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Λ_g-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract – The notion of Λ_g-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\Lambda_g}$-open sets. Also, it is established that an $\mathcal{I}_{\Lambda_g}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

Keywords – *λ-closed set, Λ_g-closed set, $\mathcal{I}_{\Lambda_g}$-closed set, $\mathcal{I}$-compact space.*

1 Introduction and Preliminaries

In 1986, Maki [14] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of A. Arenas et al [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Caldas et al [2] introduced and investigated the notion of Λ_g-closed sets in topological spaces and established several properties of such sets.

In this paper, the notion of Λ_g-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{\Lambda_g}$-closed sets and $\mathcal{I}_{\Lambda_g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\Lambda_g}$-open sets. Also, it is established that an $\mathcal{I}_{\Lambda_g}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

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1. $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$ and
2. $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \rightarrow \wp(X)$, called a local function [11] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{I},\tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x)=\{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [[8], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A)=A \cup A^\star(\mathcal{I},\tau)$ [24]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I},\tau)$.

If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed [8] (resp. $\star$-dense in itself [6]) if $A^\star \subseteq A$ (resp. $A \subseteq A^\star$). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [3] if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and $int^\star(A)$ will denote the interior of A in (X, $\tau^\star$).

A subset A of a space (X, τ) is an α-open [19] (resp. semi-open [12], preopen [15], regular open [23]) set if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$, $A = int(cl(A))$).

The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by $cl_\alpha(A)$.

Definition 1.1. A subset A of a space (X, τ) is said to be

1. *g*-closed [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
2. *g*-open [13] if its complement is *g*-closed.
3. λ-closed [1] if $A = L \cap D$, where L is a Λ-set and D is a closed set.
4. λ-open [1] if its complement is λ-closed.
5. Λ_g-closed [2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open.
6. $\hat{g}$-closed [25] or ω-closed [22] or $s^\star g$-closed [10, 16, 20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.

Definition 1.2. An ideal $\mathcal{I}$ is said to be

1. codense [4] or τ-boundary [18] if $\tau \cap \mathcal{I}=\{\phi\}$,
2. completely codense [4] if $PO(X) \cap \mathcal{I}=\{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ).

Lemma 1.3. Every completely codense ideal is codense but not conversely [4].

The following Lemmas, Result and Definition will be useful in the sequel.

Lemma 1.4. [8] Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:

1. A⊆B⇒$A^\star$⊆$B^\star$,
2. $A^\star$=cl($A^\star$)⊆cl(A),
3. $(A^\star)^\star \subseteq A^\star$,
4. $(A\cup B)^\star = A^\star \cup B^\star$,
5. $(A\cap B)^\star \subseteq A^\star \cap B^\star$.

Lemma 1.5. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. If A⊆$A^\star$, then $A^\star$=cl($A^\star$)=cl(A) =$cl^\star$(A) [[21], Theorem 5].

Lemma 1.6. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then $\mathcal{I}$ is codense if and only if G⊆$G^\star$ for every semi-open set G in X [[21], Theorem 3].

Lemma 1.7. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^\star \subseteq \tau^\alpha$ [[21], Theorem 6].

Result 1.8. For a subset of a topological space, the following properties hold:

1. Every closed set is Λ_g-closed but not conversely [2].
2. Every Λ_g-closed set is g-closed but not conversely [2].
3. Every closed set is λ-closed but not conversely [1, 2].
4. Every closed set is $\hat{g}$-closed but not conversely [25].
5. Every $\hat{g}$-closed set is g-closed but not conversely [25].

Definition 1.9. An ideal space (X, τ, $\mathcal{I}$) is said to be a $T_{\mathcal{I}}$-space [3] if every $\mathcal{I}_g$-closed subset of X is a $\star$-closed set.

Lemma 1.10. If (X, τ, $\mathcal{I}$) is a T_1-space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[17], Corollary 2.2].

Lemma 1.11. Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[3], Theorem 2.1].

Lemma 1.12. [1] Let $A_i (i \in \mathcal{I})$ be subsets of a topological space (X, τ). The following properties hold:

1. If A_i is λ-closed for each $i \in I$, then $\cap_{i\in I} A_i$ is λ-closed.
2. If A_i is λ-open for each $i \in I$, then $\cup_{i\in I} A_i$ is λ-open.

Recall that the intersection of a λ-closed set and a closed set is λ-closed.

2 Ideal Topological View of Λ_g-closed Sets

Definition 2.1. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be

1. $\mathcal{I}_{\Lambda_g}$-closed if A*⊆U whenever A⊆U and U is λ-open,
2. $\mathcal{I}_{\Lambda_g}$-open if its complement is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.2. If (X, τ, $\mathcal{I}$) is any ideal topological space, then every $\mathcal{I}_{\Lambda_g}$-closed set is $\mathcal{I}_g$-closed but not conversely.

Proof. It follows from the fact that every open set is λ-open.

Example 2.3. Let X={a, b, c}, τ={ϕ, X, {a}, {a, b}} and $\mathcal{I}$={ϕ}. It is clear that {a, c} is $\mathcal{I}_g$-closed but it is not $\mathcal{I}_{\Lambda_g}$-closed.

The following Theorem gives characterizations of $\mathcal{I}_{\Lambda_g}$-closed sets.

Theorem 2.4. If (X, τ, $\mathcal{I}$) is any ideal topological space and A⊆X, then the following are equivalent.

1. A is $\mathcal{I}_{\Lambda_g}$-closed,
2. cl*(A)⊆U whenever A⊆U and U is λ-open in X,
3. cl*(A)−A contains no nonempty λ-closed set,
4. A*−A contains no nonempty λ-closed set.

Proof. (1) ⇒ (2) Let A ⊆ U where U is λ-open in X. Since A is $\mathcal{I}_{\Lambda_g}$-closed, A* ⊆ U and so cl*(A) = A ∪ A* ⊆ U.

(2) ⇒ (3) Let F be a λ-closed subset such that F ⊆ cl*(A)−A. Then F ⊆ cl*(A). Also F ⊆ cl*(A)−A ⊆ X − A and hence A ⊆ X − F where X − F is λ-open. By (2) cl*(A) ⊆ X − F and so F ⊆ X − cl*(A). Thus F ⊆ cl*(A) ∩ X − cl*(A) = ϕ.

(3) ⇒ (4) A* − A = A ∪ A* − A = cl*(A) − A which has no nonempty λ-closed subset by (3).

(4) ⇒ (1) Let A ⊆ U where U is λ-open. Then X − U ⊆ X − A and so A* ∩ (X − U) ⊆ A* ∩ (X − A) = A* − A. Since A* is always a closed subset and X − U is λ-closed, A* ∩ (X − U) is a λ-closed set contained in A* − A and hence A* ∩ (X − U) = ϕ by (4). Thus A* ⊆ U and A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.5. Every $\star$-closed set is $\mathcal{I}_{\Lambda_g}$-closed but not conversely.

Proof. Let A be a $\star$-closed. To prove A is $\mathcal{I}_{\Lambda_g}$-closed, let U be any λ-open set such that A ⊆ U. Since A is $\star$-closed, A* ⊆ A ⊆ U. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.6. Let X={a, b, c}, τ={ϕ, X, {a}} and $\mathcal{I}$={ϕ}. It is clear that {b} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not $\star$-closed.

Theorem 2.7. Let (X, τ, $\mathcal{I}$) be an ideal topological space. For every A∈$\mathcal{I}$, A is $\mathcal{I}_{\Lambda_g}$-closed.

Proof. Let A ∈ $\mathcal{I}$ and let A ⊆ U where U is λ-open. Since A ∈ $\mathcal{I}$, A* = ϕ ⊆ U. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.8. If (X, τ, $\mathcal{I}$) is an ideal topological space, then A* is always $\mathcal{I}_{\Lambda_g}$-closed for every subset A of X.

Proof. Let $A^\star \subseteq U$ where U is λ-open. Since $(A^\star)^\star \subseteq A^\star$ [8], we have $(A^\star)^\star \subseteq U$. Hence $A^\star$ is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.9. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every $\mathcal{I}_{\Lambda_g}$-closed, λ-open set is $\star$-closed.

Proof. Let A be $\mathcal{I}_{\Lambda_g}$-closed and λ-open. We have $A \subseteq A$ where A is λ-open. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq A$. Thus A is $\star$-closed.

Corollary 2.10. If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space and A is an $\mathcal{I}_{\Lambda_g}$-closed set, then A is $\star$-closed set.

Proof. By assumption A is $\mathcal{I}_{\Lambda_g}$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.9, A is $\star$-closed.

Corollary 2.11. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an $\mathcal{I}_{\Lambda_g}$-closed set. Then the following are equivalent.

1. A is a $\star$-closed set,
2. $cl^\star(A) - A$ is a λ-closed set,
3. $A^\star - A$ is a λ-closed set.

Proof. (1) $\Rightarrow$ (2) By (1) A is $\star$-closed. Hence $A^\star \subseteq A$ and $cl^\star(A) - A = (A \cup A^\star) - A = \phi$ which is a λ-closed set.

(2) $\Rightarrow$ (3) $A^\star - A = A \cup A^\star - A = cl^\star(A) - A$ which is a λ-closed set by (2).

(3) $\Rightarrow$ (1) Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.4 $A^\star - A$ contains no non-empty λ-closed set. By assumption (3) $A^\star - A$ is λ-closed and hence $A^\star - A = \phi$. Thus $A^\star \subseteq A$ and A is $\star$-closed.

Theorem 2.12. Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every Λ_g-closed set is an $\mathcal{I}_{\Lambda_g}$-closed set but not conversely.

Proof. Let A be a Λ_g-closed set. Let U be any λ-open set such that $A \subseteq U$. Since A is Λ_g-closed, $cl(A) \subseteq U$. So, by Lemma 1.4, $A^\star \subseteq cl(A) \subseteq U$ and thus A is $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.13. Let X={a, b, c}, τ={ϕ, X, {a}, {c}, {a, b}, {a, c}} and $\mathcal{I}$={ϕ, {a}}. It is clear that {a} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not Λ_g-closed.

Theorem 2.14. If (X, τ, $\mathcal{I}$) is an ideal topological space and A is a $\star$-dense in itself, $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is Λ_g-closed.

Proof. Let $A \subseteq U$ where U is λ-open. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.5, $cl(A) = A^\star$. Thus $cl(A) \subseteq U$ and hence A is Λ_g-closed.

Corollary 2.15. If (X, τ, $\mathcal{I}$) is any ideal topological space where $\mathcal{I}$={ϕ}, then A is $\mathcal{I}_{\Lambda_g}$-closed if and only if A is Λ_g-closed.

Proof. In (X, τ, $\mathcal{I}$), if $\mathcal{I} = \{\phi\}$ then $A^\star = cl(A)$ for the subset A. A is $\mathcal{I}_{\Lambda_g}$-closed $\Leftrightarrow$ $A^\star \subseteq U$ whenever $A \subseteq U$ and U is λ-open $\Leftrightarrow$ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open $\Leftrightarrow$ A is Λ_g-closed.

Corollary 2.16. In an ideal topological space (X, τ, $\mathcal{I}$) where $\mathcal{I}$ is codense, if A is a semi-open and $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is Λ_g-closed.

Proof. By Lemma 1.6, A is $\star$-dense in itself. By Theorem 2.14, A is Λ_g-closed.

Example 2.17. In Example 2.3, it is clear that {a, c} is g-closed set but it is not $\mathcal{I}_{\Lambda_g}$-closed.

Example 2.18. In Example 2.13, it is clear that {a} is $\mathcal{I}_{\Lambda_g}$-closed set but it is not g-closed.

Example 2.19. In Example 2.6, it is clear that {b} is Λ_g-closed but it is not $\hat{g}$-closed.

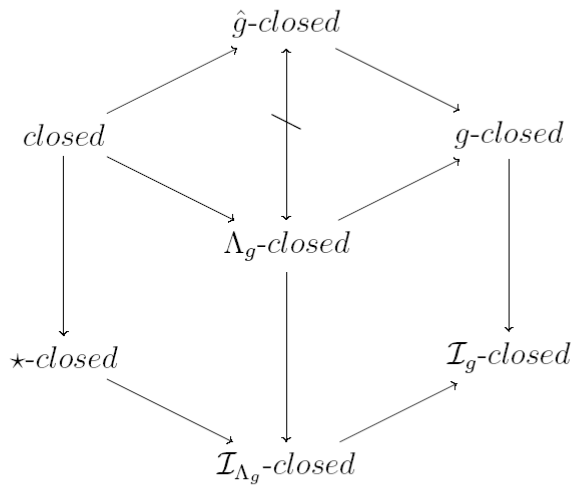
Example 2.20. In Example 2.6, it is clear that {a} is $\hat{g}$-closed but it is not Λ_g-closed.

Remark 2.21. We see that

1. From Examples 2.17 and 2.18, g-closed sets and $\mathcal{I}_{\Lambda_g}$-closed sets are independent.

2. From Examples 2.19 and 2.20, Λ_g-closed sets and $\hat{g}$-closed sets are independent.

Remark 2.22. We have the following implications for the subsets stated above.



Theorem 2.23. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. Then A is $\mathcal{I}_{\Lambda_g}$-closed if and only if A=F−N where F is $\star$-closed and N contains no nonempty λ-closed set.

Proof. If A is $\mathcal{I}_{\Lambda_g}$-closed, then by Theorem 2.4 (4), N=$A^\star$−A contains no nonempty λ-closed set. If F=$cl^\star$(A), then F is $\star$-closed such that F−N=$(A\cup A^\star)-(A^\star-A)=(A\cup A^\star)\cap(A^\star\cap A^c)^c=(A\cup A^\star)\cap((A^\star)^c\cup A)=(A\cup A^\star)\cap(A\cup(A^\star)^c)= A\cup(A^\star\cap(A^\star)^c)=A$.

Conversely, suppose A=F−N where F is $\star$-closed and N contains no nonempty λ-closed set. Let U be an λ-open set such that A⊆U. Then F−N⊆U which implies that F∩(X−U)⊆N. Now A⊆F and $F^\star\subseteq F$ then $A^\star\subseteq F^\star$ and so $A^\star\cap(X-U)\subseteq F^\star\cap(X-U)\subseteq F\cap(X-U)\subseteq N$. Since $A^\star\cap(X-U)$ is λ-closed, by hypothesis $A^\star\cap(X-U)=\phi$ and so $A^\star\subseteq U$. Hence A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.24. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. If $A\subseteq B\subseteq A^\star$, then $A^\star=B^\star$ and B is $\star$-dense in itself.

Proof. Since A⊆B, then $A^\star\subseteq B^\star$ and since $B\subseteq A^\star$, then $B^\star\subseteq(A^\star)^\star\subseteq A^\star$. Therefore $A^\star=B^\star$ and $B\subseteq A^\star\subseteq B^\star$. Hence proved.

Theorem 2.25. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A⊆B⊆cl*(A) and A is $\mathcal{I}_{\Lambda_g}$-closed, then B is $\mathcal{I}_{\Lambda_g}$-closed.

Proof. Since A is $\mathcal{I}_{\Lambda_g}$-closed, then by Theorem 2.4 (3), cl*(A)−A contains no nonempty λ-closed set. But cl*(B)−B⊆cl*(A)−A and so cl*(B)−B contains no nonempty λ-closed set. Hence B is $\mathcal{I}_{\Lambda_g}$-closed.

Corollary 2.26. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A⊆B⊆A* and A is $\mathcal{I}_{\Lambda_g}$-closed, then A and B are Λ_g-closed sets.

Proof. Let A and B be subsets of X such that A⊆B⊆A*. Then A⊆B ⊆A*⊆cl*(A). Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.25, B is $\mathcal{I}_{\Lambda_g}$-closed. Since A⊆B⊆A*, we have A* = B*. Hence A ⊆ A* and B ⊆ B*. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.14, A and B are Λ_g-closed.

The following Theorem gives a characterization of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.27. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. Then A is $\mathcal{I}_{\Lambda_g}$-open if and only if F⊆int*(A) whenever F is λ-closed and F⊆A.

Proof. Suppose A is $\mathcal{I}_{\Lambda_g}$-open. If F is λ-closed and F⊆A, then X−A⊆X−F and so cl*(X−A)⊆X−F by Theorem 2.4(2). Therefore F⊆X−cl*(X−A)=int*(A). Hence F⊆int*(A).

Conversely, suppose the condition holds. Let U be a λ-open set such that X−A⊆U. Then X−U⊆A and so X−U⊆int*(A). Therefore cl*(X−A)⊆U. By Theorem 2.4(2), X−A is $\mathcal{I}_{\Lambda_g}$-closed. Hence A is $\mathcal{I}_{\Lambda_g}$-open.

Corollary 2.28. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. If A is $\mathcal{I}_{\Lambda_g}$-open, then F⊆int*(A) whenever F is closed and F⊆A.

The following Theorem gives a property of $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.29. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. If A is $\mathcal{I}_{\Lambda_g}$-open and int*(A)⊆B⊆A, then B is $\mathcal{I}_{\Lambda_g}$-open.

Proof. Since int*(A) ⊆ B ⊆ A, we have X − A ⊆ X − B ⊆ X − int*(A) = cl*(X − A). By assumption A is $\mathcal{I}_{\Lambda_g}$-open and so X − A is $\mathcal{I}_{\Lambda_g}$-closed. Hence by Theorem 2.25, X − B is $\mathcal{I}_{\Lambda_g}$-closed and B is $\mathcal{I}_{\Lambda_g}$-open.

The following Theorem gives a characterization of $\mathcal{I}_{\Lambda_g}$-closed sets in terms of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.30. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A⊆X. Then the following are equivalent.

1. A is $\mathcal{I}_{\Lambda_g}$-closed,
2. A∪(X−A*) is $\mathcal{I}_{\Lambda_g}$-closed,
3. A*−A is $\mathcal{I}_{\Lambda_g}$-open.

Proof. (1)⇒(2) Let U be any λ-open set such that A∪(X−A*)⊆U. Then $U^c \subseteq [A\cup(X-A^\star)]^c = [A \cup (A^\star)^c]^c = A^\star \cap A^c = A^\star - A$ where U^c is λ-closed. Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Theorem 2.4(4), $U^c = \phi$ and X=U. Thus X is the only λ-open set containing A∪(X−A*) and hence A∪(X−A*) is $\mathcal{I}_{\Lambda_g}$-closed.

(2)⇒(3) $(A^\star - A)^c = (A^\star \cap A^c)^c = A \cup A^{\star c} = A \cup (X - A^\star)$ which is $\mathcal{I}_{\Lambda_g}$-closed by (2). Hence A* − A is $\mathcal{I}_{\Lambda_g}$-open.

(3) ⇒ (1) Since $A^\star - A$ is $\mathcal{I}_{\Lambda_g}$-open, $(A^\star - A)^c = A \cup A^{\star c}$ is $\mathcal{I}_{\Lambda_g}$-closed. Hence by Theorem 2.4(4) $(A \cup (A^\star)^c)^\star - (A \cup A^{\star c})$ contains no nonempty λ-closed subset. But $(A \cup (A^\star)^c)^\star - (A \cup (A^\star)^c) = (A \cup (A^\star)^c)^\star \cap (A \cup (A^\star)^c)^c = (A \cup (A^\star)^c)^\star \cap (A^\star \cup A^c) = (A^\star \cup ((A^\star)^c)^\star) \cap (A^\star \cap A^c) = A^\star \cap A^c = A^\star - A$. Thus $A^\star - A$ has no nonempty λ-closed subset. Hence by Theorem 2.4(4), A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 2.31. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is $\mathcal{I}_{\Lambda_g}$-closed if and only if every λ-open set is $\star$-closed.

Proof. Suppose every subset of X is $\mathcal{I}_{\Lambda_g}$-closed. Let U be λ-open in X. Then $U \subseteq U$ and U is $\mathcal{I}_{\Lambda_g}$-closed by assumption implies $U^\star \subseteq U$. Hence U is $\star$-closed.

Conversely, let $A \subseteq X$ and U be λ-open such that $A \subseteq U$. Since U is $\star$-closed by assumption, we have $A^\star \subseteq U^\star \subseteq U$. Thus A is $\mathcal{I}_{\Lambda_g}$-closed.

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\Lambda_g}$-open sets.

Theorem 2.32. Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.

1. X is normal,
2. For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{\Lambda_g}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,
3. For any closed set A and open set V containing A, there exists an $\mathcal{I}_{\Lambda_g}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.

Proof. (1)⇒(2) The proof follows from the fact that every open set is $\mathcal{I}_{\Lambda_g}$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and $X-V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{\Lambda_g}$-open sets U and W such that $A \subseteq U$ and $X-V \subseteq W$. Since $X-V$ is λ-closed and W is $\mathcal{I}_{\Lambda_g}$-open, $X-V \subseteq int^\star(W)$. Then $X-int^\star(W) \subseteq V$. Again $U \cap W = \phi$ which implies that $U \cap int^\star(W) = \phi$ and so $U \subseteq X- int^\star (W)$. Then $cl^\star(U) \subseteq X-int^\star(W) \subseteq V$ and thus U is the required $\mathcal{I}_{\Lambda_g}$-open sets with $A \subseteq U \subseteq cl^\star(U) \subseteq V$.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and $X - B$ an open set containing A. By hypothesis, there exists an $\mathcal{I}_{\Lambda_g}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq X-B$. Since U is $\mathcal{I}_{\Lambda_g}$-open and A is λ-closed we have $A \subseteq int^\star(U)$. Since $\mathcal{I}$ is completely codense, by Lemma 1.7, $\tau^\star \subseteq \tau^\alpha$ and so $int^\star(U)$ and $X-cl^\star(U) \in \tau^\alpha$. Hence $A \subseteq int^\star(U) \subseteq int(cl(int(int^\star(U)))) = G$ and $B \subseteq X-cl^\star(U) \subseteq int(cl(int(X-cl^\star(U)))) = H$. G and H are the required disjoint open sets containing A and B respectively, which proves (1).

Definition 2.33. A subset A of a topological space (X, τ) is said to be an $\Lambda_{g\alpha}$-closed set if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open. The complement of $\Lambda_{g\alpha}$-closed is said to be an $\Lambda_{g\alpha}$-open set.

If $\mathcal{I}=\mathcal{N}$, it is not difficult to see that $\mathcal{I}_{\Lambda_g}$-closed sets coincide with $\Lambda_{g\alpha}$-closed sets and so we have the following Corollary.

Corollary 2.34. Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}=\mathcal{N}$. Then the following are equivalent.

1. X is normal,

2. For any disjoint closed sets A and B, there exist disjoint $\Lambda_{g\alpha}$-open sets U and V such that A⊆U and B⊆V,

3. For any closed set A and open set V containing A, there exists an $\Lambda_{g\alpha}$-open set U such that A⊆U⊆cl_α(U)⊆V.

Definition 2.35. A subset A of an ideal topological space is said to be $\mathcal{I}$-compact [5] or compact modulo $\mathcal{I}$ [18] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A - \cup\{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space (X, τ, $\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.

Theorem 2.36. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[17], Theorem 2.17].

Corollary 2.37. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_{\Lambda_g}$-closed subset of X, then A is $\mathcal{I}$-compact.

Proof. The proof follows from the fact that every $\mathcal{I}_{\Lambda_g}$-closed is $\mathcal{I}_g$-closed.

3 λ-$\mathcal{I}$-locally Closed Sets

Definition 3.1. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a λ-$\mathcal{I}$-locally closed set (briefly, λ-$\mathcal{I}$-LC) if A=U∩V where U is λ-open and V is $\star$-closed.

Definition 3.2. [9] A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a weakly $\mathcal{I}$-locally closed set (briefly, weakly $\mathcal{I}$-LC) if A=U∩V where U is open and V is $\star$-closed.

Proposition 3.3. Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.

1. If A is λ-open, then A is λ-$\mathcal{I}$-LC-set.

2. If A is $\star$-closed, then A is λ-$\mathcal{I}$-LC-set.

3. If A is a weakly $\mathcal{I}$-LC-set, then A is a λ-$\mathcal{I}$-LC-set.

The converses of the above Proposition 3.3 need not be true as shown in the following examples.

Example 3.4.

1. In Example 2.6, it is clear that {a} is a λ-$\mathcal{I}$-LC-set but it is not $\star$-closed.

2. In Example 2.3, it is clear that {b} is a λ-$\mathcal{I}$-LC-set but it is not λ-open.

Example 3.5. In Example 2.3, it is clear that {a, c} is a λ-$\mathcal{I}$-LC-set but it is not a weakly $\mathcal{I}$-LC-set.

Theorem 3.6. Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a λ-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A∩B is a λ-$\mathcal{I}$-LC-set.

Proof. Let B be $\star$-closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is $\star$-closed. Hence A∩B is a λ-$\mathcal{I}$-LC-set.

Theorem 3.7. A subset of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed if and only if it is

1. weakly $\mathcal{I}$-LC and $\mathcal{I}_g$-closed [7]
2. λ-$\mathcal{I}$-LC and $\mathcal{I}_{\Lambda_g}$-closed.

Proof. (2) Necessity is trivial. We prove only sufficiency. Let A be λ-$\mathcal{I}$-LC-set and $\mathcal{I}_{\Lambda_g}$-closed set. Since A is λ-$\mathcal{I}$-LC, A=U∩V, where U is λ-open and V is $\star$-closed. So, we have A=U∩V⊆U. Since A is $\mathcal{I}_{\Lambda_g}$-closed, $A^\star \subseteq$ U. Also since A = U∩V⊆V and V is $\star$-closed, we have $A^\star \subseteq$ V. Consequently, $A^\star \subseteq$U∩V = A and hence A is $\star$-closed.

Remark 3.8.
1. The notions of weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed set are independent [7].
2. The notions of λ-$\mathcal{I}$-LC-set and $\mathcal{I}_{\Lambda_g}$-closed set are independent.

Example 3.9. In Example 2.6, it is clear that {a} is λ-$\mathcal{I}$-LC-set but not $\mathcal{I}_{\Lambda_g}$-closed.

Example 3.10. In Example 2.6, it is clear that {a, c} is $\mathcal{I}_{\Lambda_g}$-closed set but not λ-$\mathcal{I}$-LC-set.

Definition 3.11. Let A be a subset of a topological space (X, τ). Then the λ-kernel of the set A, denoted by λ-ker(A), is the intersection of all λ-open supersets of A.

Definition 3.12. A subset A of a topological space (X, τ) is called Λ_λ-set if A=λ-ker(A).

Definition 3.13. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called $\lambda^\star$-$\mathcal{I}$-closed if A=L∩F where L is a Λ_λ-set and F is $\star$-closed.

Lemma 3.14.
1. Every $\star$-closed set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.
2. Every Λ_λ-set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.
3. Every λ-$\mathcal{I}$-LC-set is $\lambda^\star$-$\mathcal{I}$-closed but not conversely.

Example 3.15. In Example 2.6, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed set but not $\star$-closed.

Example 3.16. Let X = {a, b, c, d}, τ = {ϕ, X, {c}, {d}, {a, c}, {c, d}, {a, c, d}} and $\mathcal{I}$ = {ϕ}. It is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not a Λ_λ-set.

Example 3.17. In Example 3.16, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not a λ-$\mathcal{I}$-LC-set.

Remark 3.18. The following Example supports the concepts of Λ_λ-set and $\star$-closed set are independent. Let X = {a, b, c}, τ = {ϕ, X, {b, c}} and $\mathcal{I}$ = {ϕ, {a}, {b}, {a, b}}. It is clear that {b, c} is a Λ_λ-set but not a $\star$-closed whereas {b} is $\star$-closed but not a Λ_λ-set.

Lemma 3.19. For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.

1. A is $\lambda^\star$-$\mathcal{I}$-closed.
2. A=L∩$cl^\star$(A) where L is a Λ_λ-set.
3. A=λ-ker(A)∩$cl^\star$(A).

Lemma 3.20. A subset A⊆(X, τ, $\mathcal{I}$) is $\mathcal{I}_{\Lambda_g}$-closed if and only if $cl^\star$(A)⊆ λ-ker(A).

Proof. Suppose that A $\subseteq$ X is an $\mathcal{I}_{\Lambda_g}$-closed set. Suppose x $\notin$ λ-ker(A). Then there exists an λ-open set U containing A such that x $\notin$ U. Since A is an $\mathcal{I}_{\Lambda_g}$-closed set, A $\subseteq$ U and U is λ-open implies that cl*(A) $\subseteq$ U and so x $\notin$ cl*(A). Therefore cl*(A) $\subseteq$ λ-ker(A).

Conversely, suppose cl*(A) $\subseteq$ λ-ker(A). If A $\subseteq$ U and U is λ-open, then cl*(A) $\subseteq$ λ-ker(A) $\subseteq$ U. Therefore, A is $\mathcal{I}_{\Lambda_g}$-closed.

Theorem 3.21. For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.

1. A is $\star$-closed.
2. A is $\mathcal{I}_{\Lambda_g}$-closed and λ-$\mathcal{I}$-LC.
3. A is $\mathcal{I}_{\Lambda_g}$-closed and $\lambda^\star$-$\mathcal{I}$-closed.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) Obvious.

(3)$\Rightarrow$(1) Since A is $\mathcal{I}_{\Lambda_g}$-closed, by Lemma 3.20, cl*(A)$\subseteq$$\lambda$-ker(A). Since A is $\lambda^\star$-$\mathcal{I}$-closed, by Lemma 3.19, A=λ-ker(A)$\cap$cl*(A)=cl*(A). Hence A is $\star$-closed.

The following two Examples show that the concepts of $\mathcal{I}_{\Lambda_g}$-closedness and $\lambda^\star$-$\mathcal{I}$-closedness are independent.

Example 3.22. In Example 2.6, it is clear that {b} is $\mathcal{I}_{\Lambda_g}$-closed set but not $\lambda^\star$-$\mathcal{I}$-closed.

Example 3.23. In Example 2.6, it is clear that {a} is $\lambda^\star$-$\mathcal{I}$-closed but not $\mathcal{I}_{\Lambda_g}$-closed.

4 Decompositions of $\star$-continuity

Definition 4.1. A function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ) is said to be $\star$-continuous [7] (resp. $\mathcal{I}_g$-continuous [7], λ-$\mathcal{I}$-LC-continuous, $\lambda^\star$-$\mathcal{I}$-continuous, $\mathcal{I}_{\Lambda_g}$-continuous, weakly $\mathcal{I}$-LC-continuous [9]) if f^{-1}(A) is $\star$-closed (resp. $\mathcal{I}_g$-closed, λ-$\mathcal{I}$-LC-set, $\lambda^\star$-$\mathcal{I}$-closed, $\mathcal{I}_{\Lambda_g}$-closed, weakly $\mathcal{I}$-LC-set) in (X, τ, $\mathcal{I}$) for every closed set A of (Y, σ).

Theorem 4.2. A function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ) is $\star$-continuous if and only if it is

1. weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [7].
2. λ-$\mathcal{I}$-LC-continuous and $\mathcal{I}_{\Lambda_g}$-continuous.

Proof. It is an immediate consequence of Theorem 3.7.

Theorem 4.3. For a function f : (X, τ, $\mathcal{I}$)$\rightarrow$(Y, σ), the following are equivalent.

1. f is $\star$-continuous.
2. f is $\mathcal{I}_{\Lambda_g}$-continuous and λ-$\mathcal{I}$-LC-continuous.
3. f is $\mathcal{I}_{\Lambda_g}$-continuous and $\lambda^\star$-$\mathcal{I}$-continuous.

Proof. It is an immediate consequence of Theorem 3.21.

References

[1] F. G. Arenas, J. Dontchev and M. Ganster, On λ-sets and dual of generalized continuity, Questions Answer Gen. Topology, 15(1997), 3-13.

[2] M. Caldas, S. Jafari and T. Noiri, On Λ-generalized closed sets in topological spaces, Acta Math. Hungar., 118(4)(2008), 337-343.

[3] J. Dontchev, M. Ganster and T. Noiri, Unified operation approach of generalized closed sets via topological ideals, Math. Japonica, 49(1999), 395-401.

[4] J. Dontchev, M. Ganster and D. Rose, Ideal resolvability, Topology and its Applications, 93(1999), 1-16.

[5] T. R. Hamlett and D. Jankovic, Compactness with respect to an ideal, Boll. U. M. I., (7) 4-B(1990), 849-861.

[6] E. Hayashi, Topologies defined by local properties, Math. Ann., 156(1964), 205-215.

[7] V. Inthumathi, S. Krishnaprakash and M. Rajamani, Strongly-$\mathcal{I}$-Locally closed sets and decompositions of $\star$-continuity, Acta Math. Hungar., 130(4)(2011), 358-362.

[8] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(4)(1990), 295-310.

[9] A. Keskin, S. Yuksel and T. Noiri, Decompositions of $\mathcal{I}$-continuity and continuity, Commun. Fac. Sci. Univ. Ank. Series A, 53(2004), 67-75.

[10] M. Khan, T. Noiri and M. Hussain, On s*g-closed sets and s*-normal spaces, J. Natur. Sci. Math., 48(1-2)(2008), 31-41.

[11] K. Kuratowski, Topology, Vol. I, Academic Press (New York, 1966).

[12] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.

[13] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), 19(1970), 89-96.

[14] H. Maki, Generalized Λ-sets and the associated closure operator, The special issue in commemoration of Prof. Kazusada IKEDA' Retirement, 1. Oct. (1986), 139-146.

[15] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53.

[16] M. Murugalingam, A study of semi generalized topology, Ph.D Thesis, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India, (2005).

[17] M. Navaneethakrishnan and J. Paulraj Joseph, g-closed sets in ideal topological spaces, Acta Math. Hungar., 119(4)(2008), 365-371.

[18] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph.D Dissertation, Univ. of Cal. at Santa Barbara (1967).

[19] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(1965), 961-970.

[20] K. C. Rao and K. Joseph, Semi-star generalized closed sets, Bull. Pure Appl. Sci., 19(E)(2)(2002), 281-290.

[21] V. Renuka Devi, D. Sivaraj and T. Tamizh Chelvam, Codense and Completely codense ideals, Acta Math. Hungar., 108(2005), 197-205.

[22] M. Sheik John, A study on generalizations of closed sets and continuous maps in topological and bitopological spaces, Ph.D Thesis, Bharathiar University, Coimbatore, (2002).

[23] M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 375-481.

[24] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company (1946).

[25] M. K. R. S. Veerakumar, $\hat{g}$-closed sets in topological spaces, Bull. Allah. Math. Soc., 18(2003), 99-112.