(\lambda,\mu^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then δ_ξ is a perfect (L,M)-fuzzy quasi-proximity space on X. Moreover, if ξ is symmetric, then δ_ξ is a bi-perfect (L,M)-fuzzy proximity space on X.

(3) If (X,ξ) is a co-perfect (L,M)-fuzzy co-topogenous space and we define δ_ξ : $L^X\times L^X\to M$ by

$$\delta_\xi(\lambda,\mu)=\xi^*(\mu,\lambda^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then δ_ξ is a co-perfect (L,M)-fuzzy quasi-proximity space on X. Moreover, if ξ is symmetric, then δ_ξ is a bi-perfect (L,M)-fuzzy proximity space on X.

(4) If δ is an (resp. perfect) (L,M)-fuzzy quasi-proximity space and we define $\xi_\delta : L^X\times L^X\to M$ by

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\ \ \forall\ \lambda,\mu\in L^X,$$

then ξ_δ is an (resp. perfect) (L,M)-fuzzy topogenous space such that $\delta_{\xi_\delta}=\delta$. Moreover, if ξ is an (resp. perfect) (L,M)-fuzzy topogenous space, then $\xi_{\delta_\xi}=\xi$.

(5) If δ is an (resp. co-perfect) (L,M)-fuzzy quasi-proximity space and we define $\xi_\delta : L^X\times L^X\to M$ by

$$\xi_\delta(\lambda,\mu)=\delta^*(\mu^*,\lambda)\ \ \forall\ \lambda,\mu\in L^X,$$

then ξ_δ is an (resp. co-perfect) (L,M)-fuzzy co-topogenous space such that $\delta_{\xi_\delta}=\delta$. Moreover, if ξ is an (resp. co-perfect) (L,M)-fuzzy co-topogenous space, then $\xi_{\delta_\xi}=\xi$.

Proof. (1) It is easily proved.

(2) (QP1) $\delta_\xi(1_X,0_X)=\xi^*(1_X,0_X^*)=\xi^*(1_X,0_X)=0_M$. Similarly, $\delta_\xi(0_X,1_X)=0$.

(QP2) By Definition 3.1 (ST2) and Lemma 2.3 (16), we have

$$\delta_\xi(\lambda,\mu) \geq (S(\lambda,\mu^*))^* = (\bigwedge_{x\in X}(\lambda(x)\to\mu^*(x))^* = \bigvee_{x\in X}(\lambda(x)\to\mu^*(x))^* = \bigvee_{x\in X}(\lambda\odot\mu)(x).$$

(QP3) If $\lambda \geq \mu$, then

$$\xi(\lambda,\rho^*) \leq \xi(\mu,\rho^*) \text{ iff } \xi^*(\mu,\rho^*) \leq \xi^*(\lambda,\rho^*), \text{ then } \delta_\xi(\mu,\rho) \leq \delta_\xi(\lambda,\rho).$$

(QP4)

$$\begin{aligned}\delta_\xi(\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2) &= \xi^*(\lambda_1\odot\lambda_2,(\rho_1\oplus\rho_2)^*) = \xi^*(\lambda_1\odot\lambda_2,\rho_1^*\odot\rho_2^*)\\ &\leq \xi^*(\lambda_1,\rho_1^*)\oplus\xi^*(\lambda_2,\rho_2^*) = \delta_\xi(\lambda_1,\rho_1)\oplus\delta_\xi(\lambda_2,\rho_2).\end{aligned}$$

(QP5) Since $\xi\circ\xi \geq \xi$ by definition 3.7, then

$$\begin{aligned}\delta_\xi(\lambda,\mu) = \xi^*(\lambda,\mu^*) \geq (\xi\circ\xi)^*(\lambda,\mu^*) &= \big(\bigvee_{\gamma\in L^X}\xi(\lambda,\gamma)\odot\xi(\gamma,\mu^*)\big)^*\\ = \bigwedge_{\gamma\in L^X}\xi^*(\lambda,\gamma)\oplus\xi^*(\gamma,\mu^*) &= \bigwedge_{\gamma\in L^X}\delta_\xi(\lambda,\gamma^*)\oplus\delta_\xi(\gamma,\mu).\end{aligned}$$

(PP) $\delta_\xi(\bigvee_{i\in\Gamma}\lambda_i,\mu) = \xi^*(\bigvee_{i\in\Gamma}\lambda_i,\mu^*) = \bigvee_{i\in\Gamma}\xi^*(\lambda_i,\mu^*) = \bigvee_{i\in\Gamma}\delta_\xi(\lambda_i,\mu).$

Let $\xi = \xi^s$ be given, then ξ is co-perfect by

$$\xi(\lambda,\bigwedge_{i\in\Gamma}\rho_i) = \xi^s(\lambda,\bigwedge_{i\in\Gamma}\rho_i) = \xi(\bigvee_{i\in\Gamma}\rho_i^*,\lambda^*) = \bigwedge_{i\in\Gamma}\xi(\rho_i^*,\lambda^*) = \bigwedge_{i\in\Gamma}\xi^s(\lambda,\rho_i) = \bigwedge_{i\in\Gamma}\xi(\lambda,\rho_i).$$

(P) $\delta_\xi(\lambda,\mu) = \xi^*(\lambda,\mu^*) = (\xi^s)^*(\lambda,\mu^*) = \xi^*(\mu,\lambda^*) = \delta_\xi(\mu,\lambda).$

(CPP) $\delta_\xi(\lambda,\bigvee_{i\in\Gamma}\rho_i) = \xi^*(\lambda,\bigwedge_{i\in\Gamma}\rho_i^*) = \bigvee_{i\in\Gamma}\xi^*(\lambda,\rho_i^*) = \bigvee_{i\in\Gamma}\delta_\xi(\lambda,\rho_i)$. Hence δ_ξ is a biperfect (L,M)-fuzzy proximity space on X.

(3) It is similarly proved as (2).

(QP4)

$$\begin{aligned}\delta_\xi(\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2) &= \xi^*(\rho_1\oplus\rho_2,(\lambda_1\odot\lambda_2)^*) = \xi^*(\rho_1\oplus\rho_2,\lambda_1^*\oplus\lambda_2^*)\\ &\leq \xi^*(\rho_1,\lambda_1^*)\oplus\xi^*(\rho_2,\lambda_2^*) = \delta_\xi(\lambda_1,\rho_1)\oplus\delta_\xi(\lambda_2,\rho_2).\end{aligned}$$

Other cases are similarly proved as (2).

(4) (ST1) $\xi_\delta(1_X,1_X) = \delta^*(1_X,1_X^*) = \delta^*(1_X,0_X) = 0^* = 1_M,$

$$\xi_\delta(0_X,0_X) = \delta^*(0_X,0_X^*) = \delta^*(0_X,1_X) = 0^* = 1_M.$$

(ST2) From Lemma 2.3 (16), we have

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\le\big(\bigvee_{x\in X}(\lambda\odot\mu^*)(x)\big)^*=\bigwedge_{x\in X}(\lambda\odot\mu^*)^*(x)$$
$$=\bigwedge_{x\in X}(\lambda(x)\to\mu(x))=S(\lambda,\mu).$$

(ST3) If $\lambda_1\le\lambda,\ \mu\le\mu_1$, then from (QP3) and (QP6)

$$\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)\ge\delta^*(\lambda_1,\mu^*)=\delta^*(\mu^*,\lambda_1)\ge\delta^*(\mu_1^*,\lambda_1)=\delta^*(\lambda_1,\mu_1^*)=\xi_\delta(\lambda_1,\mu_1).$$

(ST4) Obviously, $\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)=\delta^*(\mu^*,\lambda)=\xi_\delta(\mu^*,\lambda^*)=\xi_\delta^s(\lambda,\mu)$.

(T)

$$\xi_\delta(\lambda_1,\mu_1)\odot\xi_\delta(\lambda_2,\mu_2)=\delta^*(\lambda_1,\mu_1^*)\odot\delta^*(\lambda_2,\mu_2^*)=\big(\delta(\lambda_1,\mu_1^*)\oplus\delta(\lambda_2,\mu_2^*)\big)^*$$
$$\le\delta^*(\lambda_1\odot\lambda_2,\mu_1^*\oplus\mu_2^*)=\delta^*(\lambda_1\odot\lambda_2,(\mu_1\odot\mu_2)^*)=\xi_\delta(\lambda_1\odot\lambda_2,\mu_1\odot\mu_2).$$

$$\delta_{\xi_\delta}(\lambda,\mu)=\xi_\delta^*(\lambda,\mu^*)=\delta(\lambda,\mu),\quad \xi_{\delta_\xi}(\lambda,\mu)=\delta_\xi^*(\lambda,\mu^*)=\xi(\lambda,\mu).$$

(5) (CT)

$$\xi_\delta(\lambda_1,\mu_1)\odot\xi_\delta(\lambda_2,\mu_2)=\delta^*(\mu_1^*,\lambda_1)\odot\delta^*(\mu_2^*,\lambda_2)=\big(\delta(\mu_1^*,\lambda_1)\oplus\delta(\mu_2^*,\lambda_2)\big)^*$$
$$\le\delta^*(\mu_1^*\odot\mu_2^*,\lambda_1\oplus\lambda_2)=\delta^*((\mu_1\oplus\mu_2)^*,\lambda_1\oplus\lambda_2)=\xi_\delta(\lambda_1\oplus\lambda_2,\mu_1\oplus\mu_2).$$

$$\delta_{\xi_\delta}(\lambda,\mu)=\xi_\delta^*(\mu,\lambda^*)=\delta(\lambda,\mu),\quad \xi_{\delta_\xi}(\lambda,\mu)=\delta_\xi^*(\mu^*,\lambda)=\xi(\lambda,\mu).$$

Theorem 4.3. Let δ be an (L,M)-fuzzy quasi-proximity on X, then

(1) If δ is perfect and the mapping $\mathcal{T}_\delta:L^X\to M$ defined by $\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)$, then $\mathcal{T}_\delta$ is an (L,M)-fuzzy topology on X.

(2) If δ is co-perfect and the mapping $\mathcal{F}_\delta:L^X\to M$ defined by $\mathcal{F}_\delta(\lambda)=\delta^*(\lambda^*,\lambda)$, then $\mathcal{F}_\delta$ is an (L,M)-fuzzy cotopology on X.

(3) If δ is a perfect (L,M)-fuzzy proximity on X, then $\mathcal{T}_\delta(\lambda)=\mathcal{F}_\delta(\lambda)$.

Proof. (1) Let δ be a perfect (L,M)-fuzzy quasi-proximity on X and define $\xi_\delta(\lambda,\mu)=\delta^*(\lambda,\mu^*)$, then ξ_δ a perfect (L,M)-fuzzy topogenuous and

$$\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)=\xi_\delta(\lambda,\lambda).$$

Hence $\mathcal{T}_\delta$ is an (L,M)-fuzzy topology on X.

(2) It is easily proved as $\mathcal{T}_\delta(\lambda)=\delta^*(\lambda,\lambda^*)=\delta^*(\lambda^*,\lambda)=\mathcal{T}_\delta(\lambda)$.

Theorem 4.4. Let $\mathcal{F}$ be an (L, M)-fuzzy co-topology on X, then

(1) The mapping $\delta_{\mathcal{F}} : L^X \times L^X \to M$ defined by

$$\delta_{\mathcal{F}}(\lambda, \mu) = \bigwedge\{(\mathcal{F}(\gamma))^* \mid \mu \leq \gamma \leq \lambda^*\}$$

is a co-perfect (L, M)-fuzzy quasi-proximity space. Moreover, $\mathcal{F}_{\delta_{\mathcal{F}}} = \mathcal{F}$.

(2) If δ is a co-perfect (L, M)-fuzzy quasi-proximity on X, then $\delta_{\mathcal{F}_\delta} \geq \delta$.

Theorem 4.5. Let $\mathcal{T}$ be an (L, M)-fuzzy topology on X, then

(1) The mapping $\delta_{\mathcal{T}} : L^X \times L^X \to M$ defined by

$$\delta_{\mathcal{T}}(\lambda, \mu) = \bigwedge\{(\mathcal{T}(\gamma))^* \mid \lambda \leq \gamma \leq \mu^*\}$$

is a perfect (L, M)-fuzzy quasi-proximity space. Moreover, $\mathcal{T}_{\delta_{\mathcal{T}}} = \mathcal{T}$.

(2) If $\mathcal{F}_{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda^*)$ is an (L, M)-fuzzy topology on X, then $\delta_{\mathcal{F}_{\mathcal{T}}} = \delta^s_{\mathcal{T}}$.

Example 4.6. Let ξ_i be given as Example 3.13 and since $\delta_{\xi_i}(\lambda, \rho) = \xi_i^*(\lambda, \rho^*)$, then we have

$$\delta_{\xi_1}(\lambda, \rho) = S^*(\lambda, \rho^*) = \bigvee_{x \in X} (\lambda \odot \rho)(x),$$

$$\delta_{\xi_2}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X, \text{ or } \rho = 0_X, \\ 1, & \text{otherwise} \end{cases}, \delta_{\xi_3}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda \leq \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

Example 4.7. Let $\mathcal{T}, \mathcal{F}$ be given as Example 3.13.

(1) By Theorems 4.2(2) and 4.5, we obtain a perfect (L, M)-quasi-proximity $\delta_{\xi_{\mathcal{T}}} = \delta_{\mathcal{T}} : L^X \times L^X \to M$ as follows

$$\delta_{\xi_{\mathcal{T}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.4, & \text{if } u \odot u \not\geq \lambda \leq u \leq \rho^*, \\ 0.7, & \text{if } 0_X \neq \lambda \leq u \odot u \leq \rho^*, u \not\leq \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

By Theorems 4.2(2) and 4.5, we obtain a co-perfect (L, M)-quasi-proximity $\delta_{\xi^s_{\mathcal{T}}} = \delta_{\xi_{\mathcal{T}^*}} : L^X \times L^X \to M$ with $\mathcal{T}^*(\lambda) = \mathcal{T}(\lambda^*)$ as follows

$$\delta_{\xi^s_{\mathcal{T}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.4, & \text{if } \lambda \leq u^* \leq \rho^*, \rho^* \not\geq u^* \oplus u^* \\ 0.7, & \text{if } \lambda \leq u^* \oplus u^* \leq \rho^* \neq 1_X, \lambda \not\leq u^*, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{F}_{\delta_{\xi^s_{\mathcal{T}}}}(\lambda) = \mathcal{T}(\lambda^*)$.

(2) By Theorems 4.2(2) and 4.4, we obtain co-perfect (L, M)-quasi-proximity $\delta_{\xi_{\mathcal{F}}} = \delta_{\mathcal{F}} : L^X \times L^X \to M$ as follows

$$\delta_{\xi_{\mathcal{F}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.3, & \text{if } v \oplus v \not\geq \lambda \leq v \leq \rho^*, \\ 0.5, & \text{if } 0_X \neq \lambda \leq v \oplus v \leq \rho^*, v \not\leq \rho^*, \\ 1, & \text{otherwise.} \end{cases}$$

By Theorems 4.2(2) and 4.4, we obtain perfect (L, M)-quasi-proximity $\delta_{\xi^s_{\mathcal{F}}} = \delta_{\mathcal{F}^*} : L^X \times L^X \to M$ with $\mathcal{F}^*(\lambda) = \mathcal{F}(\lambda^*)$ as follows

$$\delta_{\xi^s_{\mathcal{F}}}(\lambda, \rho) = \begin{cases} 0, & \text{if } \lambda = 0_X \text{ or } \rho = 0_X \\ 0.3, & \text{if } v \oplus v \not\geq \lambda \leq v^* \leq \rho^*, \rho^* \not\geq v^* \odot v^* \\ 0.5, & \text{if } \lambda \leq v^* \odot v^* \leq \rho^*, \lambda \not\leq v^*, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{T}_{\delta_{\xi^s_{\mathcal{F}}}}(\lambda) = \mathcal{F}(\lambda^*)$.

Definition 4.8. Let (X, δ_X) and (Y, δ_Y) be two (L, M)-fuzzy quasi-proximity spaces. A mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is said to be L-fuzzy proximally continuous if

$$\delta_X(\lambda, \mu) \leq \delta_Y(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^X,$$

or equivalently, $\quad \delta_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) \leq \delta_Y(\lambda, \mu)$.

Theorem 4.9. A mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ of two (L, M)-fuzzy quasi-proximity spaces is L-fuzzy proximally continuous iff the the mapping $f : (X, \xi_{\delta_X}) \to (Y, \xi_{\delta_Y})$ is topogenous continuous.

Conversely, a mapping $f : (X, \xi_X) \to (Y, \xi_Y)$ of (L, M)-fuzzy topogenous spaces is topogenous continuous iff the mapping $f : (X, \delta_{\xi_X}) \to (Y, \delta_{\xi_Y})$ of the corresponding (L, M)-fuzzy quasi-proximity spaces is L-fuzzy proximally continuous.

Proof. Since $f : (X, \delta_X) \to (Y, \delta_Y)$ is L-fuzzy proximally continuous, then

$$\begin{aligned} \xi_{\delta_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) &= \delta^*_X(f^{\leftarrow}(\lambda), (f^{\leftarrow}(\mu))^*) \\ &= \delta^*_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu^*)) \leq \delta^*_Y(\lambda, \mu^*) = \xi_{\delta_Y}(\lambda, \mu). \end{aligned}$$

Conversely, Since $f : (X, \xi_X) \to (Y, \xi_Y)$ is topogenous continuous, then

$$\begin{aligned} \delta_{\xi_X}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)) &= \xi^*_X(f^{\leftarrow}(\lambda), (f^{\leftarrow}(\mu))^*) \\ &= \xi^*_X(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu^*)) \leq \xi^*_Y(\lambda, \mu^*) = \delta_{\xi_Y}(\lambda, \mu). \end{aligned}$$

Theorem 4.10. Let (Y, δ) be an (L, M)-fuzzy quasi-proximity space, X be a non-empty set and $f : X \to Y$ be a mapping. We define $\delta_f : L^X \times L^X \to M$ by

$$\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^X.$$

Then,

(1) δ_f is the coarsest (L, M)-fuzzy quasi-proximity for which f is L-fuzzy proximally continuous,

(2) A mapping $g : (Z, \xi) \to (X, \delta_f)$ is L-fuzzy proximally continuous iff $f \circ g$ is L-fuzzy proximally continuous.

Proof. (QP1) $\delta_f(1_X, 0_X) = \delta(f^{\to}(1_X), f^{\to}(0_X)) \leq \delta(1_Y, 0_Y) = 0_M$. Similarly,

$$\delta_f(0_X, 1_X) = 0_M.$$

(QP2)

$$\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)) \geq \bigvee_{y \in Y} (f^{\to}(\lambda) \odot f^{\to}(\mu))(y)$$
$$\geq \bigvee_{x \in f^{\leftarrow}(y_\circ)} \lambda(x) \odot \bigvee_{x \in f^{\leftarrow}(y_\circ)} \mu(x) \geq \bigvee_{x \in X} \lambda(x) \odot \mu(x) = \bigvee_{x \in X} (\lambda \odot \mu)(x).$$

(QP3) If $\lambda \leq \mu$, then $\delta_f(\lambda, \rho) = \delta(f^{\to}(\lambda), f^{\to}(\rho)) \leq \delta(f^{\to}(\mu), f^{\to}(\rho)) = \delta_f(\mu, \rho)$.

(QP4)

$$\begin{aligned}\delta_f(\lambda_1, \rho_1) \oplus \delta_f(\lambda_2, \rho_2) &= \delta(f^{\to}(\lambda_1), f^{\to}(\rho_1)) \oplus \delta(f^{\to}(\lambda_2), f^{\to}(\rho_2))\\ &\geq \delta(f^{\to}(\lambda_1) \odot f^{\to}(\lambda_2), f^{\to}(\rho_1) \oplus f^{\to}(\rho_2))\\ &\geq \delta(f^{\to}(\lambda_1 \odot \lambda_2), f^{\to}(\rho_1 \oplus \rho_2)) = \delta_f(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2).\end{aligned}$$

(QP5) Since $\delta_f(\lambda, (f^{\leftarrow}(\rho))^*) = \delta(f^{\to}(\lambda), f^{\to}(f^{\leftarrow}(\rho^*))) \leq \delta(f^{\to}(\lambda), \rho^*)$, then we have

$$\begin{aligned}\delta_f(\lambda, \mu) = \delta(f^{\to}(\lambda), f^{\to}(\mu)) &\geq \bigwedge_{\rho \in L^X} \delta(f^{\to}(\lambda), \rho) \oplus \delta(f^{\to}(\mu), \rho^*)\\ &\geq \bigwedge_{f^{\leftarrow}(\rho) \in L^X} \delta_f(\lambda, f^{\leftarrow}(\rho)) \oplus \delta_f(\mu, (f^{\to}(\rho))^*)\\ &\geq \bigwedge_{\gamma \in L^X} \delta_f(\lambda, \gamma) \oplus \delta_f(\mu, \gamma^*).\end{aligned}$$

From the definition of δ_f, f is L-fuzzy proximally continuous. Let $f : (X, \delta_1) \to (Y, \delta)$ be L-fuzzy proximally continuous, and since

$$\delta_1(\lambda, \mu) \leq \delta(f^{\to}(\lambda), f^{\to}(\mu)) = \delta_f(\lambda, \mu).$$

Then, δ_f is coarser than δ_1.

(2) Let g be L-fuzzy proximally continuous. So,

$$\xi(\lambda, \mu) \leq \delta_f(g^{\to}(\lambda), g^{\to}(\mu)) = \delta(f^{\to}(g^{\to}(\lambda)), f^{\to}(g^{\to}(\mu))).$$

Hence, $f \circ g$ is L-fuzzy proximally continuous. Let $f \circ g$ be L-fuzzy proximally continuous, then

$$\xi(\lambda, \mu) \leq \delta(f^{\to}(g^{\to}(\lambda)), f^{\to}(g^{\to}(\mu))) = \delta_f(g^{\to}(\lambda), g^{\to}(\mu)).$$

Then g is L-fuzzy proximally continuous.

5 (L, M)-fuzzy topogenous order induced by (L, M)-fuzzy quasi uniformity

Definition 5.1. [31, 47] A mapping $\mathcal{U} : L^{X\times X} \to M$ is called an (L, M)-fuzzy quasi-uniformity on X iff it satisfies the properties.

(LU1) There exists $u \in L^{X\times X}$ such that $\mathcal{U}(u) = 1_M$,
(LU2) If $v \le u$, then $\mathcal{U}(v) \le \mathcal{U}(u)$,
(LU3) For every $u, v \in L^{X\times X}$, $\mathcal{U}(u \odot v) \ge \mathcal{U}(u) \odot \mathcal{U}(v)$,
(LU4) If $\mathcal{U}(u) \ne 0_M$, then $1_\triangle \le u$, where

$$1_\triangle(x, y) = \begin{cases} 1 & \text{if x = y} \\ 0 & \text{if } x \ne y, \end{cases}$$

(LU5) $\mathcal{U} \le \mathcal{U} \circ \mathcal{U}$, where $\mathcal{U} \circ \mathcal{U}(u) = \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \le u\}$,

$$v \circ w(x, y) = \bigvee_{z\in X} (v(z, x) \odot v(x, y)), \quad \forall\ x, y \in X.$$

Remark 5.2. Let $(X,\mathcal{U})$ be an (L, M)-fuzzy quasi-uniform space, then by (LU1) and (LU2), we have $\mathcal{U}(1_{X\times X}) = 1_M$ because $u \le 1_{X\times X}$ for all $u \in L^{X\times X}$.

Definition 5.3. [31, 47] Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be (L, M)-fuzzy uniform spaces, and $\phi : X \to Y$ ba a mapping. Then ϕ is said to be L-uniformly continuous if

$$\mathcal{V}(v) \le \mathcal{U}((\phi \times \phi)^{\leftarrow}(v)),$$

for every $v \in L^{Y\times Y}$.

Lemma 5.4. [31] Let $(X,\mathcal{U})$ be an (L, M)-fuzzy quasi-uniform space. For each $u \in L^{X\times X}$ and $\lambda \in L^X$, the image $u[\lambda]$ of λ with respect to u is the fuzzy subset of X defined by

$$u[\lambda](x) = \bigvee_{y\in X} (\lambda(y) \odot u(y, x)), \quad \forall\ x \in X.$$

For each $u, v, u_1, u_2 \in L^{X\times X}$ and $\lambda, \rho, \lambda_1, \lambda_2, \lambda_i \in L^X$, we have

(1) $\lambda \le u[\lambda]$, for each $\mathcal{U}(u) > 0_M$,
(2) $u \le u \circ u$, for each $\mathcal{U}(u) > 0_M$,
(3) $(v \circ u)[\lambda] = v[u[\lambda]]$,
(4) $u[\bigvee_i \lambda_i] = \bigvee_i u[\lambda_i]$,
(5) $(u_1 \odot u_2)[\lambda_1 \odot \lambda_2] \le u_1[\lambda_1] \odot u_2[\lambda_2]$,
(6) $(u_1 \odot u_2)[\lambda_1 \oplus \lambda_2] \le u_1[\lambda_1] \oplus u_2[\lambda_2]$.

Theorem 5.5. Let $(X,\mathcal{U})$ be an (L,M)-fuzzy quasi-uniform space. Define a mapping $\xi_{\mathcal{U}} : L^X \times L^X \to M$ by

$$\xi_{\mathcal{U}}(\lambda,\mu) = \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu\}.$$

Then $(X,\xi_{\mathcal{U}})$ is an (L,M)-fuzzy topogenous space.

Proof. . (ST1) Since $u[0_X] = 0_X$ and $u[1_X] = 1_X$, for $\mathcal{U}(u) = 1_M$, we have $\xi_{\mathcal{U}}(1_X,1_X) = \xi_{\mathcal{U}}(0_X,0_X) = 1_M$.

(ST2) Since for all $\mathcal{U}(u) > 0_M$, we have $\lambda \leq u[\lambda]$. Then if $\xi_{\mathcal{U}}(\lambda,\mu) = 1_M$, we have $\lambda \leq \mu$.

(ST3) If $\lambda_1 \leq \lambda,\ \mu \leq \mu_1$, then

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda,\mu) = \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu\} &\leq \bigvee\{\mathcal{U}(u) \mid u[\lambda] \leq \mu_1\}\\ &\leq \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} = \xi_{\mathcal{U}}(\lambda_1,\mu_1).\end{aligned}$$

(T)

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1,\mu_1) \odot \xi_{\mathcal{U}}(\lambda_2,\mu_2) &= \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} \odot \bigvee\{\mathcal{U}(v) \mid v[\lambda_2] \leq \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u) \odot \mathcal{U}(v) \mid u[\lambda_1] \odot v[\lambda_2] \leq \mu_1 \odot \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u \odot v) \mid (u \odot v)[\lambda_1 \odot \lambda_2] \leq \mu_1 \odot \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(w) \mid w[\lambda_1 \odot \lambda_2] \leq \mu_1 \odot \mu_2\}\\ &= \xi_{\mathcal{U}}(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2).\end{aligned}$$

(CT)

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1,\mu_1) \odot \xi_{\mathcal{U}}(\lambda_2,\mu_2) &= \bigvee\{\mathcal{U}(u) \mid u[\lambda_1] \leq \mu_1\} \odot \bigvee\{\mathcal{U}(v) \mid v[\lambda_2] \leq \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u) \odot \mathcal{U}(v) \mid u[\lambda_1] \oplus v[\lambda_2] \leq \mu_1 \oplus \mu_2\}\\ &\leq \bigvee\{\mathcal{U}(u \odot v) \mid u \odot v[\lambda_1 \oplus \lambda_2] \leq \mu_1 \oplus \mu_2\}\\ &= \xi_{\mathcal{U}}(\lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).\end{aligned}$$

(TS) For each $u \in L^{X\times X}$ such that $u[\lambda] \leq \mu$, by (LU5), we have

$$\mathcal{U}(u) = \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u\}.$$

Thus,

$$\begin{aligned}\bigvee\{\mathcal{U}(u) \mid u[\lambda] \le \mu\} &\le \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w[\lambda] = v[w[\lambda]] \le \mu\} \\ &\le \bigvee_{\gamma \in L^X} \{\bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid w[\lambda] \le \gamma,\ v[\gamma] \le \mu\}\} \\ &\le \bigvee_{\gamma \in L^X} \{\bigvee\{\mathcal{U}(v) \mid v[\gamma] \le \mu\} \odot \bigvee\{\mathcal{U}(w) \mid w[\lambda] \le \gamma\}\} \\ &= \bigvee_{\gamma \in L^X} \xi_{\mathcal{U}}(\lambda, \gamma) \odot \xi_{\mathcal{U}}(\gamma, \mu).\end{aligned}$$

Example 5.6. Let $(L = M = [0, 1], \odot, \to)$ be a complete residuated lattice defined as

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y, z\}$ be a set and $w \in L^{X \times X}$ such that

$$w = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}, \quad w \odot w = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}.$$

Define $\mathcal{U} : L^{X \times X} \to M$ as follows

$$\mathcal{U}(u) = \begin{cases} 1, & \text{if } u = \top_{X \times X}, \\ 0.6, & \text{if } w \le u \ne \top_{X \times X}, \\ 0.3, & \text{if } \ w \odot w \le u \not\ge w, \\ 0, & \text{otherwise.} \end{cases}$$

Since $0.3 = \mathcal{U}(w \odot w) \ge \mathcal{U}(w) \odot \mathcal{U}(w) = 0.2$ and $w \circ w = w$, $(w \odot w) \circ (w \odot w) = (w \odot w)$, then $\mathcal{U}$ is an (L, M)-fuzzy quasi-uniformity on X.

By Theorem 5.5, we obtain (L, M)-fuzzy topogenous order $\xi_{\mathcal{U}} : L^X \times L^X \to M$ as follows

$$\xi_{\mathcal{U}}(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda \le \bigvee_{x \in X} \lambda(x) \le \rho, \\ 0.6, & \text{if } 0_X \ne \lambda \le w[\lambda] \le \rho, \bigvee_{x \in X} \lambda(x) \not\le \rho, \\ 0.3, & \text{if } \ \lambda \le (w \odot w)[\lambda] \le \rho, w[\lambda] \not\le \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5.7. Let (X, ξ_X) and (Y, ξ_Y) be two (L, M)-fuzzy topogenous orders and let $f : X \to Y$ be a map. Then $f : (X, \xi_X) \to (Y, \xi_Y)$ is called an L-fuzzy open topogenous map if

$$\xi_X(\lambda, \mu) \le \xi_Y(f^{\to}(\lambda), f^{\to}(\mu)), \quad \forall\ \lambda, \mu \in L^{X \times X}.$$

Theorem 5.8. Let $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ be (L, M)-fuzzy quasi-uniform spaces. If $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is LF-uniformly continuous, then $f : (X, \xi_{\mathcal{U}}) \to (Y, \xi_{\mathcal{V}})$ is an L-fuzzy open topogenous map.

Proof. Let $v[f^{\rightarrow}(\lambda)] \leq f^{\rightarrow}(\mu)$, then

$$(f \times f)^{\leftarrow}(v)[\lambda] = f^{\leftarrow}(v[f^{\rightarrow}(\lambda)]) \leq f^{\leftarrow}f^{\rightarrow}(\mu) \leq \mu.$$

Hence,

$$\begin{aligned}
\xi_{\mathcal{V}}(f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu)) &= \bigvee\{\mathcal{V}(v) \mid v[f^{\rightarrow}(\lambda)] \leq f^{\rightarrow}(\mu)\} \\
&\geq \bigvee\{\mathcal{U}((f \times f)^{\leftarrow}(v)) \mid f^{\leftarrow}(v[f^{\rightarrow}(\lambda)]) \leq f^{\leftarrow}(f^{\rightarrow}(\mu))\} \\
&\geq \bigvee\{\mathcal{U}((f \times f)^{\leftarrow}(v)) \mid (f \times f)^{\leftarrow}(v)[\lambda] \leq \mu\} \\
&\geq \bigvee\{\mathcal{U}(w) \mid w[\lambda] \leq \mu\} = \xi_{\mathcal{U}}(\lambda, \mu).
\end{aligned}$$

Theorem 5.9. Let $(X,\mathcal{U})$ be an (L, M)-quasi uniform space. Define a mapping $\xi_{\mathcal{U}} : L^{X\times X} \rightarrow L$ such that

$$\xi_{\mathcal{U}}(\lambda, \rho) = \bigvee_{u} \big\{\mathcal{U}(u) \odot S(u[\lambda], u[\rho^*]^*)\big\},$$

then $\xi_{\mathcal{U}}$ is an (L, M)-fuzzy topogenous order.

Proof. (ST1) Since $u[0_X] = 0_X$, and $u[1_X] = 1_X$, then

$$\xi_{\mathcal{U}}(0_X, 0_X) = \xi_{\mathcal{U}}(1_X, 1_X) = \bigvee_{u} \mathcal{U}(u) = 1_M.$$

(ST2) By (QU1) and Lemma 2.3 (16), we have

$$\xi_{\mathcal{U}}(\lambda, \mu) \leq \bigwedge_{x\in X} \big(u[\lambda] \odot u[\mu^*]\big)^*(x) = \bigwedge_{x\in X} \big(u[\lambda] \rightarrow (u[\mu^*])^*\big)(x).$$

For $\mathcal{U}(u) > 0_M$, we have $\lambda \leq u[\lambda]$ and $\mu \geq (u[\mu^*])^*$. Thus, by Lemma 2.3 (2), we have

$$\bigwedge_{x\in X} \big(u[\lambda](x) \rightarrow (u[\mu^*])^*\big)(x) \leq \bigwedge_{x\in X} \big(u[\lambda](x) \rightarrow \mu(x)\big) \leq \bigwedge_{x\in X} \big(\lambda(x) \rightarrow \mu(x)\big) = S(\lambda, \mu).$$

Since $\lambda \leq u[\lambda]$, $u[\rho^*]^* \leq \rho$,

$$\xi_{\mathcal{U}}(\lambda, \rho) = \bigvee_{u} \big\{\mathcal{U}(u) \odot S(u[\lambda], u[\rho^*]^*)\big\} \leq \bigvee_{u} \big\{\mathcal{U}(u) \odot S(\lambda, \rho)\big\} \leq S(\lambda, \rho).$$

Therefore, $\xi_{\mathcal{U}}(\lambda, \mu) \leq S(\lambda, \mu)$.

(ST3) It is obvious.

(ST4) By Lemma 2.5(3) and Lemma 5.4(5), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) &= \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1 \odot \lambda_2], u[(\rho_1 \odot \rho_2)^*]^*)\} \\ &\geq \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1] \odot u[\lambda_2], u[\rho_1^*]^* \odot u[\rho_2^*]^*)\} \\ &\geq \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1], u[\rho_1^*]^*)\} \odot \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_2], u[\rho_2^*]^*)\} \\ &= \xi_{\mathcal{U}}(\lambda_1, \rho_1) \odot \xi_{\mathcal{U}}(\lambda_2, \rho_2).\end{aligned}$$

(T) By Lemma 2.5(3) and Lemma 5.4(6), we have

$$\begin{aligned}\xi_{\mathcal{U}}(\lambda_1 \oplus \lambda_2, \rho_1 \oplus \rho_2) &= \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1 \oplus \lambda_2], u[(\rho_1 \oplus \rho_2)^*]^*)\} \\ &\geq \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1] \oplus u[\lambda_2], u[\rho_1^*]^* \oplus u[\rho_2^*]^*)\} \\ &\geq \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_1], u[\rho_1^*]^*)\} \odot \bigvee_u \{\mathcal{U}(u) \odot S(u[\lambda_2], u[\rho_2^*]^*)\} \\ &= \xi_{\mathcal{U}}(\lambda_1, \rho_1) \odot \xi_{\mathcal{U}}(\lambda_2, \rho_2).\end{aligned}$$

Theorem 5.10. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two (L, M)-fuzzy quasi uniform spaces and $f : X \to Y$ be LF-uniformly continuous, then $f : (X, \xi_{\mathcal{U}}) \to (Y, \xi_{\mathcal{V}})$ is L- fuzzy topogenous continuous.

Proof. Since $(f \times f)^{\leftarrow}(v)[f^{\leftarrow}(\lambda)] = f^{\leftarrow}(v[f^{\rightarrow}(f^{\leftarrow}(\lambda))]) \leq f^{\leftarrow}(v[\lambda])$ and by Theorem 5.7 for $u = (f \times f)^{\leftarrow}(v)$, we have for all $\lambda, \mu \in L^X$

$$\begin{aligned}\xi_{\mathcal{V}}(\lambda, \mu) &= \bigvee_v \{\mathcal{V}(v) \odot S(v[\lambda], (v[\mu^*])^*)\} \\ &\leq \bigvee_v \{\mathcal{V}(v) \odot S(f^{\leftarrow}(v[\lambda]), f^{\leftarrow}((v[\mu^*])^*))\} \\ &\leq \bigvee_u \{\mathcal{U}(u) \odot S(u[f^{\leftarrow}(\lambda)], (u[f^{\leftarrow}(\mu^*)]))^*)\} \leq \xi_{\mathcal{U}}(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mu)).\end{aligned}$$

Lemma 5.11. For every $\lambda, \rho \in L^X$, we define $u_{\lambda,\rho}, u_{\lambda,\rho}^{-1} : X \times X \to L$ by

$$u_{\lambda,\rho}(x, y) = \lambda(x) \to \rho(y), \quad u_{\lambda,\rho}^{-1}(x, y) = u_{\lambda,\rho}(y, x),$$

then we have the following statements

(1) $1_{X \times X} = u_{0_X,\, 0_X} = u_{1_X,\, 1_X}$,
(2) If $\lambda_1 \leq \lambda_2$ and $\rho_1 \leq \rho_2$, then $u_{\lambda_2,\rho_1} \leq u_{\lambda_1,\rho_2}$,
(3) If $\lambda \leq \rho$, then $1_{\triangle} \leq u_{\lambda,\rho}$,
(4) For every $u_{\mu,\rho} \in L^{X \times X}$ and $\lambda \in L^X$, we have $u_{\gamma,\rho} \circ u_{\lambda,\gamma} \leq u_{\lambda,\rho}$,

(5) $u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2} \le u_{\lambda_1\odot\lambda_2}, u_{\rho_1\odot\rho_2}$,
(6) $u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2} \le u_{\lambda_1\oplus\lambda_2}, u_{\rho_1\oplus\rho_2}$,
(7) $u^{-1}_{\lambda,\rho} = u_{\rho^*,\lambda^*}$,
(8) $u^{-1}_{\lambda_1\odot\lambda_2,\rho_1\odot\rho_2} = u_{\rho_1^*\oplus\rho_2^*,\lambda_1^*\oplus\lambda_2^*}$,
(9) $u^{-1}_{\lambda_1\oplus\lambda_2,\rho_1\oplus\rho_2} = u_{\rho_1^*\odot\rho_2^*,\lambda_1^*\odot\lambda_2^*}$.

Proof.

(1) $1_{X\times X}(x,y) = 1 = u_{0_X,0_X}(x,y) = 0_X(x) \to 0_X(y) = 1_X(x) \to 1_X(y) = u_{1_X,1_X}(x,y)$.

(2) Let $\lambda_1 \le \lambda_2$ and $\rho_1 \le \rho_2$, then

$$u_{\lambda_2,\rho_1}(x,y) = \lambda_2(x) \to \rho_1(y) \le \lambda_1(x) \to \rho_2(y) = u_{\lambda_1,\rho_2}(x,y).$$

(3) Since $1_\triangle[\lambda] = \lambda \le \rho$, then $1_\triangle \le u_{\lambda,\rho}$.

(4)

$$u_{\gamma,\rho}(x,z) \circ u_{\lambda,\gamma}(x,z) = \bigvee_{y\in X}((\gamma(y) \to \rho(z))\odot(\lambda(x) \to \gamma(y))) \le \lambda(x) \to \rho(y) = u_{\lambda,\rho}(x,z).$$

(5)

$$\begin{aligned}(u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2})(x,z) &= u_{\lambda_1,\rho_1}(x,z) \odot u_{\lambda_2,\rho_2}(x,z)\\ &\le (\lambda_1(x) \to \rho_1(y)) \odot (\lambda_2(x) \to \rho_2(y))\\ &\le \lambda_1(x) \odot \lambda_2(x) \to \rho_1(y) \odot \rho_2(y) = u_{\lambda_1\odot\lambda_2}, u_{\rho_1\odot\rho_2}(x,y).\end{aligned}$$

(6)

$$\begin{aligned}(u_{\lambda_1,\rho_1} \odot u_{\lambda_2,\rho_2})(x,y) &= u_{\lambda_1,\rho_1}(x,y) \odot u_{\lambda_2,\rho_2}(x,y)\\ &\le (\lambda_1(x) \to \rho_1(y)) \odot (\lambda_2(x) \to \rho_2(y))\\ &\le \lambda_1(x) \oplus \lambda_2(x) \to \rho_1(y) \oplus \rho_2(y) = u_{\lambda_1\oplus\lambda_2}, u_{\rho_1\oplus\rho_2}(x,y).\end{aligned}$$

(7) $u^{-1}_{\lambda,\rho}(x,y) = u_{\lambda,\rho}(y,x) = \lambda(y) \to \rho(x) = \rho^*(x) \to \lambda^*(y) = u_{\rho^*,\lambda^*}(x,y)$.

(8),(9) are similarly proved.

In the following theorem, we obtain an (L,M)-fuzzy quasi uniform space from an (L,M)-fuzzy topogenous order.

Theorem 5.12. Let (X,ξ) be an (L,M)-fuzzy quasi topogenous space. Define $\mathcal{U}_\xi : L^{X\times X} \to M$ by

$$\mathcal{U}_\xi(u) = \bigvee\{\odot_{i=1}^n \xi(\mu_i,\rho_i) \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \le u\},$$

where $\bigvee$ is taken over every finite family $\{u_{\mu_i,\rho_i} \mid i = 1,2,3,....,n\}$. Then,

(1) $\mathcal{U}_\xi$ is an (L,M)-fuzzy quasi uniformity on X,

(2) $\xi_{\mathcal{U}_\xi} = \xi$.

Proof. (1) (LU1) Since $\xi(0_X, 0_X) = \xi(1_X, 1_X) = 1_M$, there exists $1_{X\times X} = u_{0_X,0_X} = u_{1_X,1_X} \in L^{X\times X}$. It follows $\mathcal{U}_\xi(1_{X\times X}) = 1_M$.

(LU2) It is trivial from the definition of $\mathcal{U}_\xi$.

(LU3) For every $u, v \in L^{X\times X}$, each two families $\{u_{\mu_i,\rho_i} \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u\}$ and $\{u_{\nu_j,w_j} \mid \odot_{j=1}^k u_{\nu_j,w_j} \leq v\}$, we have

$$\begin{aligned}\mathcal{U}_\xi(u) \odot \mathcal{U}_\xi(v) &= \Big(\bigvee\{\odot_{i=1}^n \xi(\mu_i, \rho_i) | \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u\}\Big) \odot \Big(\bigvee\{\odot_{j=1}^k \xi(\nu_i, w_i) | \odot_{j=1}^k u_{\nu_i,w_i} \leq v\}\Big) \\ &\leq \bigvee\{(\odot_{i=1}^n \xi(\mu_i, \rho_i)) \odot (\odot_{j=1}^k \xi(\nu_i, w_i)) \mid \odot_{i=1}^n u_{\mu_i,\rho_i} \leq u, \ \odot_{j=1}^k u_{\nu_i,w_i} \leq v\} \\ &\leq \bigvee\{(\odot_{i=1}^n \xi(\mu_i, \rho_i)) \odot (\odot_{j=1}^k \xi(\nu_i, w_i)) \mid (\odot_{i=1}^n u_{\mu_i,\rho_i}) \odot (\odot_{j=1}^k u_{\nu_i,w_i}) \leq u \odot v\} \\ &\leq \mathcal{U}_\xi(u \odot v).\end{aligned}$$

(LU4) If $\mathcal{U}(u) \neq 0_M$, there exists a family $\{u_{\lambda_i,\rho_i} \mid \odot_{i=1}^m u_{\lambda_i,\rho_i} \leq u\}$ such that $\odot_{j=1}^m \xi(\lambda_i, \rho_i) \neq 0_M$. Since $\xi(\lambda_i, \rho_i) \neq 0_M$, for $i = 1, 2, ..., m$, then $\lambda_i \leq \rho_i$ for $i = 1, 2, ..., m$, i.e. $1_\triangle \leq u_{\lambda_i,\rho_i}$. Thus $1_\triangle \leq \odot_{i=1}^m u_{\lambda_i,\rho_i} \leq u$.

(LU5) Suppose there exists $u \in L^{X\times X}$ such that

$$\bigvee\{\mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\} \not\geq \mathcal{U}_\xi(u).$$

Put $t = \bigvee\{ \mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \mid v \circ w \leq u\}$. From the Definition of $\mathcal{U}_\xi(u)$, there exists family $\{u_{\mu_i,\rho_i} \mid \odot_{i=1}^m u_{\mu_i,\rho_i} \leq u\}$ such that

$$t \not\geq \odot_{i=1}^m \xi(\lambda_i, \rho_i).$$

Since $\xi \circ \xi \geq \xi$, $t \not\geq \odot_{i=1}^m \xi \circ \xi(\lambda_i, \rho_i) = \odot_{i=1}^m \{\bigvee_{\gamma \in L^X} \{\xi(\gamma, \rho_i) \odot (\xi(\lambda_i, \gamma))\}\}$ and L is a stsc-quantal, then there exist $\gamma_i \in L^X$ such that

$$t \not\geq \odot_{i=1}^m (\xi(\gamma_i, \rho_i) \odot \xi(\lambda_i, \gamma_i)).$$

On the other hand $v_i = u_{\gamma_i,\rho_i}, w_i = u_{\lambda_i,\gamma_i}$, then it satisfies

$$v_i \circ w_i \leq u_{\gamma_i,\ \rho_i} \circ u_{\lambda_i,\gamma_i} \leq u_{\lambda_i,\rho_i}, \ \ \mathcal{U}_\xi(v_i) \geq \xi(\gamma_i, \rho_i), \ \mathcal{U}_\xi(w_i) \geq \xi(\lambda_i, \gamma_i).$$

Let $v = \odot_{i=1}^m v_i$ and $w = \odot_{i=1}^m w_i$ be given. Since $v_i \circ w_i \leq u_{\lambda_i,\rho_i}$, for each $i = 1, 2, 3, ..., m$, we have

$$\big(\odot_{i=1}^m \ v_i\big) \circ \big(\odot_{i=1}^m \ v_i\big) = \odot_{i=1}^m \ (v_i \circ w_i) \leq \odot_{i=1}^m \ u_{\lambda_i,\rho_i} \leq u.$$

Then, we have $v \circ w \leq u$, $\mathcal{U}_\xi(v) \geq \odot_{i=1}^m \mathcal{U}_\xi(v_i)$ and $\mathcal{U}_\xi(w) \geq \odot_{i=1}^m \mathcal{U}_\xi(w_i)$. Thus,

$$t = \bigvee\{\mathcal{U}_\xi(w) \odot \mathcal{U}_\xi(w) \mid w \circ w \leq u\} \geq \mathcal{U}_\xi(v) \odot \mathcal{U}_\xi(w) \geq \odot_{i=1}^m \ (\xi(\gamma_i, \rho_i) \odot \xi(\lambda_i, \gamma_i)).$$

It is a contradiction. Thus, $\mathcal{U}_\xi$ is an (L, M)-fuzzy quasi uniformity on X.

(2) Since $u[\lambda] \leq \rho$, then $u \leq u_{\lambda,\rho}$. Hence,

$$\xi_{\mathcal{U}_\xi}(\lambda, \rho) = \bigvee\{\mathcal{U}_\xi(u) \mid u[\lambda] \leq \rho\} = \mathcal{U}_\xi(u_{\lambda,\rho}) = \xi(\lambda, \rho).$$

6 Conclusion

The main purpose of this paper is to introduce concepts in fuzzy set theory, namely that an (L, M)-fuzzy semi-topogenous order, (L, M)-fuzzy topogenous space, (L, M)-fuzzy uniform space and the (L, M)-fuzzy proximity space in strictly two sided, commutative quantales. On the other hand, we study some relationships between previous spaces and we give their examples. As a special case our (L, M)-fuzzy topogenous structures contain classical Cs*ász*è r topogenous structures, Katasaras fuzzy topogenous structures and Čimoka L-fuzzy topogenous structures.

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